Homework Assignment No. 5, due Wednesday, 2/22 at 5 pm

1. Assume the standard Black-Scholes model for a stock price $S(t)$, with riskless interest rate r and volatility σ as given. Derive the no-arbitrage pricing formula for the following payoff functions:

   (a)
$$\Lambda(S(T)) = \sum_{n=1}^{N} a_n S^n(T)$$
   where a_n are real coefficients.

   (b)
$$\Lambda(S(T)) = \max(S^\alpha(T) - K, 0)$$
   where $\alpha > 0$ is a real constant.

2. The Greek letters of financial derivatives, or just the "Greeks", are sensitivities of the derivative price to various factors, such as the underlying stock price, and the volatility. In the case of European call options C in Black-Scholes model, we have the Delta and Vega
$$\Delta = \frac{\partial C}{\partial S} = N(d_1), \quad \nu = \frac{\partial C}{\partial \sigma} = S\sqrt{T}N'(d_1)$$
   where $N(x)$ is the cumulative normal distribution function, and
$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
   Suppose the stock price is \$50, and you bought 10 contracts (each contract is 100 shares) of 3-month call option with strike \$50 in the hope that the stock will go up in three months. However, the call option comes with a volatility risk, namely the option price will drop if the volatility drops. On the other hand, you do not believe that the stock is going to go above \$60 so you decide to take a short position in a call with strike \$60. How many contracts of this call should you sell in order that the vega of the portfolio is zero? Once the vega of the portfolio is zero, you will not worry about the volatility moves just for the moment. But you still want the portfolio to be delta-neutral. What can you do to maintain the portfolio to be delta-neutral? You can assume $r = 0$ and $\sigma = 30\%$.