Lecture 11: Ito Calculus

Continuous time models

- We start with the model from Chapter 3

$$\log S_j - \log S_{j-1} = \mu\Delta t + \sigma\sqrt{\Delta t} Z_j$$

- Sum it over j: $$\log S_N - \log S_0 = \sum_{j=1}^{N} \mu\Delta t + \sum_{j=1}^{N} \sigma\sqrt{\Delta t} Z_j$$
- Can we take the limit as N approaches infinity (delta t tends to zero)?
- What are the benefits?
  - last sum converges to a normal random variable, so we call it lognormal!
  - what is more important than the distribution of S at a fixed time?
  - increments: $$\log S_N - \log S_M = \sum_{j=M+1}^{N} \mu\Delta t + \sum_{j=M+1}^{N} \sigma\sqrt{\Delta t} Z_j$$

Stock price as a process

- Prices at different times: $S_0, S_1, S_2, \ldots, S_N$
- We must consider them as a collection of random variables
- Obviously the order is important - when you enter at time j and exit at time k, you care about $\log S_j - \log S_k$, another random variable
- A collection of **time indexed** random variables - a stochastic process
- Not only are we concerned about individual S_j as a random variable, we also need to consider all possible increments $\log S_j - \log S_k$
- As random variables, we ask for their distributions. But the relations between different increments can be crucial for dependence consideration
- Natural first step: independent increments. Is it appropriate for stock prices?

Increments

- Price change over a time period
- What we get from our discrete model: a sum of independent Bernoulli rv's - binomial rv
- If we further divide the time period into subintervals, we are still dealing with binomial rv's
- As the partition increases, these binomial rv's converge to normal rv's (in distribution), justified by CLT.
- Statistics: the mean and the variance (of increments) should depend on the time elapsed: $\mu(t_j - t_k)$ and $\sigma^2(t_j - t_k)$
- Independent increments: as long as individual rv's are independent!

Random walk and Markov property

- Use notation $X_j = \log S_j$
- A sum of steps, each consisting of two components (drift + Z)
- Called a random walk, X_j is the position of the walk at time j
- Increments $X_j - X_k$, independent of all the previous X's before k
- Distribution of X at j, given X at k, is unaffected by the X values before k
- Dependence of the history up to k - only through X at k
- This is called the Markov property!

From random walk to Brownian motion

- Think of the limiting process as $N \to \infty, \Delta t \to 0, \quad N\Delta t = T$
- $X_j = X_{t_j} \to X_t,$ collection of rv's indexed by a continuous time variable t
- Properties inherited or extended:
    - X at t is a normal random variable;
    - increment $X_t - X_s$ is a normal random variable: $N\left(\mu(t-s), \sigma^2(t-s)\right)$
    - increments from nonoverlapping periods are independent
    - The path, X as a function of t, is continuous, but nowhere differentiable
- Standard notation: W_t

Definition of BM

- A process W_t indexed by t for t>=0 is a Brownian motion if $W_0 = 0$, and for every t and s (s<t), we have $W_t - W_s$ distributed as a normal random variable with mean 0 and variance t-s, and the random variable $W_t - W_s$ is independent of the W random variables before s.
- The above says much more. Just compare with $X_t = \sqrt{t}Y$ where $Y = N(0,1)$
- Quadratic variations and the relevance:
    - why is it that the variance is proportional to the time elapsed?
    - why is that BM paths are so ragged?
    - how does the stock price variance grow in time?

Extending BM

- Add a (time-dependent) drift
- Allow local variance (for each step) to be time-dependent
- Discrete time: $X_j - X_{j-1} = \mu_j \Delta t + \sigma_j \sqrt{\Delta t} Z_j$
- Continuous time: $dX_t = \mu(t)\,dt + \sigma(t)\,dW_t$
- Stock return over (t,t+dt):

$$\frac{dS_t}{S_t} = \mu(t)\,dt + \sigma(t)\,dW_t$$

- This is the Black-Scholes model for stock price
- Attempt to solve - do we have $\frac{dS_t}{S_t} = d\log S_t$?

Ito's lemma

- assume that f(x) is continuously twice differentiable
- usual differential: df = f'(x) dx
- if x=x(t) is also continuously differentiable (in t): df = f'(x) x'(t) dt
- now let x=X_t from a stochastic process as described in the previous slide
- notice W_t is nowhere differentiable
- guess: $df(X_t) = f'(X_t)\, dX_t = f'(X_t)\, (\mu dt + \sigma\, dW_t)$?
- not quite! as we see $$\begin{aligned} W_{t+h}^2 - W_t^2 &= (W_{t+h} - W_t)(W_{t+h} + W_t) \\ &= 2W_t(W_{t+h} - W_t) + (W_{t+h} - W_t)^2 \end{aligned}$$
- expect $dW_t^2 = 2W_t\, dW_t + dt$

From Taylor expansion

- Assuming f(x) twice differentiable

$$f(X_{t+h}) = f(X_t) + f'(X_t)(X_{t+h} - X_t) + \frac{1}{2} f''(X_t)(X_{t+h} - X_t)^2 + \cdots$$

- Ito process: $dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$ with approximations:

$$X_{t+h} - X_t = \mu h + \sigma \sqrt{h} Z + e$$

$$(X_{t+h} - X_t)^2 = \mu^2 h^2 + \sigma^2 h Z^2 + 2\mu\sigma h^{3/2} Z + \cdots$$

- Leading term (in h) after replacing Z^2 with 1: $\sigma^2 h$

- Justifications: the difference has mean and variance:

$$\sigma^2 h E[Z^2 - 1] = 0, \quad \sigma^4 h^2 Var(Z^2 - 1) = 3\sigma^4 h^2$$

Ito's lemma

- Letting $h \to dt$
- Assuming differentiability again

$$d(f(X_t)) = \left(f'(X_t)\mu + \frac{1}{2} f''(X_t)\sigma^2 \right) dt + \sigma f'(X_t)\, dW_t$$

- If we allow f to be time dependent

$$d(f(X_t, t)) = \left(f_t(X_t, t) + f_x(X_t, t)\mu + \frac{1}{2} f_{xx}(X_t, t)\sigma^2 \right) dt + \sigma f_x(X_t, t)\, dW_t$$

- Theorem 5.1 (page 110) notations

$$\begin{aligned} dt^2 &= 0 \\ dt\, dW_t &= 0 \\ (dW_t)^2 &= dt \end{aligned}$$

Applications

- Product rule: let X_t and Y_t be Ito processes

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$$

- If

$$dX_t = \mu_1\, dt + \sigma_1\, dW_t$$

$$dY_t = \mu_2\, dt + \sigma_2\, dW_t$$

- then

$$dX_t\, dY_t = \sigma_1 \sigma_2 (dW_t)^2 = \sigma_1 \sigma_2 dt$$

- What about

$$d\left(\frac{X_t}{Y_t}\right)$$

Applications in stock price modeling

- Solving SDE $\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$
- Try $f(S_t) = \log S_t$

$$\begin{aligned} df(S_t) &= \frac{1}{S_t} dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2 S_t^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW_t \end{aligned}$$

- Integrate in t, assuming constant mu and sigma

$$\log S_T - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T$$

$$S_T = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right]$$

CEV model

- Assuming volatility is S-dependent

$$\frac{dS_t}{S_t} = \mu dt + S_t^{\beta-1}\sigma\, dW_t$$

- $0 < \beta < 1$ implies that the volatility is inverse proportional to S

- $f(S) = \frac{S^{1-\beta}}{1-\beta}$, Ito's lemma gives

$$d(f(S_t)) = \left(S^{1-\beta}\mu - \frac{\beta}{2}S^{\beta-1}\sigma^2\right)\, dt + \sigma\, dW_t$$

- No luck in explicit solution unless beta=1

Deriving Black-Scholes Equation

- Consider the pricing of a call option C, with strike K, expiration T
- Assume S follows a geometric BM
- Risk free interest rate r
- At time t<T, the price of call is a function of stock price at the time (S)
- Recognizing C=C(S,t)

$$
\begin{aligned}
dC(S_t, t) &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S}\, dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\,(dS_t)^2 \\
&= \left(C_t + \mu S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS} \right)\, dt + \sigma S C_S\, dW_t
\end{aligned}
$$

Deriving Black-Scholes Equation (continued)

- Forming a portfolio: one share of call + alpha shares of the stock
- Change of the portfolio over (t,t+dt), assuming constant alpha:

$$d(C+\alpha S) = \left(C_t + \mu S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + \alpha \mu S\right) dt + \sigma S \left(C_S + \alpha\right) dW_t$$

- If we choose $\alpha = -C_S$ (delta hedging), the random component disappears, which implies that the portfolio is hedged - no effect of stock price fluctuation
- Portfolio is iick-free, we must have $d(C+\alpha S) = r(C+\alpha S)dt$
- This leads to the Black-Scholes PDE with terminal condition

$$C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} = rC \qquad C(S_T, T) = \max(S_T - K, 0)$$

- Compare with the standard heat equation, suggest backward in time

Use of the PDE

- The PDE is parabolic, solutions will be smoothed in time (backward)
- Set up a region in (S,t): $0 < t < T$, $0 < S < S_{max}$
- Terminal condition imposed at $t=T$
- Solve **backward** in time to 0: $C(S,0)$
- Enter the observed current price $S(0)$ in place of S
- Boundary conditions: $C(0,t) = 0$, $C(S_{max},t) = (S_{max} - K) \exp(-r(T-t))$
- Advantage of the PDE approach:
  - easy to extend to time-dependent sigma
  - efficient numerical methods available

Justification of the derivation

- How do we justify this price (solution from a PDE)?
- Imagine you start with C(S,0), when the stock price is S. By following the delta hedge strategy, you want to end up with the value max(S_T-K,0), no matter what happens to the market
- Replication strategy: invest C(S,0) in stock and the risk-less bond, adjusting according to the call delta, verify at T that the total value matches the call payoff
- Composition of the portfolio: alpha shares of the stock, beta units of the bond

$$P(t) = \alpha(t)S(t) + \beta(t)B(t)$$

- $\alpha(t), \quad \beta(t)$ to be adjusted, according to the strategy

Change of value in the portfolio

- Change of portfolio value in time: $P(t+\Delta t) - P(t)$
- In differential: $dP = \alpha(t)dS(t) + \beta(t)dB(t) + S(t)d\alpha(t) + B(t)d\beta(t)$
- In discrete form:

$$
\begin{aligned}
&\alpha(t+\Delta t)S(t+\Delta t) - \alpha(t)S(t) \\
=&\alpha(t+\Delta t)S(t+\Delta t) - \alpha(t)S(t+\Delta t) + \alpha(t)S(t+\Delta t) - \alpha(t)S(t) \\
=&\left(\alpha(t+\Delta t) - \alpha(t)\right) S(t+\Delta t) + \alpha(t)\left(S(t+\Delta t) - S(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\beta(t+\Delta t)B(t+\Delta t) - \beta(t)B(t) \\
=&\beta(t+\Delta t)B(t+\Delta t) - \beta(t)B(t+\Delta t) + \beta(t)B(t+\Delta t) - \beta(t)B(t) \\
=&\left(\beta(t+\Delta t) - \beta(t)\right) B(t+\Delta t) + \beta(t)\left(B(t+\Delta t) - B(t)\right)
\end{aligned}
$$

- Total change in two parts:

$$\alpha(t)\left(S(t+\Delta t) - S(t)\right) + \beta(t)\left(B(t+\Delta t) - B(t)\right) \longrightarrow \alpha dS + \beta dB$$

$$
\begin{aligned}
&\left(\alpha(t+\Delta t) - \alpha(t)\right) S(t+\Delta t) + \left(\beta(t+\Delta t) - \beta(t)\right) B(t+\Delta t) \\
&\longrightarrow S d\alpha + B d\beta + d\alpha dS + d\beta dB
\end{aligned}
$$

Self-financing strategy

- First part in the last slide: change in stock price, bond price, holding shares fixed over time period
- Second part: adjusting the number of shares, all at the end of the time period
- Self-financing strategy: making sure the second part is zero
- This corresponds to rebalancing in such a way that no money is taken out of the portfolio, and no money is injected into the portfolio either
- Such is the name of the strategy: self-financing
- Consequence of this trading strategy:

$$dP = \alpha dS + \beta dB$$

Replicating the call

- Begin with a portfolio $P = \alpha(0)S(0) + \beta(0)B(0) = C(S(0), 0)$
- Following $\alpha = \dfrac{\partial C}{\partial S}$, and a beta such that it is a self-financing strategy
- Want to show P(T) = C(S(T),T), no matter what S(T) ends up with
- Consider the differential

$$\begin{aligned} d\left(P(S,t) - C(S,t)\right) &= dP - dC \\ &= \frac{\partial C}{\partial S} dS + \beta dB - \frac{\partial C}{\partial t} dt - \frac{\partial C}{\partial S} dS - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 \\ &= \beta r B dt - \frac{\partial C}{\partial t} dt - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt \end{aligned}$$

- We use $\dfrac{dS}{S} = \mu dt + \sigma dW, \quad dB = rBdt, \quad P = \dfrac{\partial C}{\partial S} S + \beta B$, and the BS equation
- Result: $d(P - C) = r\left(P - SC_S\right)dt - r\left(C - SC_S\right)dt = r(P - C)dt$

Matching at T

- Solving the ODE: $P(t) - C(t) = (P(0) - C(0))\, e^{rt} = 0$
- We have $P(S,t) = C(S,t), \quad \text{for } 0 < t \leq T$, **the call is replicated!**
- Need to check the self-financing condition
- Theorem 5.3:
    - A unique beta exists, given alpha is a smooth function of S and an initial portfolio value P(0), such that $P = \alpha S + \beta B$ is a self-financing portfolio with initial value P(0).
- Implication on the hedging practice: by the end of the trading adjustment period, the rebalancing needs to observe the following condition: there can only be transfer of money within the stock and bond accounts

Solving the PDE

- Linear PDE, variable coefficients
- A series of changes of variables introduced to reduce to the heat equation
- First, $S = e^Z$, we arrive at a constant-coefficient equation

$$\frac{\partial C}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial C}{\partial Z} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial Z^2} = rC$$

- Change of time variable $\tau = T - t$

$$\frac{\partial C}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial C}{\partial Z} - \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial Z^2} + rC = 0$$

- $C = e^{-r\tau} D$

$$\frac{\partial D}{\partial \tau} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial D}{\partial Z} - \frac{1}{2}\sigma^2\frac{\partial^2 D}{\partial Z^2} = 0$$

Heat Equation

- Eliminate the first-order term:

$$y = Z + \left(r - \frac{1}{2}\sigma^2\right)\tau$$

- Standard heat equation

$$\frac{\partial D}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial y^2}$$

- Initial condition is also likewise transformed
- Solution transformed into the original variables
- Black-Scholes formula reproduced

Dividend-paying stock

- The previous model assumes no dividend paying stocks
- Many stocks do pay dividends
- FX products - foreign currency as the underlying and it grows at its rf rate
- This model assumes reinvestment
- If the dividend rate is d, one share at t will grow to exp(d(T-t)) shares at T
- Buying exp(-d(T-t)) shares is equivalent to one futures contract:
- Price of a futures contract: $S(t)e^{-d(T-t)} - Ke^{-r(T-t)}$
- or delivery contract price $X(t) = S(t)e^{-d(T-t)}$, the price at t to have one share delivered at T

Call option on X

- An option on X with expiration T must have the same value as an option on S
- But the delivery contract pays no dividend (X is its price)
- Process for X: $$\frac{dX_t}{X_t} = (\mu + d)\, dt + \sigma\, dW_t$$
- Drift does not matter!
- Call price: $$\begin{aligned} C(S,t) &= X N(d_1) - K e^{-r(T-t)} N(d_2) \\ &= S e^{-d(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \end{aligned}$$
- with $$d_1 = \frac{\log(S/K) + \left(r - d + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$
- Applies to commodity options - it costs money to hold commodities (d=-q), this is the cost of carry.