Solutions to Practice Final Problems

1. (a) $6!$
   (b) $3! \times 3! \times 2! \times 1!$
   (c) $5! \times 2!$
2. (a) 70 percent
   (b) 23 percent
3. (a)

$$\begin{aligned} P(I \,|\, V) &= \frac{P(V \,|\, I)P(I)}{P(V \,|\, I)P(I) + P(V \,|\, L)P(L) + P(V \,|\, C)P(C)} \\ &= \frac{0.35 \times 0.46}{0.35 \times 0.46 + 0.62 \times 0.3 + 0.58 \times 0.24} \\ &= 0.3311 \end{aligned}$$

   (b)

$$\begin{aligned} P(L \,|\, V) &= \frac{P(V \,|\, L)P(L)}{P(V \,|\, I)P(I) + P(V \,|\, L)P(L) + P(V \,|\, C)P(C)} \\ &= \frac{0.62 \times 0.3}{0.35 \times 0.46 + 0.62 \times 0.3 + 0.58 \times 0.24} \\ &= 0.3826 \end{aligned}$$

   (c)

$$\begin{aligned} P(C \,|\, V) &= \frac{P(V \,|\, C)P(C)}{P(V \,|\, I)P(I) + P(V \,|\, L)P(L) + P(V \,|\, C)P(C)} \\ &= \frac{0.58 \times 0.24}{0.35 \times 0.46 + 0.62 \times 0.3 + 0.58 \times 0.24} \\ &= 0.2863 \end{aligned}$$

   (d)

$$P(V) = P(V \,|\, I)P(I) + P(V \,|\, L)P(L) + P(V \,|\, C)P(C) = 0.4862$$

4. (a)

$$p(u^3s) = p^3,\ p(us) = 3p^2(1-p),\ p(ds) = 3p(1-p)^2,\ p(d^3s) = (1-p)^3$$

(b)

$$E[\max(X-s,0)] = \left[(u^3-1)p^3 + 3(u-1)p^2(1-p)\right] s$$

5. The stock price after 100 time periods can be expressed as

$$X = su^N d^{100-N} = su^{2N} u^{-100}$$

where N is the number of up movements in 100 moves and it is assumed to be a binomial random variable with mean $100p$ and variance $100p(1-p)$. As 100 is large enough, this binomial random variable can be approximated by a normal random variable $Y \sim N(100p, 100p(1-p))$. The probability in question is

$$\begin{aligned} P\{X/s \geq 1.1\} &= P\left\{u^{2N} \geq 1.1u^{100}\right\} \\ &= P\left\{N \geq \frac{\log 1.1 + 100 \log u}{2 \log u}\right\} \\ &\approx 1 - \Phi\left(\frac{\frac{\log 1.1 + 100\log u}{2\log u} - 100p}{\sqrt{100p(1-p)}}\right) \end{aligned}$$

6. (a) Verify that $f(x,y) \geq 0$, and $\frac{6}{7}\int_0^1 \int_0^2 \left(x^2 + \frac{xy}{2}\right) dy, dx = 1$.

   (b)

$$P\{X < Y\} = \int_0^1 \int_x^2 f(x,y)\, dy\, dx = \frac{41}{56}$$

   (c)

$$f_X(x) = \int_0^2 f(x,y)dy = \frac{6}{7}\int_0^2 \left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7}\left(2x^2 + x\right), \quad 0 < x < 1$$

   (d)

$$E[X] = \int_0^1 \int_0^2 xf(x,y)\, dy\, dx = \frac{6}{7}\int_0^1 \int_0^2 \left(x^3 + \frac{x^2 y}{2}\right) dy\, dx = \frac{5}{7}$$

   (e) We only need $f_{X|Y}(x|y=1) = \frac{f(x,1)}{f_Y(1)}$, and

$$f_Y(1) = \int_0^1 f(x,1)\, dx = \frac{1}{2}$$

   So

$$f_{X|Y}(x|y=1) = \frac{12}{7}(x^2 + \frac{x}{2})$$

$$E[X^2 \,|\, Y=1] = \int_0^1 x^2 f_{X|Y}(x\,|y=1)\, dx = \frac{12}{7}\int_0^1 \left(x^4 + \frac{x^3}{2}\right) dx = \frac{39}{70}$$

7. (a)

$$P\{X > 2\} = 1 - F(2) = e^{-\frac{1}{2}\times 2} = e^{-1}$$

(b)

$$P\{X \geq 10 \,|\, X \geq 5\} = P\{X \geq 5\} = e^{-5/2}$$

8. (a)

$$P\{X = i, Y = j\} = \frac{n!}{i!j!(n-i-j)!}\frac{1}{6^{i+j}}\left(\frac{2}{3}\right)^{n-i-j}$$

(b) Let

$$X_i = \begin{cases} 1 & i\text{-th roll is 3} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_j = \begin{cases} 1 & j\text{-th roll is 2} \\ 0 & \text{otherwise} \end{cases}$$

Therefore $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{j=1}^{n} Y_j$.

$$\begin{aligned} \text{Cov}(X,Y) &= E[XY] - E[X]\cdot E[Y] \\ &= \sum_i \sum_j E[X_i Y_j] - \frac{n^2}{36} \\ &= \sum_{i\neq j} \frac{1}{36} - \frac{n^2}{36} \\ &= \frac{1}{36}\left(n^2 - n - n^2\right) \\ &= -\frac{n}{36} \end{aligned}$$

9. (See solution note for Problem 7.19)

(a)

$$E[N-1] = \frac{1}{1/3} - 1 = 3 - 1 = 2$$

(b)

$$E\left[\sum_{j=2}^{3} X_j\right] = \sum_{j=2}^{3} \frac{1/3}{1/3 + 1/3} = 1$$

10. $n = 10,000$ is large enough to apply the central limit theorem. Here $\mu = 240,\ \sigma = 800$.

$$
\begin{aligned}
& P\left\{\sum_{i=1}^{10000} X_i > 2500000\right\} \\
= & \ P\left\{\frac{\sum_{i=1}^{10000} X_i - 240 \times 10000}{100 \times 800} > \frac{2500000 - 2400000}{100 \times 800}\right\} \\
\approx & \ P\{Z > 1.25\} \\
= & \ 1 - \Phi(1.25)
\end{aligned}
$$

Here we used the central limit theorem based on that n is large enough, and assume individual claims are i.i.d.'s.