Math 2280 - Assignment 13

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Spring 2014

Section 9.1 - 1, 8, 11, 13, 21

Section 9.2 - 1, 9, 15, 17, 20

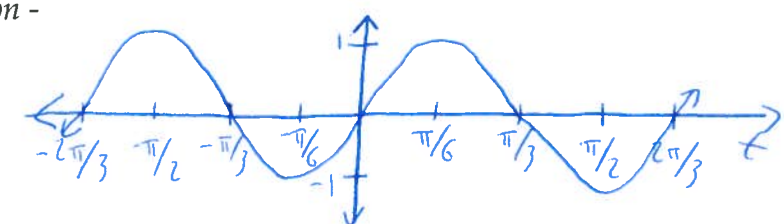
Section 9.3 - 1, 5, 8, 13, 20

Section 9.1 - Periodic Functions and Trigonometric Series

9.1.1 - Sketch the graph of the function f defined for all t by the given formula, and determine whether it is periodic. If so, find its smallest period.

$$f(t) = \sin 3t.$$

Solution -

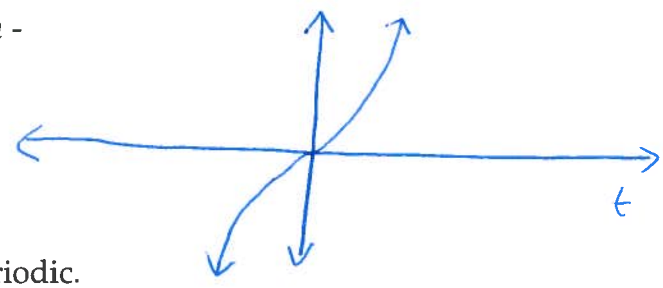


Periodic with period $\frac{2\pi}{3}$.

9.1.8 - Sketch the graph of the function f defined for all t by the given formula, and determine whether it is periodic. If so, find its smallest period.

$$f(t) = \sinh \pi t.$$

Solution -

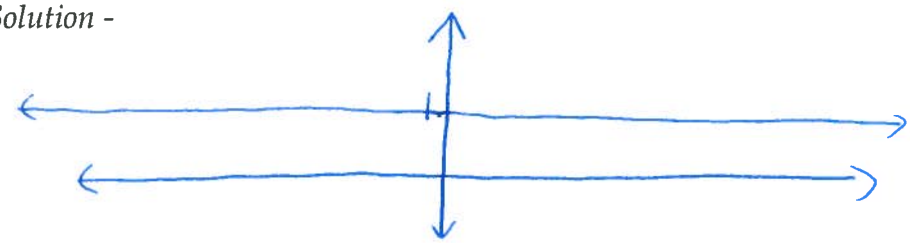


Not periodic.

9.1.11 - The value of a period 2π function $f(t)$ in one full period is given below. Sketch several periods of its graph and find its Fourier series.

$$f(t) = 1, \qquad -\pi \leq t \leq \pi.$$

Solution -



Just the constant function 1. The Fourier series for this will be just 1, but let's formally calculate it anyways.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dt = \frac{1}{\pi}(\pi - (-\pi)) = 2.$$

For $n \geq 1$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(nt)dt = \frac{1}{n\pi}\sin(nt)\Big|_{-\pi}^{\pi} = 0 - 0 = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nt)dt = \frac{-1}{n\pi}\cos(nt)\Big|_{-\pi}^{\pi} = -\frac{1}{n\pi}(\cos(n\pi) - \cos(n\pi)) = 0.$$

So, the Fourier series for 1 is:

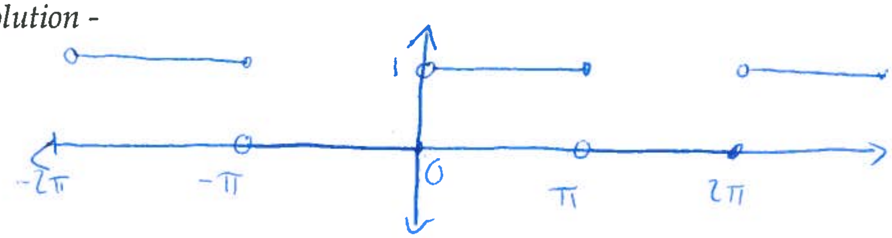
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + \sum_{n=1}^{\infty} b_n\sin(nt) = \frac{2}{2} = 1.$$

Wow!

9.1.13 - The value of a period 2π function $f(t)$ in one full period is given below. Sketch several periods of its graph and find its Fourier series.

$$f(t) = \begin{cases} 0 & -\pi < t \leq 0 \\ 1 & 0 < t \leq \pi \end{cases}$$

Solution -



The Fourier coefficients are:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_0^{\pi} dt = \frac{1}{\pi}(\pi - 0) = 1.$$

For $n \geq 1$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \frac{1}{\pi}\int_0^{\pi} cos(nt)dt = \frac{1}{n\pi}\sin(nt)\Big|_0^{\pi} = 0 - 0 = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = \frac{1}{\pi}\int_0^{\pi}\sin(nt)dt = \frac{-1}{n\pi}\cos(nt)\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos(n\pi) - 1) = \begin{cases} \frac{2}{n\pi} & n\ odd \\ 0 & n\ even \end{cases}.$$

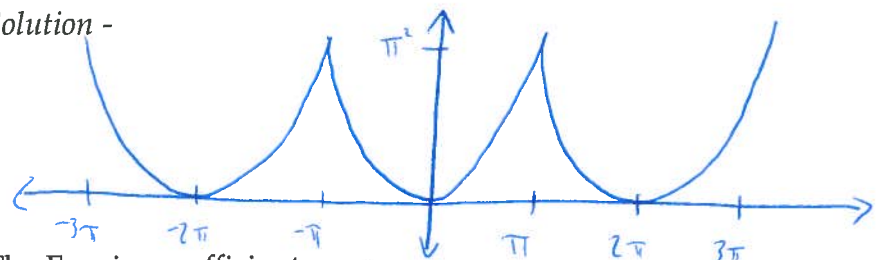
So, the Fourier series is:

$$\frac{1}{2} + \frac{2}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right).$$

9.1.21 - The value of a period 2π function $f(t)$ in one full period is given below. Sketch several periods of its graph and find its Fourier series.

$$f(t) = t^2, \qquad -\pi \leq t < \pi$$

Solution -



The Fourier coefficients are:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{2}{\pi}\int_0^{\pi} t^2 dt = \frac{2}{3\pi}t^3\bigg|_0^{\pi} = \frac{2\pi^2}{3}.$$

For $n \geq 1$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(nt) dt =$$

$$\frac{2}{\pi}\left(\frac{t^2 \sin(nt)}{n} + \frac{2t\cos(nt)}{n^2} - \frac{2\sin(nt)}{n^3}\right)\bigg|_0^{\pi} = \frac{2}{\pi}\left(\frac{2\pi\cos(n\pi)}{n^2}\right) =$$

$$\frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin(nt) dt = 0 \quad \text{(Odd function.)}$$

So, the Fourier series is:

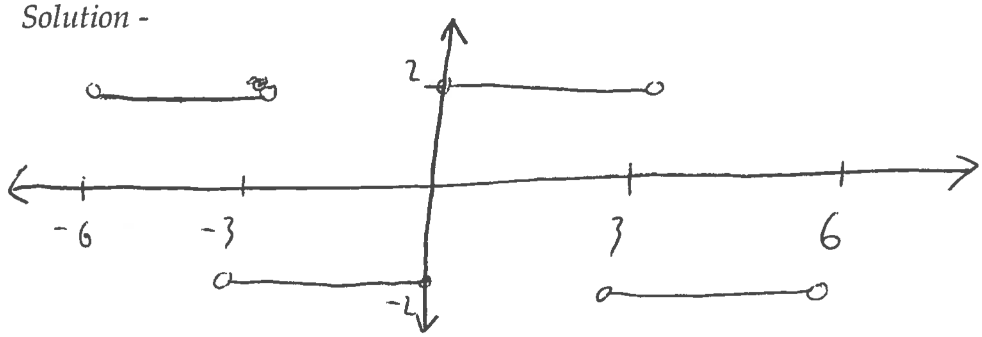
$$\frac{\pi^2}{3} - 4\left(\cos(t) - \frac{\cos(2n)}{4} + \frac{\cos(3t)}{9} - \frac{\cos(4t)}{16} + \cdots\right).$$

Section 9.2 - General Fourier Series and Convergence

9.2.1 - The values of a periodic function $f(t)$ in one full period are given below; at each discontinuity the value of $f(t)$ is that given by the average value condition. Sketch the graph of f and find its Fourier series.

$$f(t) = \begin{cases} -2 & -3 < t < 0 \\ 2 & 0 < t < 3 \end{cases}$$

Solution -



The function $f(t)$ is odd, so all the cosine terms will be 0. The sine terms are:

$$b_n = \frac{1}{3}\int_{-3}^{3} f(t)\sin\left(\frac{n\pi t}{3}\right)dt = \frac{2}{3}\int_0^3 2\sin\left(\frac{n\pi t}{3}\right)dt$$

$$= \frac{4}{3}\int_0^3 \sin\left(\frac{n\pi t}{3}\right)dt = \frac{4}{n\pi}(-\cos(n\pi) - (-1)) = \begin{cases} \frac{8}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

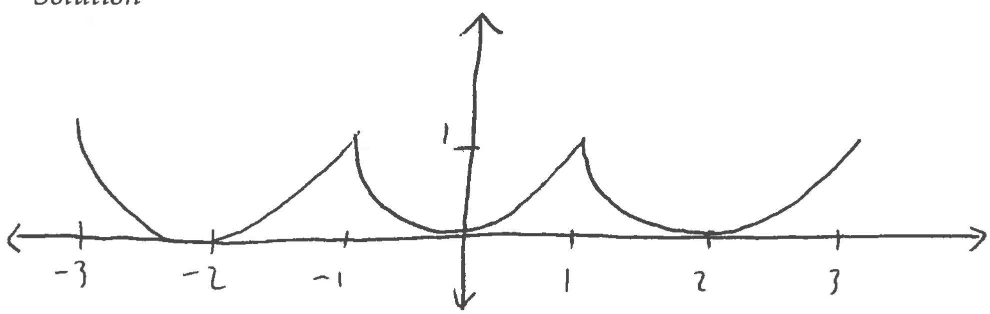
The Fourier series for $f(t)$ is:

$$\frac{8}{\pi}\left(\sin\left(\frac{\pi t}{3}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{3}\right) + \cdots\right).$$

9.2.9 - The values of a periodic function $f(t)$ in one full period are given below; at each discontinuity the value of $f(t)$ is that given by the average value condition. Sketch the graph of f and find its Fourier series.

$$f(t) = t^2, \qquad -1 < t < 1$$

Solution -



The function $f(t)$ is even, so all the sine terms in its Fourier series are zero. As for the cosine terms we have:

$$a_0 = \int_{-1}^{1} t^2 dt = \left.\frac{t^3}{3}\right|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$$

$$a_n = \int_{-1}^{1} t^2 \cos(n\pi t)dt = 2\int_0^1 t^2 \cos(n\pi t)dt$$

$$= 2\left(\frac{t^2 \sin(n\pi t)}{n\pi} + \frac{2t\cos(n\pi t)}{n^2\pi^2} - \frac{2\sin(n\pi t)}{n^3\pi^3}\right)\Bigg|_0^1 = \frac{4}{n^2\pi^2}\cos(n\pi)$$

$$= \frac{4(-1)^n}{n^2\pi^2}.$$

So, the Fourier series for the function $f(t)$ is:

$$\frac{1}{3} - \frac{4}{\pi^2}\left(\cos(\pi t) - \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{9} - \cdots\right).$$

9.2.15 -

(a) - Suppose that f is a function of period 2π with $f(t) = t^2$ for $0 < t < 2\pi$. Show that

$$f(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nt}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

and sketch the graph of f, indicating the value at each discontinuity.

(b) - Deduce the series summations

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

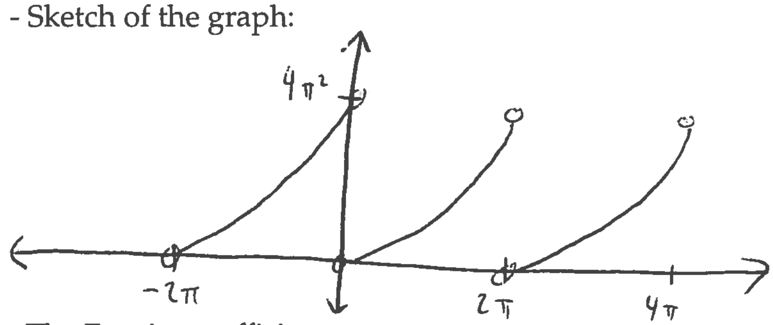
and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

from the Fourier series in part (a).

Solution -

(a) - Sketch of the graph:



The Fourier coefficients are:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2 dt = \frac{1}{\pi}\left(\frac{t^3}{3}\right)\Big|_0^{2\pi} = \frac{8\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos{(nt)} dt =$$
$$\frac{1}{\pi}\left(\frac{t^2\sin{(nt)}}{n} + \frac{2t\cos{(nt)}}{n^2} - \frac{2\sin{(nt)}}{n^3}\right)\Big|_0^{2\pi}$$
$$= \frac{9\pi\cos{(2n\pi)}}{\pi n^2} = \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin{(nt)} dt =$$
$$\frac{1}{\pi}\left(-\frac{t^2\cos{(nt)}}{n} - \frac{2t\sin{(nt)}}{n^2} + \frac{2\sin{(nt)}}{n^3}\right)\Big|_0^{2\pi}$$
$$= -\frac{4\pi\cos{(2n\pi)}}{n} = -\frac{4\pi}{n}.$$

So, the Fourier series for the function $f(t)$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos{(nt)} + \sum_{n=1}^{\infty} b_n \sin{(nt)}$$
$$= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos{(nt)}}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin{(nt)}}{n}.$$

(b) - The averaged value of our function at $t = 0$ is $\frac{4\pi^2 + 0}{2} = 2\pi^2$.

So,

$$2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

At $t = \pi$ we get:

$$
\begin{aligned}
\pi^2 &= \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\
\Rightarrow -\frac{\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\
\Rightarrow -\frac{\pi^2}{12} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\
\Rightarrow \frac{\pi^2}{12} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.
\end{aligned}
$$

9.2.17 -

(a) - Supose that f is a funciton of period 2 with $f(t) = t$ for $0 < t < 2$. Show that

$$f(t) = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}$$

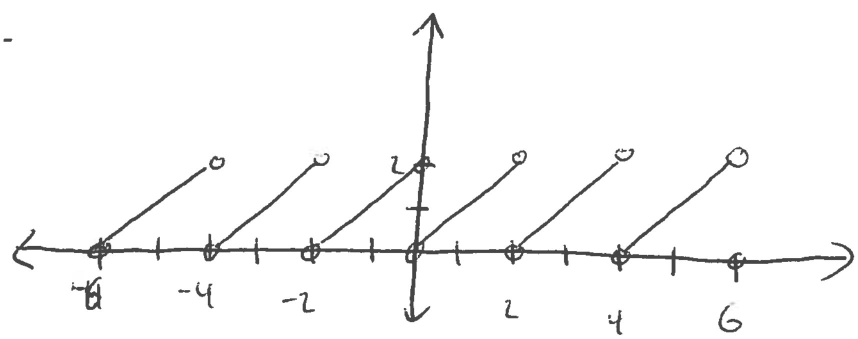
and sketch the graph of f, indicating the value at each discontinuity.

(b) - Substitute an appropriate value of t to deduce *Leibniz's series*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Solution - Sketch of the graph:

(a) -



The Fourier coefficients will be:

$$a_0 = \int_0^2 t dt = \left. \frac{t^2}{2} \right|_0^2 = 2.$$

$$a_n = \int_0^2 t \cos(n\pi t) = \left. \frac{t \sin(n\pi t)}{n\pi} - \frac{\cos(n\pi t)}{n^2\pi^2} \right|_0^2$$

$$= \left(\frac{2 \sin(2n\pi)}{n\pi} - \frac{\cos(2n\pi)}{n^2\pi^2} \right) - \left(0 + \frac{1}{n^2\pi^2} \right) = -\frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2} = 0.$$

$$b_n = \int_0^2 t \sin(n\pi t) dt = - \left. \left(\frac{t \cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2} \right) \right|_0^2 = -\frac{2}{n\pi}.$$

So, the Fourier series for $f(t)$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$
$$= 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi t)}{n}.$$

(b) - If we plug in $t = \frac{1}{2}$ we get:

$$\frac{1}{2} = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n}$$
$$\Rightarrow \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n} = \frac{1}{2}$$
$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

9.2.20 - Derive the Fourier series given below, and graph the period 2π function to which the series converges.

$$\sum_{n=1}^{\infty} \frac{\cos nt}{n^2} = \frac{3t^2 - 6\pi t + 2\pi^2}{12} \qquad (0 < t < 2\pi)$$

Solution - If we take the even extension of $f(t) = \dfrac{3t^2 - 6\pi t + 2\pi^2}{12}$ we must have $b_n = 0$ for all n, and

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3t^2 - 6\pi t + 2\pi^2}{12}\right)dt = \frac{1}{\pi}\left(\frac{t^3 - 3\pi t^2 + 2\pi^2 t}{12}\right)\Big|_0^{2\pi}$$
$$= \frac{1}{\pi}\left(\frac{8\pi^2 - 12\pi^2 + 4\pi^2}{12}\right) = 0.$$

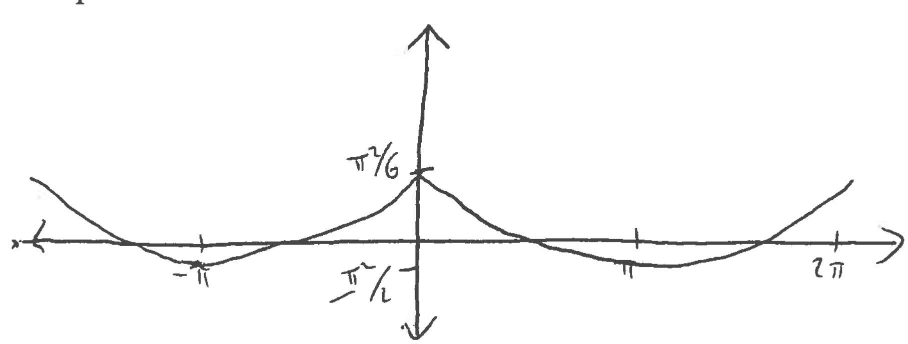
As for the a_n terms, these are (after a nasty integral, which you can do using Mathematica)

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3t^2 - 6\pi t + 2\pi^2}{12}\right)\cos(nt)dt = \frac{1}{n^2}.$$

So, the Fourier series for the even extension is:

$$\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2}.$$

Graph:

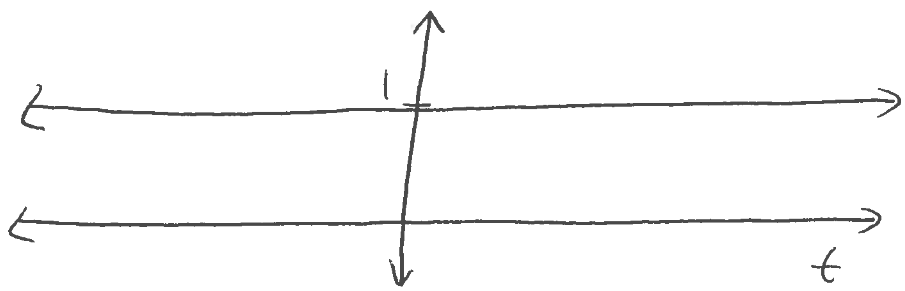


Section 9.3 - Fourier Sine and Cosine Series

9.3.1 - For the given function $f(t)$ defined on the given interval find the Fourier cosine and sine series of f and sketch the graphs of the two extensions of f to which these two series converge.

$$f(t) = 1, \qquad 0 < t < \pi.$$

Solution - Even extension



Cosine series:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} dt = 2,$$

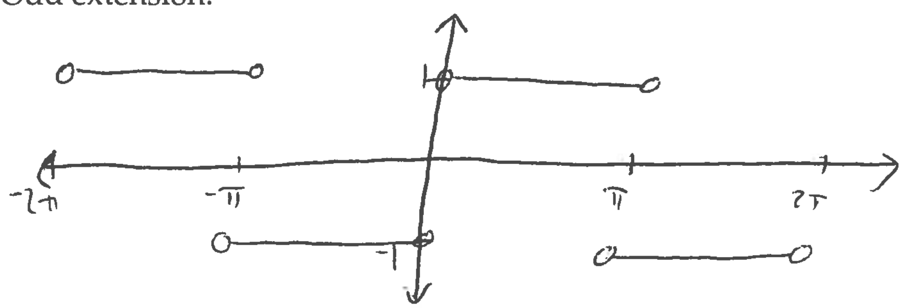
$$a_n = 0. \quad \text{(See problem 9.1.11)}$$

So, the cosine series is:

$$\frac{a_0}{2} = 1.$$

Wow!

Odd extension:



The sine series has coefficients:

$$\frac{2}{\pi}\int_0^{\pi} \sin(nt)dt = -\frac{2}{n\pi}\cos(nt)\Big|_0^{\pi}$$

$$= \begin{cases} \frac{4}{n\pi} & n\ odd \\ 0 & n\ even \end{cases}$$

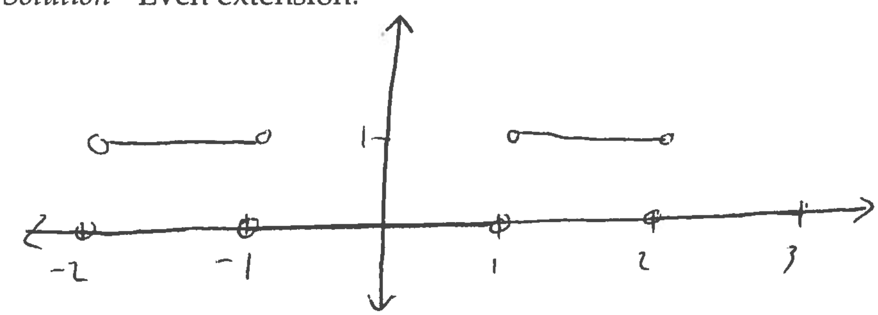
So, the sine series is:

$$\frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right).$$

9.3.5 - For the given function $f(t)$ defined on the given interval find the Fourier cosine and sine series of f and sketch the graphs of the two extensions of f to which these two series converge.

$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

Solution - Even extension:



The cosine series will have the coefficients:

$$a_0 = \frac{2}{3}\int_0^3 f(t)dt = \frac{2}{3}\int_1^2 dt = \frac{2}{3}.$$

$$a_n = \frac{2}{3}\int_0^3 f(t)\cos\left(\frac{n\pi t}{3}\right)dt = \frac{2}{3}\int_1^2 \cos\left(\frac{n\pi t}{3}\right)$$

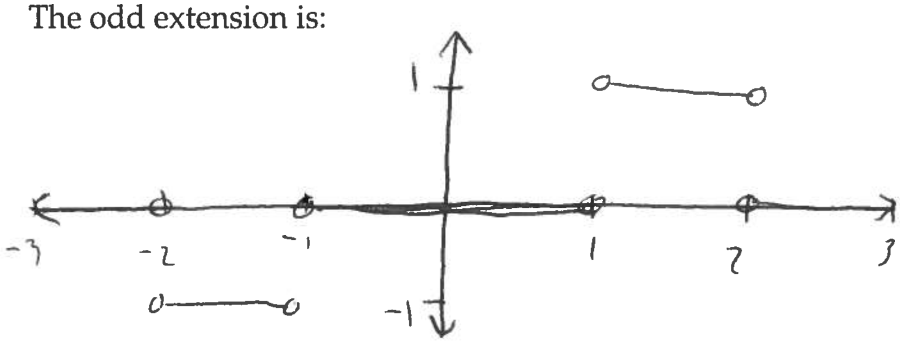
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi t}{3}\right)\Bigg|_1^2 = \frac{2}{n\pi}\left(\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{2n\pi}{3}\right)\right)$$

$$= \begin{cases} 0 & n = 6k, 6k+1, 6k+3, 6k+5 \\ -\frac{2\sqrt{3}}{n\pi} & n = 6k+2 \\ \frac{2\sqrt{3}}{n\pi} & n = 6k+4 \end{cases}$$

So, the cosine series is:

$$\frac{1}{3} - \frac{2\sqrt{3}}{\pi}\left(\frac{1}{2}\cos\left(\frac{2\pi t}{3}\right) - \frac{1}{4}\cos\left(\frac{4\pi t}{3}\right) + \frac{1}{8}\cos\left(\frac{8\pi t}{3}\right) - \cdots\right).$$

The odd extension is:



The sine series will have coefficients

$$b_n = \frac{2}{3}\int_0^3 f(t)\sin\left(\frac{n\pi t}{3}\right)dt = \frac{2}{3}\int_1^2 \sin\left(\frac{n\pi t}{3}\right)dt$$

$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi t}{3}\right)\Big|_1^2 = -\frac{2}{n\pi}\left(\cos\left(\frac{2n\pi}{3}\right) - \cos\left(\frac{n\pi}{3}\right)\right)$$

$$= \begin{cases} 0 & n = 6k, 6k+2, 6k+4 \\ \frac{2}{n\pi} & n = 6k+1, 6k+5 \\ -\frac{4}{n\pi} & n = 6k+3 \end{cases}$$

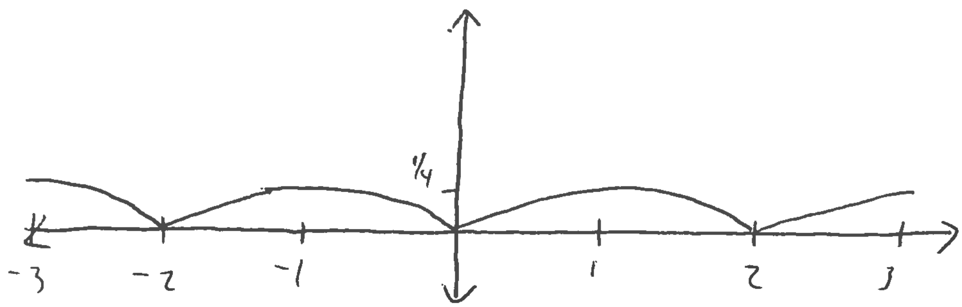
The sine series is:

$$\frac{2}{\pi}\left(\sin\left(\frac{\pi t}{3}\right) - \frac{2}{3}\sin\left(\frac{3\pi t}{3}\right) + \frac{1}{5}\sin\left(\frac{5\pi t}{3}\right) + \cdots\right).$$

9.3.8 - For the given function $f(t)$ defined on the given interval find the Fourier cosine and sine series of f and sketch the graphs of the two extensions of f to which these two series converge.

$$f(t) = t - t^2, \qquad 0 < t < 1$$

Solution - The even extension is:



The cosine series has the coefficients:

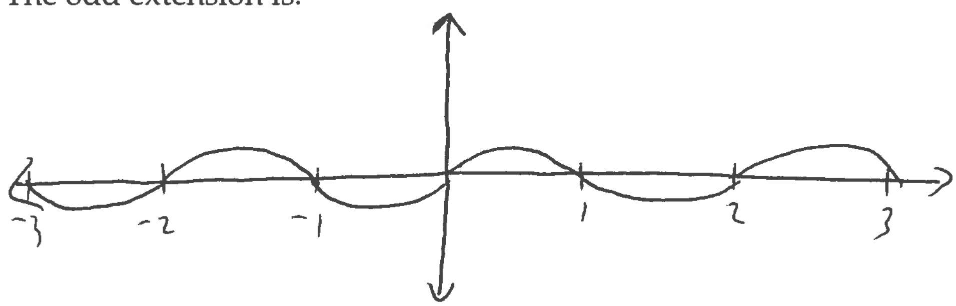
$$a_0 = 2\int_0^1 (t - t^2)dt = 2\left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}.$$

$$a_n = 2\int_0^1 (t - t^2)\cos(n\pi t)dt = \begin{cases} -\frac{4}{n^2\pi^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

The integral above can be evaluated with a good graphing calculator, a computer, Wolfram alpha, integrals.com, or by hand with a couple integrations by parts. I just cut to the chase. The corresponding Fourier cosine series will be:

$$\frac{1}{6} - \frac{4}{\pi^2}\left(\frac{\cos(2\pi t)}{4} + \frac{\cos(4\pi t)}{16} + \cdots\right).$$

The odd extension is:



The sine series has the coefficients:

$$b_n = 2\int_0^1 (t - t^2)\sin(n\pi t)dt = \begin{cases} \frac{8}{n^3\pi^3} & n\ odd \\ 0 & n\ even \end{cases}$$

The corresponding Fourier sine series will be:

$$\frac{8}{\pi^3}\left(\sin(\pi t) + \frac{\sin(3\pi t)}{27} + \frac{\sin(5\pi t)}{125} + \cdots\right).$$

The odd extension is:

The sine series has the coefficients:

$$b_n = 2\int_0^1 (t - t^2)\sin(n\pi t)dt = \begin{cases} \frac{8}{n^3\pi^3} & n\ odd \\ 0 & n\ even \end{cases}$$

The corresponding Fourier sine series will be:

$$\frac{8}{\pi^3}\left(\sin(\pi t) + \frac{\sin(3\pi t)}{27} + \frac{\sin(5\pi t)}{125} + \cdots\right).$$

9.3.13 - Find a formal Fourier series solution to the endpoint value problem

$$x'' + x = t \qquad\qquad x(0) = x(1) = 0.$$

Solution - We note that $\sin(n\pi t)$ has value 0 at $t = 0$ and $t = 1$ for all n, so we'll want to use the sine series.

$$x(t) = \sum_{n=1}^{\infty} a_n \sin(n\pi t)$$

$$x''(t) = \sum_{n=1}^{\infty} a_n(-n^2\pi^2)\sin(n\pi t).$$

The Fourier coefficients for the sine series of t are:

$$b_n = 2\int_0^1 t\sin(n\pi t)dt = 2\left(-\frac{t\cos(n\pi t)}{n\pi} + \frac{\sin(n\pi t)}{n^2\pi^2}\right)\Bigg|_0^1 = \frac{2(-1)^{n+1}}{n\pi}.$$

Plugging these into our differential equation we get:

$$a_n(1 - n^2\pi^2) = \frac{2(-1)^{n+1}}{n\pi}$$
$$\Rightarrow a_n = \frac{2(-1)^{n+1}}{n\pi(1 - n^2\pi^2)}.$$

So,

$$x(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi t)}{n(n^2\pi^2 - 1)}.$$

9.3.20 - Substitute $t = \pi/2$ and $t = \pi$ in the series

$$\frac{1}{24}t^4 = \frac{\pi^2 t^2}{12} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos nt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}, \quad -\pi < t < \pi,$$

to obtain the summations

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720},$$

and

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

For $t = \frac{\pi}{2}$ we get:

$$\frac{1}{24}\left(\frac{\pi^4}{16}\right) = \frac{\pi^2}{12}\left(\frac{\pi^2}{4}\right) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos\left(\frac{n\pi}{2}\right) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}.$$

Now,

$$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & n = 4k \\ -1 & n = 4k+2 \\ 0 & n = 4k+1, 4k+3 \end{cases}$$

So,