Problem 1: The Water Dilemma

I have 2 containers of water, each with identical volume V. Container A is at a temperature of 273 K, and Container B is at a temperature of 373 K.

I want to cool Container B to as low of a temperature as possible.

There are 2 tools that I can use to cool Container B:

1) **I can split Container A into any finite number of sub containers {A1, A2, … , An} with volumes {V1, V2, …, Vn} such that $\sum_{i=1}^{n} V_n = V$**

2) **I can bring any sub container of A into thermal equilibrium with Container B and then separate them.**

For example, I can split A into 2 containers of equal volume V/2. Then bring one of these sub containers into contact with Container B, and allow them to reach equilibrium (which is ~339.667 K). I can then separate them, and bring the second sub container into contact with Container B. This will lower Container B's temperature to ~317.444 K.

To how low of a Temperature can Container B be cooled?

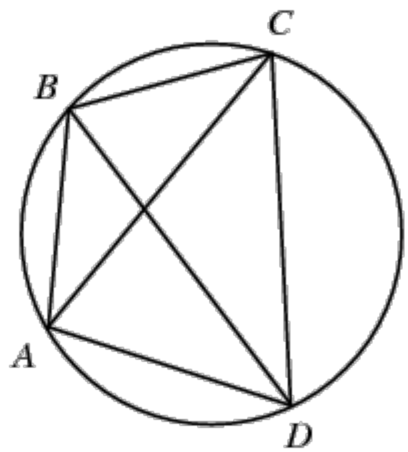
Note: The Equilibrium Temperature T_f reached when 2 boxes are placed in thermal equilibrium is a volume-weighted average of the temperature of both containers, following the equation

$$T_f = \frac{T_A V_A + T_b V_b}{V_T}$$

This formula assumes a constant specific heat of water between 273- and 373-degrees Kelvin. (which you can also, definitely assume for this problem!)

Problem 2: Ptolemy's Theorem

In Euclidean geometry, Ptolemy's theorem is a relation between the four sides and two diagonals of a cyclic quadrilateral (a quadrilateral whose vertices lie on a common circle). The Theorem states that for any cyclic quadrilateral ABCD, the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of the opposite sides:



$$AC * BD = AB * CD + AD * BC$$

Prove that Ptolemy's Theorem is true and show that the Pythagorean Theorem follows as a direct result.

Problem 3: The Mean Problem

Prove the Arithmetic Mean – Geometric Mean (AM-GM) Inequality, which states:

If $\boldsymbol{x_1, x_2, \dots, x_n \geq 0\ ,\ x \in \mathbb{R}\ ,\ n \in \mathbb{N}\ \ then}$

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}$$

With equality if and only if $\boldsymbol{x_1 = x_2 = \cdots = x_n}$