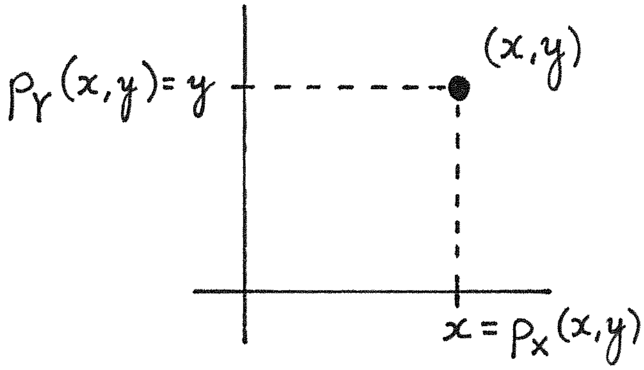
Sine and Cosine

Recall that $p_X : \mathbb{R}^2 \to \mathbb{R}$ where $p_X(x,y) = x$ is the projection onto the x-axis, and that $p_Y : \mathbb{R}^2 \to \mathbb{R}$ where $p_Y(x,y) = y$ is the projection onto the y-axis.



Examples:

- $p_X(2,8) = 2$
- $p_Y(-3,5) = 5$
- $p_X\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}$

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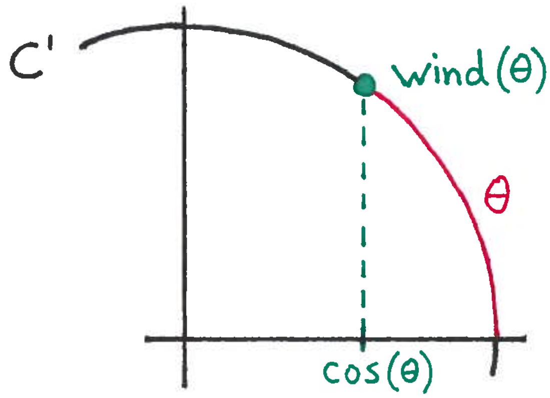
Definition of cosine

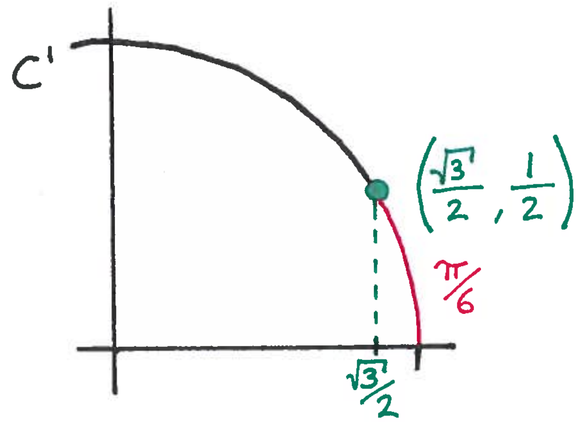
The *cosine* function is the function $\cos : \mathbb{R} \to \mathbb{R}$ defined as

$$\cos(\theta) = p_X \circ \text{wind}(\theta)$$

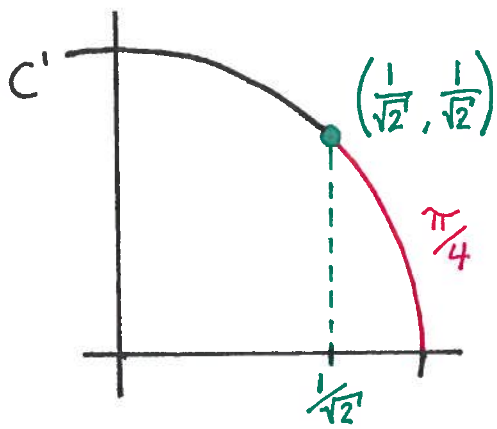


Examples.

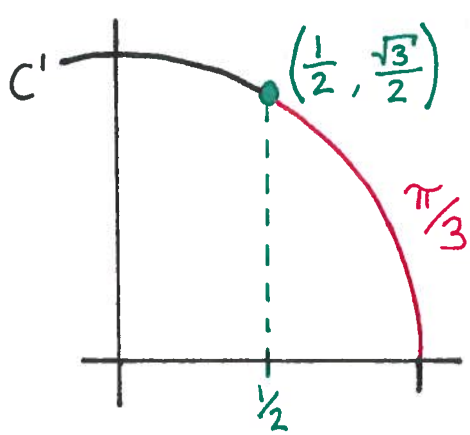
- $\cos\left(\frac{\pi}{6}\right) = p_X \circ \text{wind}\left(\frac{\pi}{6}\right) = p_X\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{\sqrt{3}}{2}$



- $\cos\left(\frac{\pi}{4}\right) = p_X \circ \text{wind}\left(\frac{\pi}{4}\right) = p_X\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$



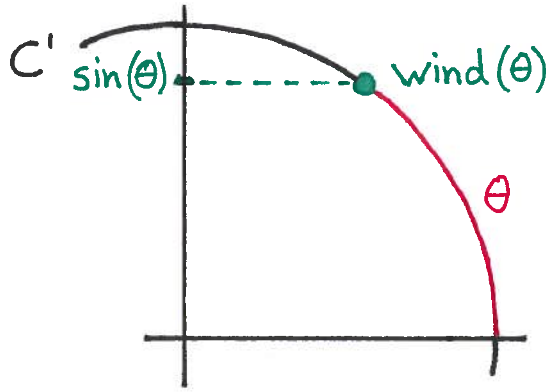
- $\cos\left(\frac{\pi}{3}\right) = p_X \circ \text{wind}\left(\frac{\pi}{3}\right) = p_X\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}$



Definition of sine

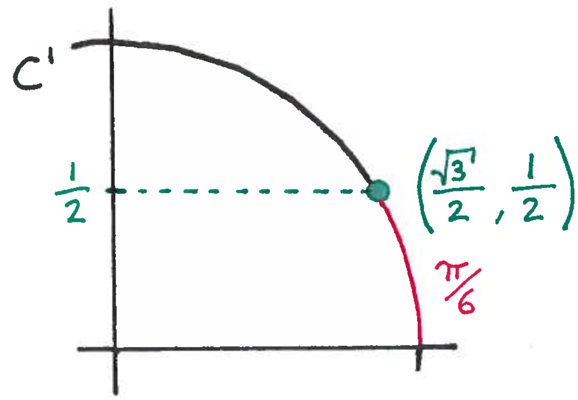
The *sine* function is the function $\sin : \mathbb{R} \to \mathbb{R}$ defined as

$$\sin(\theta) = p_Y \circ \text{wind}(\theta)$$

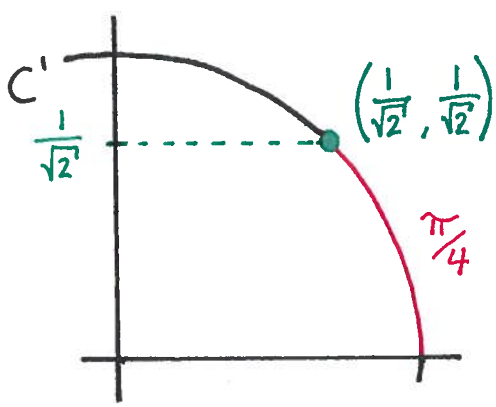


Examples.

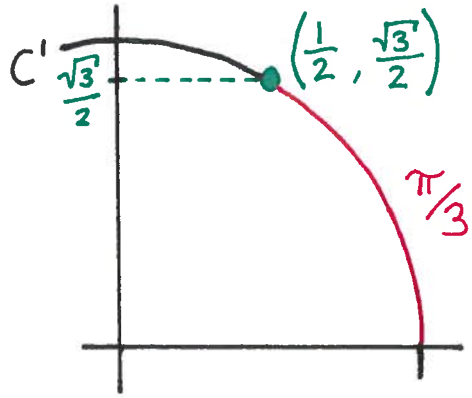
- $\sin\left(\frac{\pi}{6}\right) = p_Y \circ \text{wind}\left(\frac{\pi}{6}\right) = p_Y\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{1}{2}$



- $\sin\left(\frac{\pi}{4}\right) = p_Y \circ \text{wind}\left(\frac{\pi}{4}\right) = p_Y\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$



- $\sin\left(\frac{\pi}{3}\right) = p_Y \circ \text{wind}\left(\frac{\pi}{3}\right) = p_Y\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$

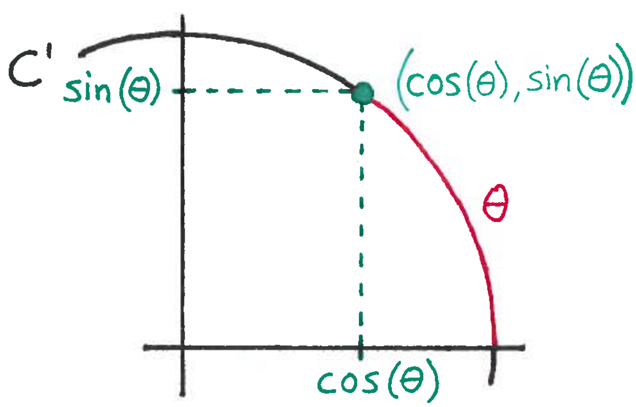


Cosine and sine are the coordinates of wind

If $\theta \in \mathbb{R}$, then $\cos(\theta)$ is $p_X \circ \text{wind}(\theta)$. That is, $\cos(\theta)$ is the x-coordinate of the point $\text{wind}(\theta)$. Similarly, $\sin(\theta)$ is the y-coordinate of the point $\text{wind}(\theta)$. Taken together, we have

$$\text{wind}(\theta) = \big(\cos(\theta), \sin(\theta)\big)$$

Throughout mathematics, the point on the unit circle obtained by beginning at the point $(1, 0)$ and winding a length of θ is usually written as $\big(\cos(\theta), \sin(\theta)\big)$, and that's the way we'll usually write it from now on.

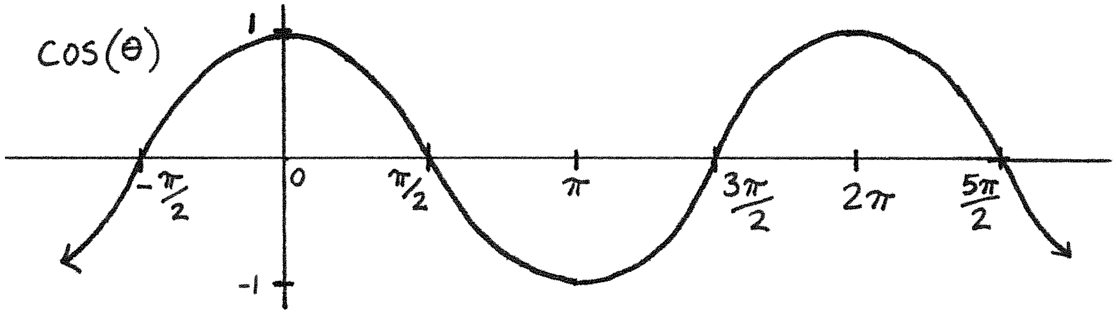


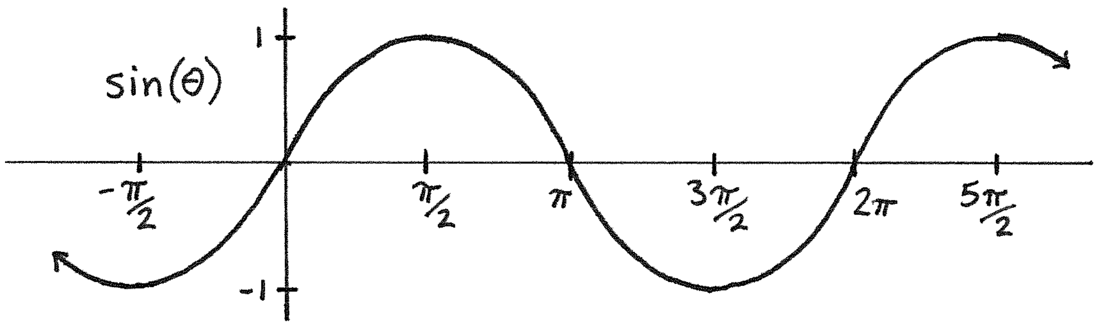
It will be important to keep in mind that a point on the unit circle is a point of the form $\big(\cos(\theta), \sin(\theta)\big)$, and that any point of the form $\big(\cos(\theta), \sin(\theta)\big)$ is a point on the unit circle.

The next page contains a list of some common values of θ that arise in trigonometry, along with their values from cos and sin.

θ	$\text{wind}(\theta)$	$\cos(\theta)$	$\sin(\theta)$
$\frac{-\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
0	$\big(1, 0\big)$	1	0
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	$\big(0, 1\big)$	0	1
$\frac{2\pi}{3}$	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{5\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
π	$\big(-1, 0\big)$	-1	0
$\frac{7\pi}{6}$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

Graphs of sine and cosine





Identities for sine and cosine

An *identity* is an equation in one variable that is true for every possible value of the variable. For example, $x + x = 2x$ is an identity because it's always true. It does't matter whether x equals 1, or 5, or $-\frac{3}{5}$; it's always true that $x + x = 2x$.

The remainder of this chapter contains an assortment of important identities for the functions sine and cosine.

Lemma (7). (The Pythagorean identity) For any number θ,

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

Proof: The equation for the unit circle is $x^2 + y^2 = 1$. Since $(\cos(\theta), \sin(\theta))$ is a point on the unit circle, it is a solution of this equation. That is,

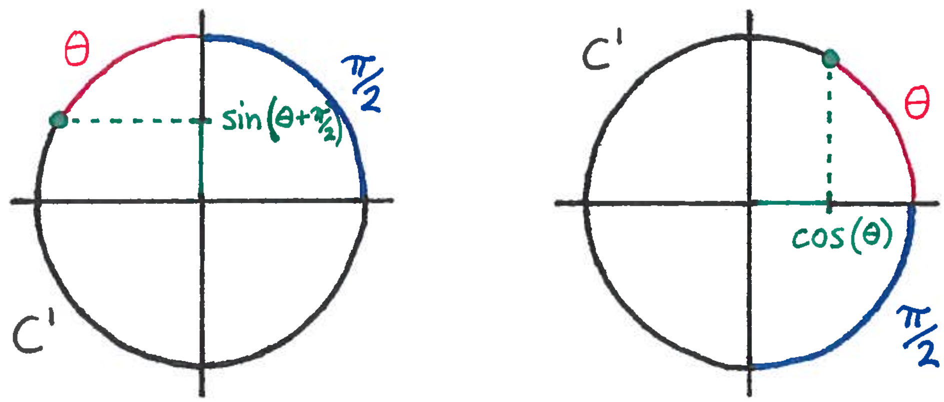
$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

■

Lemma (8). For any number θ,

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$$

Proof: The marked point on the y-axis in the picture on the left is $\sin\left(\theta + \frac{\pi}{2}\right)$. It's the y-coordinate of the point obtained by winding around the circle a distance of $\frac{\pi}{2}$ and then winding another θ more. We can rotate the picture on the left by a quarter turn clockwise, which would match $\sin\left(\theta + \frac{\pi}{2}\right)$ with the x-coordinate of the point obtained by winding around the circle a distance of θ, the number $\cos(\theta)$. Thus, $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$.



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Lemma (9). For any number θ,

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$$

Proof: We'll use Lemma 8 to prove this lemma. Notice that Lemma 8 tells us

$$\sin\left(\left[\theta - \frac{\pi}{2}\right] + \frac{\pi}{2}\right) = \cos\left(\left[\theta - \frac{\pi}{2}\right]\right)$$

Simplifying, we have

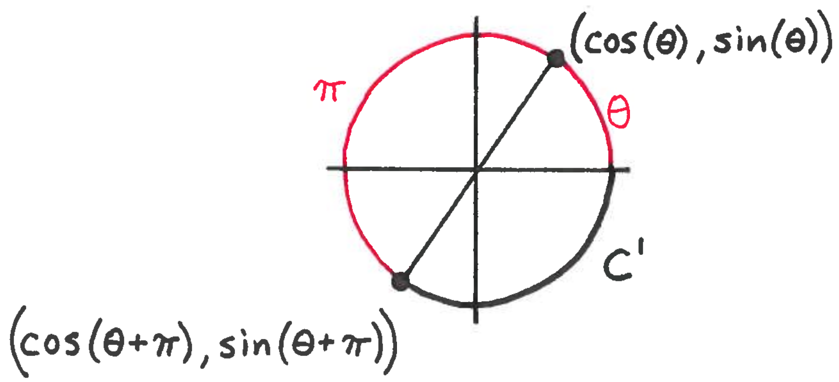
$$\sin(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)$$

which is what we had wanted to show. ■

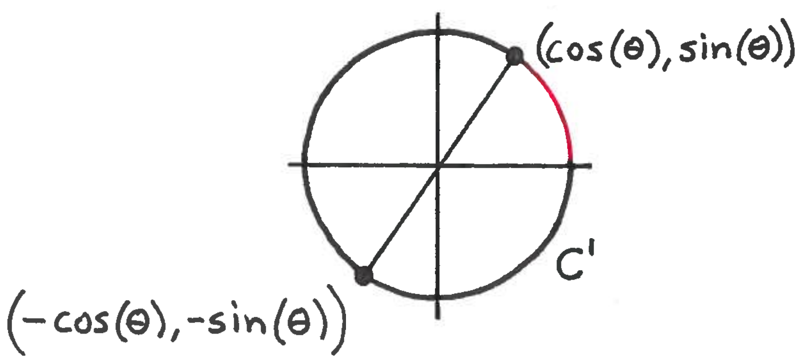
Lemma (10). For any number θ,

$$\cos(\theta+\pi) = -\cos(\theta) \qquad \text{and} \qquad \sin(\theta+\pi) = -\sin(\theta)$$

Proof: The number π is exactly half the length of the unit circle. Therefore, the point $\big(\cos(\theta+\pi)\,,\,\sin(\theta+\pi)\big)$ is the point on the unit circle that is exactly halfway around the unit circle from the point $\big(\cos(\theta)\,,\,\sin(\theta)\big)$.



Also notice that the negative of the vector $\big(\cos(\theta)\,,\,\sin(\theta)\big)$, which is the vector $\big(-\cos(\theta)\,,\,-\sin(\theta)\big)$, is the vector that points in the opposite direction of $\big(\cos(\theta)\,,\,\sin(\theta)\big)$.



We can see in the two pictures above that the vectors drawn are the same. That is,

$$\big(\cos(\theta+\pi)\,,\,\sin(\theta+\pi)\big) = \big(-\cos(\theta)\,,\,-\sin(\theta)\big)$$

Because these vectors are equal, their first coordinates are equal

$$\cos(\theta+\pi) = -\cos(\theta)$$

and their second coordinates are equal.

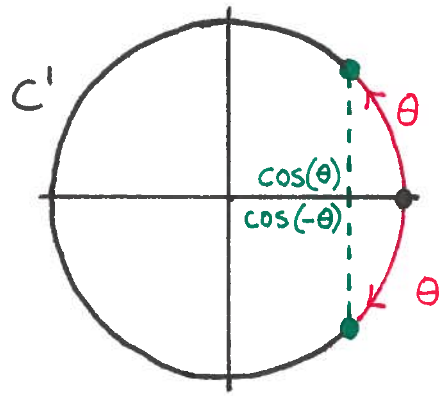
$$\sin(\theta+\pi) = -\sin(\theta)$$

■

Lemma (11). For any number θ,

$$\cos(-\theta) = \cos(\theta)$$

Proof: Whether we wind clockwise around the circle a length of θ, or counterclockwise a length of θ, the x-coordinates will be the same. Which is to say that $\cos(\theta) = \cos(-\theta)$.



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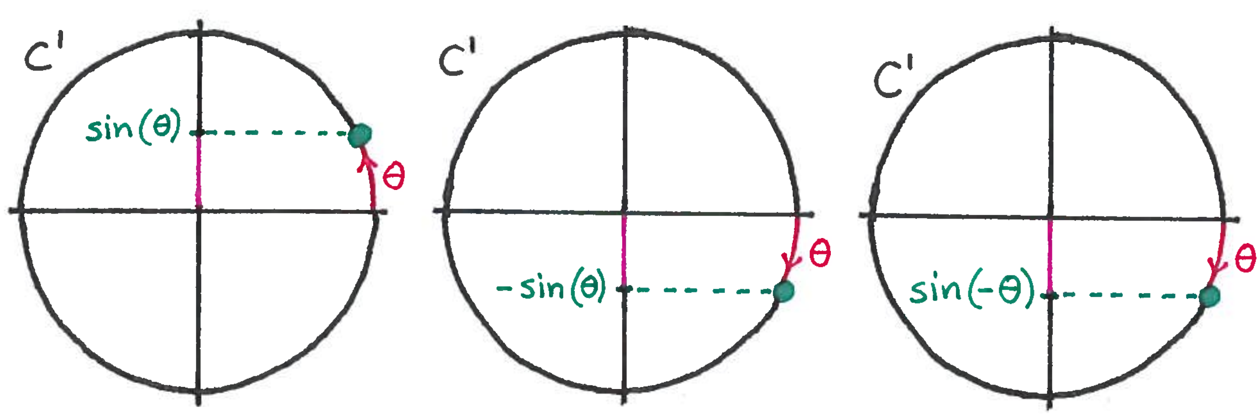
Lemma (12). For any number θ,

$$\sin(-\theta) = -\sin(\theta)$$

Proof: The picture on the left shows $\sin(\theta)$. The picture in the middle is the first picture flipped over the x-axis. All of the y-coordinates are exchanged with their negatives, so the picture in the middle shows $-\sin(\theta)$. The picture on the right shows $\sin(-\theta)$. Notice that the rightmost picture is the same picture as the one in the middle, and thus, $-\sin(\theta) = \sin(-\theta)$.



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Even and odd functions

An *even* function is a function $f(x)$ that has the property $f(-x) = f(x)$ for every value of x. Examples of such functions include x^2, x^4, x^6, and by Lemma 11, $\cos(x)$.

An *odd* function is a function $g(x)$ that has the property $g(-x) = -g(x)$ for every value of x. Examples of such functions include x^3, x^5, x^7, and by Lemma 12, $\sin(x)$.

Period of sine and cosine

The period of the winding function is 2π, meaning that

$$\text{wind}(\theta) = \text{wind}(\theta + 2\pi)$$

Therefore,

$$\big(\, \cos(\theta)\, ,\, \sin(\theta)\, \big) = \big(\, \cos(\theta + 2\pi)\, ,\, \sin(\theta + 2\pi)\, \big)$$

Because these vectors are equal, their first coordinates are equal

$$\cos(\theta) = \cos(\theta + 2\pi)$$

and their second coordinates are equal

$$\sin(\theta) = \sin(\theta + 2\pi)$$

These last two identities show that sine and cosine are, just as the winding function, *periodic* functions. Their period is 2π.

Exercises

For #1-14, identify the given value.

1.) $\cos\left(\frac{5\pi}{4}\right)$

2.) $\cos\left(\frac{4\pi}{3}\right)$

3.) $\cos\left(\frac{3\pi}{2}\right)$

4.) $\cos\left(\frac{5\pi}{3}\right)$

5.) $\cos\left(\frac{7\pi}{4}\right)$

6.) $\cos\left(\frac{11\pi}{6}\right)$

7.) $\cos\left(2\pi\right)$

8.) $\sin\left(\frac{5\pi}{4}\right)$

9.) $\sin\left(\frac{4\pi}{3}\right)$

10.) $\sin\left(\frac{3\pi}{2}\right)$

11.) $\sin\left(\frac{5\pi}{3}\right)$

12.) $\sin\left(\frac{7\pi}{4}\right)$

13.) $\sin\left(\frac{11\pi}{6}\right)$

14.) $\sin\left(2\pi\right)$

Suppose that α is a real number, that $0 \leq \alpha \leq \frac{\pi}{2}$, and that $\cos(\alpha) = \frac{2}{3}$. Use Lemmas 7-12, and that the period of sine and cosine is 2π to find the following values.

15.) $\sin(\alpha)$

16.) $\sin(\alpha + \frac{\pi}{2})$

17.) $\cos(\alpha - \frac{\pi}{2})$

18.) $\cos(\alpha + \pi)$

19.) $\sin(\alpha + \pi)$

20.) $\cos(-\alpha)$

21.) $\sin(-\alpha)$

22.) $\cos(\alpha + 2\pi)$

23.) $\sin(\alpha + 2\pi)$

Match the numbered piecewise defined functions with their lettered graphs below.

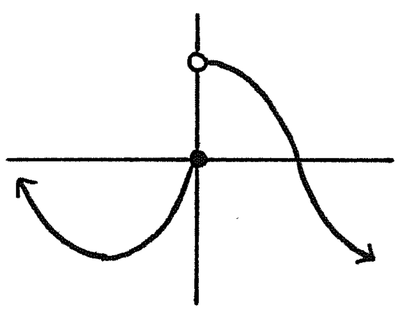
24.) $f(x) = \begin{cases} \cos(x) & \text{if } x \geq 0; \text{ and} \\ \sin(x) & \text{if } x < 0. \end{cases}$

25.) $g(x) = \begin{cases} \sin(x) & \text{if } x \geq 0; \text{ and} \\ \cos(x) & \text{if } x < 0. \end{cases}$

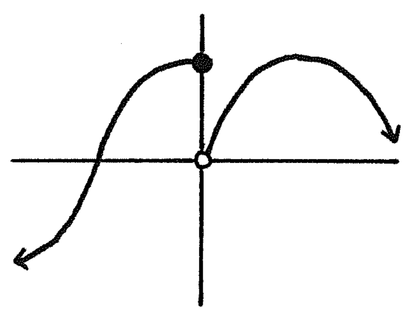
26.) $h(x) = \begin{cases} \cos(x) & \text{if } x > 0; \text{ and} \\ \sin(x) & \text{if } x \leq 0. \end{cases}$

27.) $p(x) = \begin{cases} \sin(x) & \text{if } x > 0; \text{ and} \\ \cos(x) & \text{if } x \leq 0. \end{cases}$

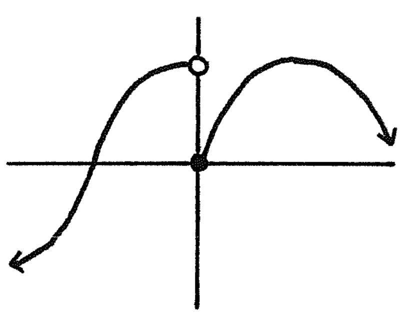
A.)



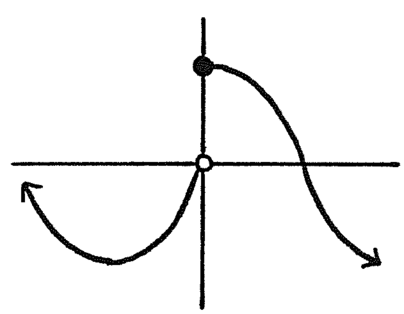
B.)



C.)



D.)



For #28-39, identify the given value.

28.) $\log_2(8)$ 29.) $\log_5(125)$ 30.) $\log_3(9)$ 31.) $\log_{13}(13)$

32.) $\log_9(3)$ 33.) $\log_2\left(\frac{1}{4}\right)$ 34.) $\log_6\left(\frac{1}{6}\right)$ 35.) $\log_{10}(10,000)$

36.) $\log_7(49)$ 37.) $\log_e(e^7)$ 38.) $\log_e(\sqrt{e})$ 39.) $\log_e\left(\frac{1}{e}\right)$

Find the solutions of the equations given in #40-42.

40.) $\frac{x-1}{x} - 2x = 6$ 41.) $\frac{x-1}{3x+2} = 4$ 42.) $x + \frac{1}{x} = 4$