UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Algebraic Topology
Aug 18, 2020.

Instructions. Answer as many questions as you can. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a low pass you need to solve *completely* at least two problems and score at least 25 points.

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1. Let X be a connected cell complex and $f : \mathbb{R}P^{2n} \to X$ a covering map from the real projective space of dimension $2n$. Show that f is a homeomorphism.

2. Let $x_0 \in \mathbb{R}P^2$ be a basepoint and view the wedge $\mathbb{R}P^2 \vee \mathbb{R}P^2$ as the subspace of the product $\mathbb{R}P^2 \times \mathbb{R}P^2$ where at least one coordinate is x_0.
   (a) Use the Seifert-van Kampen theorem to compute $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$.
   (b) Prove that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is not a retract of $\mathbb{R}P^2 \times \mathbb{R}P^2$.

3. Describe a cell structure on the real projective space $\mathbb{R}P^n$. For each cell define its attaching map.

4. Recall that the *mapping torus* of a map $f : X \to X$ is the quotient space T_f obtained from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for every $x \in X$. Compute the homology groups of T_f when $f : S^n \to S^n$ is a map of degree d and $n > 1$.

5. Let G be a finite group of order d and $\phi : F_n \to G$ an epimorphism from the free group of rank n. Show that $Ker(\phi)$ is a free group and compute its rank.

6. Let S_g denote an orientable surface of genus g. So for example S_0 is the 2-sphere and S_1 is the torus.
   (a) If $g > 0$ show that every map $S_0 \to S_g$ is null-homotopic.
   (b) If $g > h > 0$ show that every map $S_h \to S_g$ has degree 0, but show by example that there exist maps that are not null-homotopic.