Math 2280 - Assignment 3 Solutions

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Section 2.3 - 1, 2, 4, 10, 24

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Section 3.1 - 1, 16, 18, 24, 30

Section 2.3 - Acceleration-Velocity Models

2.3.1 The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If the machine can accelerate from rest to 100 km/h in 10s, how long will it take for the car to accelerate from rest to 200 km/h?

Solution - The differential equation governing the car's movement will be:

$$\frac{dv}{dt} = k(250 - v).$$

This is a separable differential equation. We can rewrite it as:

$$\frac{dv}{250 - v} = kdt.$$

Integrating both sides of this equation we get:

$$\int \frac{dv}{250 - v} = \int kdt.$$

$$\Rightarrow -\ln(250 - v) = kt + C$$

$$\Rightarrow 250 - v = Ce^{-kt}$$

$$\Rightarrow v(t) = 250 - Ce^{-kt}.$$

Using the initial condition $v(0) = 0 = 250 - C$ we get $C = 250$. Using the given $v(10) = 100$ we get:

$$v(10) = 250(1 - e^{-10k}) = 100$$

$$\Rightarrow 1 - e^{-10k} = \frac{2}{5}$$

$$\Rightarrow \ln(e^{-10k}) = \ln\left(\frac{3}{5}\right)$$

$$\Rightarrow k = \frac{\ln 5 - \ln 3}{10} \approx .05108.$$

Using this value of k we want to find the value of t for which $v(t) = 200$. We do this by solving:

$$200 = 250(1 - e^{-.05108t_*})$$

$$\Rightarrow \frac{4}{5} = 1 - e^{-.05108t_*}$$

$$\Rightarrow e^{-.05108t_*} = \frac{1}{5}$$

$$\Rightarrow t_* = \frac{\ln 5}{.05108} \approx 31.5 \text{ seconds.}$$

2.3.2 Suppose that a body moves through a resisting medium with resistance proportional to its velocity v, so that $dv/dt = -kv$.

(a) Show that its velocity and position at time t are given by

$$v(t) = v_0 e^{-kt}$$

and

$$x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt}).$$

(b) Conclude that the body travels only a finite distance, and find that distance.

Solution

(a) - The differential equation

$$\frac{dv}{dt} = -kv$$

is separable, and can be rewritten as

$$\frac{dv}{v} = -kdt.$$

If we integrate both sides of the above differential equation we get:

$$\ln v = -kt + C$$

$$\Rightarrow v(t) = Ce^{-kt}.$$

Using the initial value $v(0) = v_0 = C$ we get:

$$v(t) = v_0 e^{-kt}.$$

Integrating this function to get the position function gives us:

$$x(t) = -\frac{v_0}{k}e^{-kt} + C_*.$$

Using $x(0) = x_0 = -\frac{v_0}{k} + C_*$ we get $C_* = x_0 + \frac{v_0}{k}$. This gives us:

$$x(t) = x_0 + \frac{v_0}{k} - \frac{v_0}{k}e^{-kt} = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt}).$$

(b) - If we take the limit of our position function as $t \to \infty$ we get:

$$\lim_{t\to\infty} x(t) = x_0 + \frac{v_0}{k}(1 - e^{-k\infty}) = x_0 + \frac{v_0}{k}.$$

2.3.4 Consider a body that moves horizontally through a medium whose resistance is proportional to the *square* of the velocity v, so that

$$dv/dt = -kv^2.$$

Show that

$$v(t) = \frac{v_0}{1 + v_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1 + v_0 kt).$$

Note that, in contrast with the result of Problem 2, $x(t) \to \infty$ as $t \to \infty$. Which offers less resistance when the body is moving fairly slowly - the medium in this problem or the one in Problem 2? Does your answer seem to be consistent with the observed behaviors of $x(t)$ as $t \to \infty$?

Solution - The differential equation

$$\frac{dv}{dt} = -kv^2$$

is separable. We can rewrite it as:

$$\frac{dv}{v^2} = -kdt.$$

Integrating both sides of this equation gives us:

$$-\frac{1}{v} = -kt + C.$$

Solving for $v(t)$ gives us:

$$v(t) = \frac{1}{kt + C}.$$

Using the initial condition $v(0) = v_0 = \frac{1}{C}$ we have $C = \frac{1}{v_0}$. Plugging this into our velocity equation gives us:

$$v(t) = \frac{1}{kt + \frac{1}{v_0}} = \frac{v_0}{1 + v_0 kt}.$$

Integrating this we get:

$$x(t) = \int \frac{v_0}{1 + v_0 kt} dt = \frac{v_0}{kv_0} \ln(1 + v_0 kt) + C = C + \frac{1}{k} \ln(1 + v_0 kt).$$

Our initial condition $x(0) = x_0 = C$ gives us:

$$x(t) = x_0 + \frac{1}{k} \ln(1 + v_0 kt).$$

Indeed $\lim_{t \to \infty} x(t) = \infty$.

For $|v| < 1$ we have $v^2 < |v|$ and so the drag is *smaller* for fairly small values of v. This is why the distance can go forever and is not finite.

2.3.10 A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance $\rho v \; ft/s^2$, taking $\rho = .15$ without the parachute and $\rho = 1.5$ with the parachute. (*Suggestion*: First determine her height above the ground and velocity when the parachute opens.)

Solution - We have:

$$v(t) = \left(v_0 + \frac{g}{\rho} \right) e^{-\rho t} - \frac{g}{\rho}.$$

If we integrate this to find $x(t)$ we get:

$$x(t) = -\frac{g}{\rho}t - \frac{1}{\rho}\left(v_0 + \frac{g}{\rho} \right) e^{-\rho t} + C.$$

Plugging in the initial condition $x(0) = x_0$ we get:

$$C = x_0 + \frac{1}{\rho}\left(v_0 + \frac{g}{\rho} \right) = x_0 + \frac{1}{\rho}(v_0 - v_\tau).$$

Using this value for C after a little algebra our equation for $x(t)$ becomes:

$$x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

Now, the initial distance is $x_0 = 10,000$ ft, the initial velocity is $v_0 = 0$ ft/s, the terminal velocity is $v_\tau = -\dfrac{32.2}{.15}$ ft/s, and $\rho = .15$/s. The total distance traveled in the first 20 seconds is:[1]

[1]Leaving out units on the intermediate steps. Trust me, they work out.

$$x(20) = 10,000 - \left(\frac{32.2}{.15}\right)(20) = \frac{1}{.15}\left(0 + \frac{32.2}{.15}\right)(1 - e^{-.15(20)})$$
$$\approx 7,067 ft.$$

The velocity of the skydiver after 20 seconds is:

$$v(20) = \left(0 + \frac{32.2}{.15}\right)e^{-.15(20)} - \frac{32.2}{.15} \approx 204 \text{ ft/s.}$$

Now, to find the total time for the rest of the trip down we want to solve for t_f in the equation:

$$0 = 7,067 - \left(\frac{32.2}{1.5}\right)t_f + \left(\frac{1}{1.5}\right)\left(-204 + \frac{32.2}{1.5}\right)(1 - e^{-1.5t_f}).$$

Using a calculator to find this we get:

$$t_f \approx 340 \text{ seconds.}$$

So, the total skydive time is about $340s + 20s = 360s$, or about 6 minutes.

2.3.24 The mass of the sun is 329,320 times that of the earth and its radius is 109 times the radius of the earth.

(a) To what radius (in meters) would the earth have to be compressed in order for it to become a *black hole* - the escape velocity from its surface equal to the velocity $c = 3 \times 10^8 m/s$ of light?

(b) Repeat part (a) with the sun in place of the earth.

Solution -

(a) - We have $3 \times 10^8 \dfrac{m}{s} = \sqrt{\dfrac{2GM_e}{R}}$.

Solving this for R we get:

$$R = \frac{2GM_e}{c^2} = \frac{2\left(6.67 \times 10^{-11}\frac{Nm^2}{kg^2}\right)(5.972 \times 10^{24}kg)}{\left(3 \times 10^8\frac{m}{s}\right)^2}$$

$$= .00885m = .885cm.$$

WOW!

(b) - Using the same equation we get:

$$R = \frac{2GM_s}{c^2} = \frac{2GM_e}{c^2}(329,320) \approx 2,915m = 2.915km.$$

Section 2.4 - Numerical Approximation: Euler's Method

2.4.1 Apply Euler's method twice to approximate the solution to the initial value problem below on the interval $[0, \frac{1}{2}]$, first with step size $h = .25$, then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = \frac{1}{2}$ with the value $y(\frac{1}{2})$ of the actual solution, also given below.

$$y' = -y,$$

$$y(0) = 2;$$

$$y(x) = 2e^{-x}.$$

Solution - If we apply Euler's method with a step size of $h = .25$ we get:

$$y_0 = 2 \qquad x_0 = 0,$$

$$y_1 = y_0 + h * f(x_0, y_0) = 2 + (.25)(-2) = \frac{3}{2} \qquad x_1 = .25,$$

$$y_2 = y_1 + h * f(x_1, y_1) = \frac{3}{2} + (.25)\left(-\frac{3}{2}\right) = \frac{9}{8} \qquad x_2 = .5.$$

If we apply Euler's method with a step size of $h = .1$ we get:

$$y_0 = 2 \qquad x_0 = 0,$$

$$y_1 = 2 + (.1)(-2) = 1.8 \qquad x_1 = .1,$$

$$y_2 = 1.8 + (.1)(-1.8) = 1.62 \qquad x_2 = .2,$$

$$y_3 = 1.62 + (.1)(1.62) = 1.458 \qquad x_3 = .3,$$

$$y_4 = 1.458 + (.1)(-1.458) = 1.3122 \qquad x_4 = .4,$$

$$y_5 = 1.3122 + (.1)(-1.3122) = 1.18098 \qquad x_5 = .5.$$

The exact value is $y(.5) = 2e^{-.5} = 1.213$. So, with $h = .25$ our estimate is off by $.088$, and with $h = .1$ our estimate is off by $.032$. So, $h = .1$ gives a better estimate, which is what we'd expect.

2.4.5 Apply Euler's method twice to approximate the solution to the initial value problem below on the interval $[0, \frac{1}{2}]$, first with step size $h = .25$, then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = \frac{1}{2}$ with the value $y(\frac{1}{2})$ of the actual solution, also given below.

$$y' = y - x - 1,$$

$$y(0) = 1;$$

$$y(x) = 2 + x - e^x.$$

Solution - If we apply Euler's method with a step size of $h = .25$ we get:

$$y_0 = 1 \qquad\qquad x_0 = 0,$$

$$y_1 = y_0 + h * f(x_0, y_0) = 1 + (.25)(1 - 0 - 1) = 1$$
$$x_1 = .25,$$

$$y_2 = y_1 + h * f(x_1, y_1) = 1 + (.25)(1 - .25 - 1) = \frac{15}{16}$$
$$x_2 = .5.$$

If we apply Euler's method with a step size of $h = .1$ we get:

$$y_0 = 1 \qquad\qquad x_0 = 0,$$

$$y_1 = 2 + (.1)(1 - 0 - 1) = 1 \qquad\qquad x_1 = .1,$$

$$y_2 = 1 + (.1)(1 - .1 - 1) = .99 \qquad\qquad x_2 = .2,$$

$$y_3 = .99 + (.1)(.99 - .2 - 1) = .969 \qquad\qquad x_3 = .3,$$

$$y_4 = .969 + (.1)(.969 - .3 - 1) = .9359 \qquad\qquad x_4 = .4,$$

$$y_5 = .9359 + (.1)(.9359 - .4 - 1) = .88949 \qquad\qquad x_5 = .5.$$

The exact value is $y(.5) = 2 + .5 - e^{.5} = .851$. So, with $h = .25$ our estimate is off by $.087$, and with $h = .1$ our estimate is off by $.038$.

2.4.9 Apply Euler's method twice to approximate the solution to the initial value problem below on the interval $[0, \frac{1}{2}]$, first with step size $h = .25$, then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = \frac{1}{2}$ with the value $y(\frac{1}{2})$ of the actual solution, also given below.

$$y' = \frac{1}{4}(1 + y^2),$$

$$y(0) = 1;$$

$$y(x) = \tan\left(\frac{1}{4}(x + \pi)\right).$$

Solution - If we apply Euler's method with a step size of $h = .25$ we get:

$$y_0 = 1 \qquad x_0 = 0,$$

$$y_1 = y_0 + h * f(x_0, y_0) = 1 + (.25)(.25(1 + 1^2)) = \frac{9}{8}$$
$$x_1 = .25,$$

$$y_2 = y_1 + h * f(x_1, y_1) = \frac{9}{8} + (.25)(.25(1 + (\frac{9}{8})^2)) = 1.27$$
$$x_2 = .5.$$

If we apply Euler's method with a step size of $h = .1$ we get:

$$y_0 = 1 \qquad x_0 = 0,$$

$$y_1 = 1 + (.1)(.25(1 + 1^2)) = 1.05 \qquad x_1 = .1,$$

$$y_2 = 1.05 + (.1)(.25(1 + 1.05^2)) = 1.10 \qquad x_2 = .2,$$

$$y_3 = 1.10 + (.1)(.25(1 + 1.10^2)) = 1.158 \qquad x_3 = .3,$$

$$y_4 = 1.158 + (.1)(.25(1 + 1.158^2)) = 1.217 \qquad x_4 = .4,$$

$$y_5 = 1.217 + (.1)(.25(1 + 1.217^2)) = 1.279 \qquad x_5 = .5.$$

The exact value is $y(.5) = \tan\left(\frac{1}{4}(.5 + \pi)\right) = 1.287$. So, with $h = .25$ our estimate is off by $.017$, and with $h = .1$ our estimate is off by $.008$. Pretty close!

2.4.26 Suppose the deer population $P(t)$ in a small forest initially numbers 25 and satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2$$

(with t in months.) Use Euler's method with a programmable calculator or computer to approximate the solution for 10 years, first with step size $h = 1$ and then with $h = .5$, rounding off approximate P-values to integrals numbers of deer. What percentage of the limiting population of 75 deer has been attained after 5 years? After 10 years?

Solution - With $h = 1$ month we get the values:

t	P	Rounded P
0	25	25
1 Month	25.375	25
2 Months	25.753	26
$\vdots$	$\vdots$	$\vdots$
1 year	29.667	30
$\vdots$	$\vdots$	$\vdots$
5 years	49.389	49
$\vdots$	$\vdots$	$\vdots$
10 years	66.180	66

With $h = .5$ month we get the values:

t	P	Rounded P
0	25	25
1 Month	25.376	25
2 Months	25.755	26
$\vdots$	$\vdots$	$\vdots$
1 year	29.675	30
$\vdots$	$\vdots$	$\vdots$
5 years	49.390	49
$\vdots$	$\vdots$	$\vdots$
10 years	66.235	66

The percentage change after 5 years is 66%. The percentage change after 10 years is 88%. Note that $h = 1$ and $h = .5$ are almost identical.

2.4.30 Apply Euler's method with successively smaller step sizes on the interval $[0, 2]$ to verify empirically that the solution of the initial value problem

$$\frac{dy}{dx} = y^2 + x^2, \quad y(0) = 0$$

has vertical asymptote near $x = 2.003147$.

Solution - The estimates for successively smaller step sizes are:

x-value	y-values			
	$h = .5$	$h = .1$	$h = .01$	$h = .001$
.5	0	.030	.041	.042
1	.125	.401	.344	.350
1.5	.633	1.213	1.479	1.518
2	1.958	5.842	28.393	142.627

We see as h gets small when we approach $x = 2$ we get *very* large y-values. This is caused by the asymptote. Note: I had to write a Java program to do this.

Section 3.1 - Second-Order Linear Equations

3.1.1 Verify that the functions y_1 and y_2 given below are solutions to the second-order ODE also given below. Then, find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$y'' - y = 0;$$

$$y_1 = e^x \quad y_2 = e^{-x};$$

$$y(0) = 0 \quad y'(0) = 5.$$

Solution - We first verify that the two functions we're given are, in fact, solutions to the ODE. For the first function we have:

$$y_1 = e^x,$$

$$y_1' = e^x,$$

$$y_1'' = e^x.$$

Plugging these into the ODE we get:

$$y_1'' - y_1 = e^x - e^x = 0.$$

So, it checks out. As for the second function we have:

$$y_2 = e^{-x},$$

$$y_2' = -e^{-x},$$

$$y_2'' = e^{-x}.$$

Plugging these into the ODE we get:

$$y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

So, it checks out too. Combining these functions we have:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^x + e_2 e^{-x},$$

$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x) = c_1 e^x - c_2 e^{-x}.$$

Using our initial conditions we have:

$$y(0) = 0 = c_1 + c_2,$$

$$y'(0) = 5 = c_1 - c_2.$$

Solving these for c_1 and c_2 we get $c_1 = \frac{5}{2}$, and $c_2 = -\frac{5}{2}$. So, the solution to the initial value problem is:

$$y(x) = \frac{5}{2} e^x - \frac{5}{2} e^{-x}.$$

3.1.16 Verify that the functions y_1 and y_2 given below are solutions to the second-order ODE also given below. Then, find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$x^2y'' + xy' + y = 0;$$

$$y_1 = \cos(\ln x), \quad y_2 = \sin(\ln x);$$

$$y(1) = 2, \quad y'(1) = 3.$$

Solution - We first need to verify that the given functions are, in fact, solutions to the ODE. For our first function we have:

$$y_1 = \cos(\ln x),$$

$$y_1' = -\frac{1}{x}\sin(\ln x),$$

$$y_1'' = \frac{1}{x^2}\sin(\ln x) - \frac{1}{x^2}\cos(\ln x).$$

Plugging these into our ODE we get:

$$x^2y_1'' + xy_1' + y = \sin(\ln x) - \cos(\ln x) - \sin(\ln x) + \cos(\ln x) = 0.$$

So, y_1 checks out. As for y_2 we have:

$$y_2 = \sin(\ln x),$$

$$y_2' = \frac{1}{x}\cos(\ln x),$$

$$y_2'' = -\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x).$$

Plugging these into our ODE we get:

$$x^2y_2'' + xy_2' + y_2 = -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) = 0.$$

So, y_2 checks out as well. Our general solution will be of the form:

$$y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x),$$

and so

$$y'(x) = -\frac{c_1}{x}\sin(\ln x) + \frac{c_2}{x}\cos(\ln x).$$

If we plug in the initial conditions we get:

$$y(1) = c_1 = 2,$$

$$y'(1) = c_2 = 3.$$

So, the solution to our initial value problem is:

$$y(x) = 2\cos(\ln x) + 3\sin(\ln x).$$

3.1.18 Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution.

Solution - We first check that $y = x^3$ is indeed a solution. It's derivative is $y' = 3x^2$, and its second derivative is $y'' = 6x$, so

$$yy'' = x^3(6x) = 6x^4.$$

So, it checks out. Now, suppose $y = cx^3$. Then its first and second derivatives are $y' = 3cx^2$ and $y'' = 6cx$, and therefore

$$yy'' = 6c^2x^4.$$

This is only a solution if $c^2 = 1$, so $c = \pm 1$.

3.1.24 Determine if the functions

$$f(x) = \sin^2 x, \quad g(x) = 1 - \cos(2x)$$

are linearly dependent on the real line $\mathbb{R}$.

Solution - If we use the trig identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

we see $2f(x) = g(x)$. So, $f(x)$ and $g(x)$ are linearly dependent.

3.1.30 **(a)** Show that $y_1 = x^3$ and $y_2 = |x^3|$ are linearly independent solutions on the real line of the equation $x^2y'' - 3xy' + 3y = 0$.

(b) Verify that $W(y_1, y_2)$ is identically zero. Why do these facts not contradict Theorem 3 from the textbook?

Solution-

(a) - We first note that $y = x^3$ satisfies the ODE:

$$x^2(6x) - 3x(3x^2) + 3x^3 = 0.$$

And $y = -x^3$ does as well:

$$x^2(-6x) - 3x(-3x^2) + 3(-x^3) = 0.$$

As $|x^3|$ has continuous first and second derivatives on the real line we note that the above calculations imply $|x^3|$ satisfies the ODE on the entire real line as well.

If $x < 0$ then $y_1 = x^3$ and $y_2 = -x^3$, while if $x > 0$ we have $y_1 = x^3$ and $y_2 = x^3$. So, there is no constant c such that $y_1 = cy_2$ on the *entire* real line, and therefore y_1 and y_2 are linearly independent on the real line.

(b) - The Wronskian of these two functions is:

$$W(y_1, y_2) = \begin{vmatrix} x^3 & |x^3| \\ 3x^2 & 3x|x| \end{vmatrix} = 3x^4|x| - 3x^4|x| = 0.$$

This does not contradict theorem 3 as

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 0$$

has $-\frac{3}{x}, \frac{3}{x^2}$ as coefficient functions, which are discontinuous (undefined!) at $x = 0$.