Lecture 21: Risk Neutral and Martingale Measure

Revisiting the subjects - more analysis

- The previous chapters introduced the following approaches to express the derivative price as an expectation
  - binomial tree (multi-step) and the risk-neutral **probabilities** such that
  $$\mathrm{E}[S_N]e^{-rN\Delta t} = S_0$$
  - taking limit as $N \to \infty,\ \Delta t \to 0, N\Delta t = T.$
  - limiting probability density: lognormal, drift term $\mu = r$, leading to Black-Scholes model
- Stock price as a process
  - log of S modeled as a random walk
  - limiting process leads to geometric Brownian motion for S
  - probability measure such that $\mathrm{E}[S_T]e^{-r(T-t)} = S_t$
- Option price as the expectation of the payoff under **this** probability measure
$$V_t = \mathrm{E}[V_T]e^{-r(T-t)} = \mathrm{E}[F(S_T)]e^{-r(T-t)}$$

Justifications

- Existence of the probability measure - how do we approach it?
- Implication of Risk-neutral world - how is it like to be there?
- What the expectation requirement says about the process - **martingale**
- No-arbitrage models - what to look for?
- Final justification: this derivative price **eliminates** arbitrage opportunities
- Intricate relations among
  - risk-neutral
  - martingale
  - arbitrage-free

Developing New Tools

- Random walk to Brownian motion
- Stochastic process: discrete vs. continuous time
- concept of information - filtration - conditional expectation
- martingale as a particular type of processes, with a distinctive **no-drift** feature
- how martingale pricing leads to no-arbitrage condition

When All Steps Are Completed

- Once we finish the justifications, we can proceed to

1. Start with a process that models the stock price

2. Modify to make sure that the discounted stock price process is a martingale - achieved by a **change of measure**

3. Other derivative prices (discounted) are also martingales: therefore a formula involving an expectation is obtained to price such a derivative.

4. This price is guaranteed to be arbitrage-free.

Concepts from Stochastic Processes

- To understand the martingale pricing theory, we need some basic stochastic process concepts
- First we define a stochastic process: a collection of rv's, and the key is how they are indexed (by time)
- We can start with discrete time processes - an example is the process behind the multi-step binomial tree model
- Corresponding to a collection of rv's, each element of the sample space now corresponds to a **path**
- A probability measure: how to assign probabilities to a set of path

Information and Conditional Expectation

- The importance of conditional expectation - updated assessment of future values
- Taking expectation with respect to a certain measure
- Different measures according to the algebra (F) - the collection of the events we are using can be viewed as different restrictions
- The collection of events can be categorized based on the information available at the time
- Main question: how to describe the evolution of information?
- Filtration: a sequence of expanding algebras
- Conditional expectation is an expectation with respect to a filtration - a rv by itself!

Martingale

- A particular type of process with the feature

$$\mathrm{E}[X_k|\mathcal{F}_j] = \mathrm{E}_j[X_k] = X_j, \quad j \leq k$$

$$\mathrm{E}[X_T|\mathcal{F}_t] = \mathrm{E}_t[X_T] = X_t, \quad t \leq T$$

- No drift
- Markov process: no historical impact
- Financial interpretation: risk-neutral
- Provide a formula for X_t - pricing formula
- Next we show all everything fits together - based on **no-arbitrage**

Static (one-step) No-arbitrage Condition

- In the one-step binomial model, in order to establish no-arbitrage, we must have $S_1^d < S_0 < S_1^u$ (assuming no interest rate)

- This implies the existence of a RN measure: $S_0 = \mathrm{E}[S_1]$ where 0<p<1

- Riskless portfolio: $C_0 + \Delta \cdot S_0 = [C_1 + \Delta S_1]$

- So $C_0 = [C_1]$

- Unique p implies unique RN measure, and this is the cost to replicate the derivative, therefore **the** no-arbitrage price

- How do we extend to a multi-step model?

Dynamic No-arbitrage Model

- Still assuming zero interest, we need $S_k = \mathrm{E}_k[S_r] \quad \text{for } 0 \leq k \leq r \leq N$
- Questions:
  - Conditional expectation
  - filtration
  - sub sigma-algebra
- Conditional expectations are based on the sigma-algebra (what kind of events to consider)
- We can change the algebra to some specific algebra
- The resulting expectation is dependent on this particular “specification” - the path up to time t

Example (martingale and conditional expectation)

- Consider $$X_k = \sum_{j=1}^{k} Z_j, \quad Z_j = \begin{cases} 1 \\ -1 \end{cases}$$
- The expected value of $X_r = X_k + \sum_{j=k+1}^{r} Z_j$, observed at time k = X_k
- This is the martingale property
- The conditional expectation $\mathrm{E}[X_r|\mathcal{F}_k] = \mathrm{E}_k[X_r]$ is the expected value of X_r, given all the information up to k
- A counter example: $X_j = jX_1$, satisfying

$$\mathrm{E}[X_1] = 0 = X_0,$$
$$\mathrm{E}_1[X_2] = 2X_1$$

Martingale - No arbitrage

- Over each step: $X_{k+1} = X_k + Z_{k+1}$
- Notice that Z has positive probabilities of being positive and negative, this is the no arbitrage condition over each step
- Can find a probability (1/2 in this case) so $S_k = \mathrm{E}_k[S_{k+1}] = \mathrm{E}_k[S_N]$
- Then the option price $C_k = \mathrm{E}_k[C_{k+1}] = \mathrm{E}_k[C_N]$
- For positive interest rate

$$\frac{C_k}{B_k} = \mathrm{E}_k\left[\frac{C_N}{B_N}\right]$$

Continuous time martingale

- First we need two properties
  - Tower law: $\mathrm{E}_s\left[\mathrm{E}_t[X]\right] = \mathrm{E}_s[X], \quad s < t$
  - Independence property: $\mathrm{E}_s[X] = \mathrm{E}[X],$ if X is independent of $\mathcal{F}_s$
- Most natural example: the Brownian motion W_t
- General extension described by $dX_t = \sigma(X_t, t)\, dW_t$
- Continuous martingale pricing: $d\left(\frac{S_t}{B_t}\right) = \sigma(X_t, t)\frac{S_t}{B_t}\, dW_t$ if $\mu = r$
- This can be achieved by a **change of measure**, redistributing the probability weights
- Black-Scholes formula now is arrived again as an expectation

Martingale Pricing

- Now we have a martingale for the discounted stock price

$$\frac{S_t}{B_t} = \mathrm{E}_t\left[\frac{S_T}{B_T}\right]$$

- Option price has to be a martingale too - if we can use S and O to hedge

$$\frac{O_t}{B_t} = \mathrm{E}_t\left[\frac{O_T}{B_T}\right]$$

- Properties of this price
    - as an integral of any payoff function
    - use the same risk-neutral probability measure
    - arbitrage-free
    - call or put payoff functions - Black-Scholes formula

Connection with BS PDE

- Can verify that the BS formula satisfies the BS PDE and the terminal conditions

- Can show that if the BS PDE is satisfied by C(S,t), then the discounted option price is indeed a martingale

$$d\left(\frac{C_t}{B_t}\right) = \sigma \frac{S_t}{B_t} \frac{\partial C}{\partial S}\, dW_t$$

- PDE solution can be found for exotic options such as a barrier call option which looks like a regular call except

    - there is a barrier (B) set in the contract

    - if S reaches B at any time before T, the option disappears

    - easy to set up in the PDE problem by a proper boundary condition

Hedging and Self-financing

- Hedging portfolio: need to balance the changes in C and S
- Martingale representation theorem: any martingale is “derived” from the Brownian motion, and two martingales are therefore intricately related through their common connection with the Brownian motion
- Stock and option - how are their changes related? $d\left(\frac{C_t}{B_t}\right) = \frac{\partial C}{\partial S}\, d\left(\frac{S_t}{B_t}\right)$
- Self-financing of the replicating portfolio: $\alpha_t B_t + \beta_t S_t$
- Need to verify $d\left(\alpha_t B_t + \beta_t S_t\right) = \alpha_t\, dB_t + \beta_t\, dS_t$
- Chose $$\alpha_t = \frac{C_t}{B_t} - \frac{\partial C}{\partial S}\frac{S_t}{B_t}$$ $$\beta_t = \frac{\partial C}{\partial S}$$
- This is an important step often ignored!

PDE Advantages

- Time-dependent parameters - we can always use a numerical method to solve the PDE with time-dependent coefficients
- Impact of the sigma time-dependence: using the root mean square vol, from the time of pricing (t) to expiration (T), in both pricing and hedging
- What does the time-dependent sigma do to the implied vol curve (surface)?
  - generate certain shapes
  - not enough to match the implied vol curve (surface) observed on the market

Tradable vs. Non-tradable, Market Price of Risk

- Two **tradable** securities, driven by the same BM

$$\frac{df_1}{f_1} = \mu_1\, dt + \sigma_1 dW_t$$
$$\frac{df_2}{f_2} = \mu_2\, dt + \sigma_2 dW_t$$

- Is there a relation between the parameters?
- Observation: the risk can be eliminated by forming a portfolio
- This portfolio should be riskless, therefore with growth rate r

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda$$

- This is the market price of the risk, same for all securities driven by the same factor
- In the risk-neutral world, the market price of risk is zero