Final Practice Problems, Math 2280, Fall 2012

Instructions: The exam will be a closed book and closed note exam. You need to show all the details of your work to receive full credit. Calculators are not allowed and we will make sure that numerical calculations involved are kept at minimum.

1. For each of the following equations, determine the order of the equation, verify if it is linear or nonlinear, homogeneous or nonhomogeneous. Try to use direct integration, or the suggested change of variable, to obtain a general solution. In case an initial condition is provided, find the solution for the initial value problem.

   (a)
   $$y' + 4y^2 = 0, \quad y(0) = \frac{1}{2}$$
   for $y = y(x)$.

   (b)
   $$t^2x'' + tx' + x = 2$$
   for $x = x(t)$. Use a change of variable $z = \log t$ to obtain an equation for x as a function of z, and solve the new equation.

2. Consider the damped vibration equation for $x(t)$ with force
   $$x'' + 2x' + 4x = \cos 3t$$

   (a) Derive the characteristic equation for the homogeneous equation and find the roots of the characteristic equation. Without using the known formulas for critical damping, determine if this system is over, under, or critically damped.

   (b) Write down the general solution to the homogeneous equation, and find a particular by trying $x = A\cos 3t + B\sin 3t$.

   (c) Determine if there will be resonance with this forcing.

3. The two brine tanks system is determined by x_1 and x_2, the amounts of salt in tank 1 and tank 2 respectively. Salt water flows into the first tank at a constant rate of 800 gallons an hour, with a concentration of 1 pound of salt per 200 gallons of water. The first tank maintains a constant volume of 400 gallons by continuously pumping well-mixed water into the second tank, at the same constant rate of 800 gallons/hour. The second tank pumps well-mixed water out at this same rate, maintaining a constant volume of 800 gallons.

   (a) Show that the system is modeled by
   $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2x_1 + 4 \\ 2x_1 - x_2 \end{bmatrix}$$

(b) Solve the initial value problem with initial conditions $x_1(0) = x_2(0) = 0$ by writing it as a nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, and using the matrix theory (eigenvalue analysis) for first order systems. Notice that you can find a particular solution by considering what happens to the salt amounts as time approaches infinity.

(c) Find the matrix exponential e^{At}, and express the solution to the above initial value problem in this matrix exponential.

4. Consider the following system of differential equations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x - 2xy \\ -4y + xy \end{bmatrix}$$

(a) If x and y model two interacting populations, what kind of population dynamics would it be? Explain.

(b) Find all the critical points of this system of differential equations. For the critical point where both populations are positive, find the linearized system in terms of u and v.

(c) Write down the general solution for u and v, and determine the stability behavior of the solutions u and v near $(0, 0)$.

(d) Explain why we cannot use the conclusion for the linearized system to determine the stability for the original nonlinear system in this case.

5. Use Laplace transforms to solve the following initial value problem

$$x'' + x = u(t-1), \quad x(0) = 1,\ x'(0) = 0.$$

Here $u(t)$ is the Heaviside function. The following Laplace transform formulas and some partial fractions will be provided for your convenience:

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{f(t-a)u(t-a)\} = F(s)e^{-as}.$$

$$\frac{k^2}{s(s^2+k^2)} = \frac{1}{s} - \frac{s}{s^2+k^2}.$$

6. We consider a 2π periodic square wave function that is given over one period by the following

$$f(x) = \begin{cases} -1 & 0 < x < \pi, \\ 1 & \pi < x < 2\pi. \end{cases}$$

Determine if $f(x)$ is even, odd, or neither. Derive the Fourier series for $f(x)$.

7. Solve the heat equation

$$\frac{\partial u}{\partial t} = 4\,\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2\pi,\ t > 0,$$

with boundary conditions

$$u(0,t) = u(2\pi,t) = 0,\ t > 0,$$

and initial condition $u(x,0) = f(x)$ where $f(x)$ is given in problem 6.