Piecewise Defined Functions

Most of the functions that we've looked at this semester can be expressed as a single equation. For example, $f(x) = 3x^2 - 5x + 2$, or $g(x) = \sqrt{x-1}$, or $h(x) = e^{3x} - 1$.

Sometimes an equation can't be described by a single equation, and instead we have to describe it using a combination of equations. Such functions are called *piecewise defined functions*, and probably the easiest way to describe them is to look at a couple of examples.

First example. The function $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} x^2 - 1 & \text{if } x \in (-\infty, 0]; \\ x - 1 & \text{if } x \in [0, 4]; \\ 3 & \text{if } x \in [4, \infty). \end{cases}$$

The function g is a piecewise defined function. It is defined using three functions that we're more comfortable with: $x^2 - 1$, $x - 1$, and the constant function 3. Each of these three functions is paired with an interval that appears on the right side of the same line as the function: $(-\infty, 0]$, and $[0, 4]$, and $[4, \infty)$ respectively.

If you want to find $g(x)$ for a specific number x, first locate which of the three intervals that particular number x is in. Once you've decided on the correct interval, use the function that interval is paired with to determine $g(x)$.

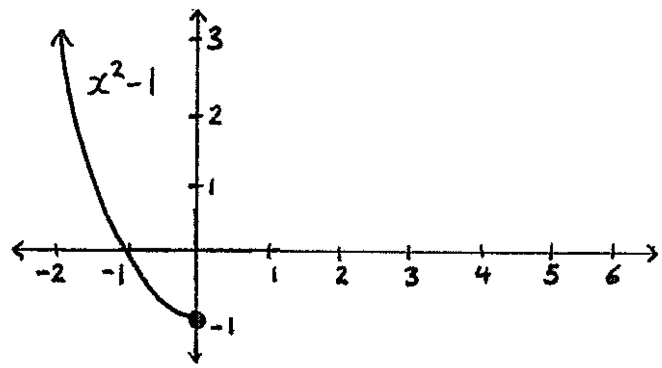
If you want to find $g(2)$, first check that $2 \in [0, 4]$. Therefore, we should use the equation $g(x) = x - 1$, because $x - 1$ is the function that the interval $[0, 4]$ is paired with. That means that $g(2) = 2 - 1 = 1$.

To find $g(5)$, notice that $5 \in [4, \infty)$. That means we should be looking at the third interval used in the definition of $g(x)$, and the function paired with that interval is the constant function 3. Therefore, $g(5) = 3$.

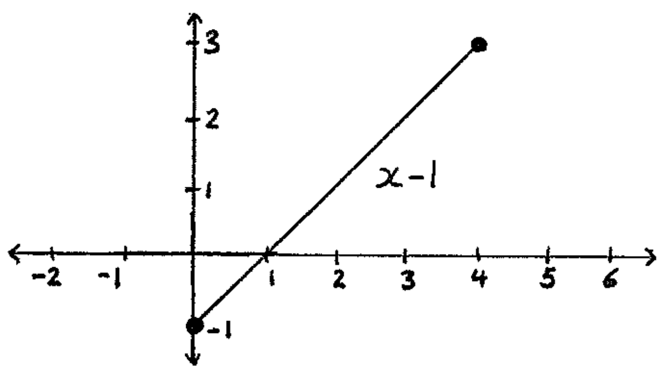
Let's look at one more number. Let's find $g(0)$. First we have to decide which of the three intervals used in the definition of $g(x)$ contains the number 0. Notice that there's some ambiguity here because 0 is contained in both the interval $(-\infty, 0]$ and in the interval $[0, 4]$. Whenever there's ambiguity, choose either of the intervals that are options. Either of the functions that these intervals are paired with will give you the same result. That is, $0^2 - 1 = -1$ is the same number as $0 - 1 = -1$, so $g(0) = -1$.

To graph $g(x)$, graph each of the pieces of g. That is, graph $g : (-\infty, 0] \to \mathbb{R}$ where $g(x) = x^2 - 1$, and graph $g : [0, 4] \to \mathbb{R}$ where $g(x) = x - 1$, and graph $g : [4, \infty) \to \mathbb{R}$ where $g(x) = 3$. Together, these three pieces make up the graph of $g(x)$.

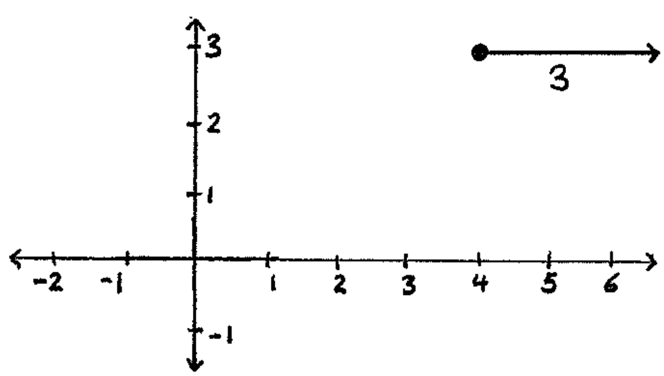
Graph of $g : (-\infty, 0] \to \mathbb{R}$ where $g(x) = x^2 - 1$.



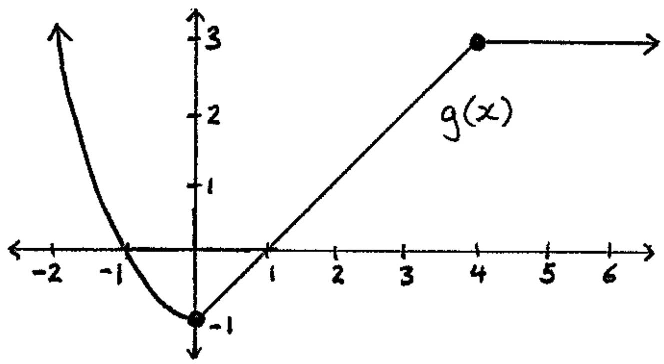
Graph of $g : [0, 4] \to \mathbb{R}$ where $g(x) = x - 1$.



Graph of $g : [4, \infty) \to \mathbb{R}$ where $g(x) = 3$.



To graph $g(x)$, draw the graphs of all three of its pieces.

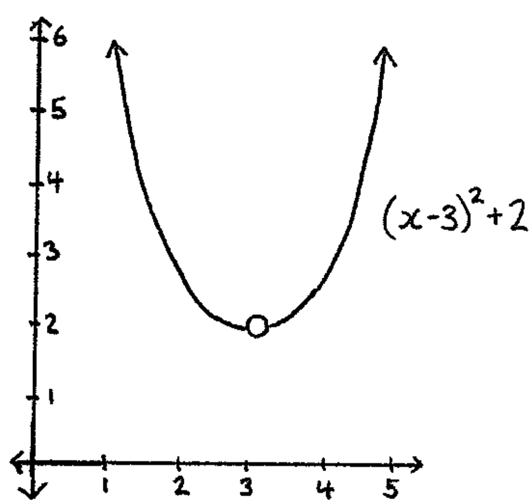


Second example. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

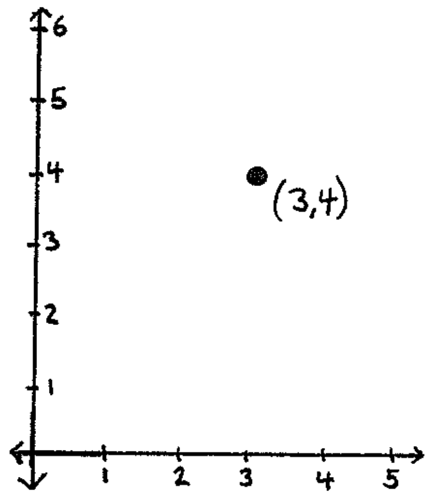
$$f(x) = \begin{cases} (x-3)^2 + 2 & \text{if } x \neq 3; \\ 4 & \text{if } x = 3. \end{cases}$$

This function is made up of two pieces. Either $x \neq 3$, in which case $f(x) = (x-3)^2 + 2$. Or $x = 3$, and then $f(3) = 4$.

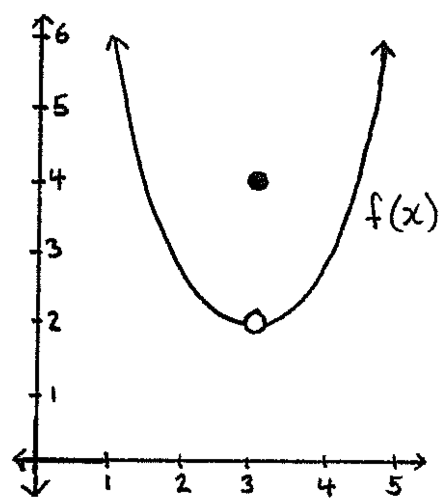
Graph of the first piece of $f(x)$: the graph of x^2 shifted right 3 and up 2 with the point of the graph whose x-coordinate equals 3 removed. (Remember that a little circle means that point is *not* a point of the graph.)



Graph of the second piece of $f(x)$: a single giant dot whose x-coordinate equals 3.



Graph of both pieces, and hence the entire graph, of $f(x)$.



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Absolute value

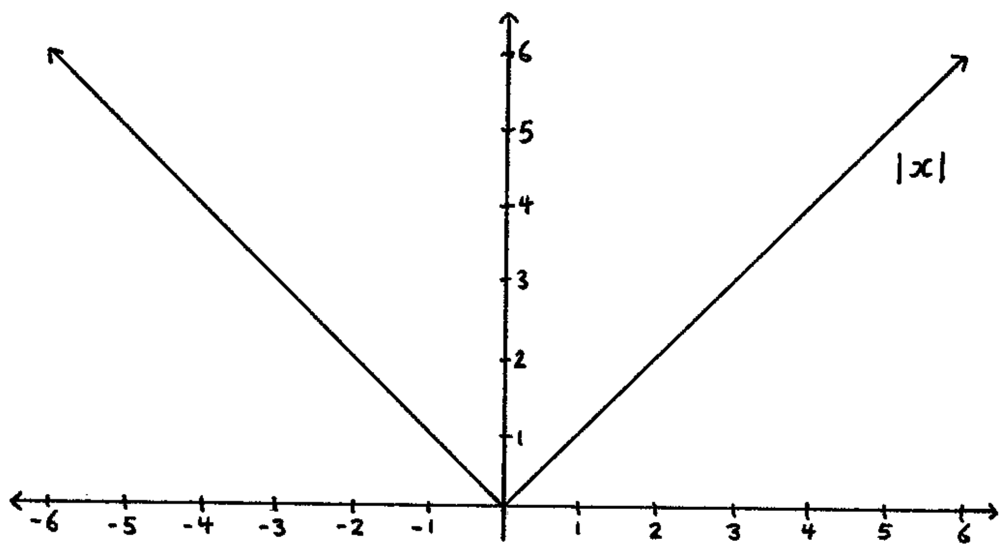
The most important piecewise defined function in calculus is the absolute value function that is defined by

$$|x| = \begin{cases} -x & \text{if } x \in (-\infty, 0]; \\ x & \text{if } x \in [0, \infty). \end{cases}$$

The domain of the absolute value function is $\mathbb{R}$. The range of the absolute value function is the set of non-negative numbers. The number $|x|$ is called the *absolute value* of x.

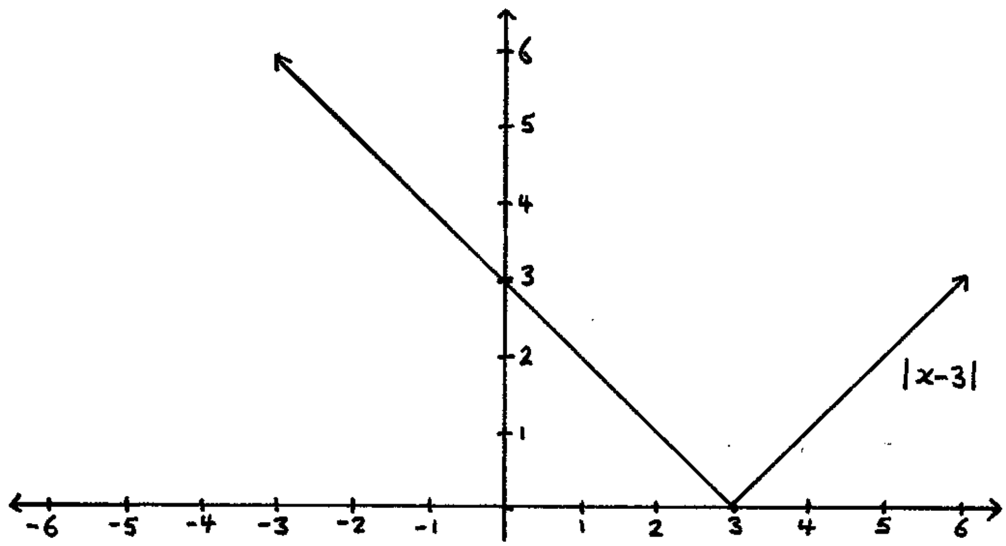
For examples of how this function works, notice that $|4| = 4$, $|0| = 0$, and $|-3| = -(-3) = 3$. If x is positive or 0, then the absolute value of x is x itself. If x is negative, then $|x|$ is the positive number that you'd get from "erasing" the negative sign: $|-10| = 10$ and $|-\frac{1}{2}| = \frac{1}{2}$.

Graph of the absolute value function.



Another interpretation of the absolute value function, and the one that's most important for calculus, is that the absolute value of a number is the same as its distance from 0. That is, the distance between 0 and 5 is $|5| = 5$, the distance between 0 and -7 is $|-7| = 7$, and the distance between 0 and 0 is $|0| = 0$.

Let's look at the graph of say $|x-3|$. It's the graph of $|x|$ shifted right by 3.



You might guess from the graph of $|x-3|$, that $|x-3|$ is the function that measures the distance between x and 3, and that's true. Similarly, $|x-6|$ is the distance between x and 6, $|x+2|$ is the distance between x and -2, and more generally, $|x-y|$ is the distance between x and y.

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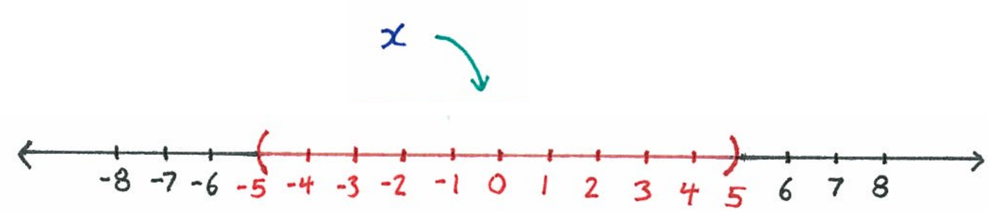
Solving inequalities involving absolute values

The inequality $|x| < 5$ means that the distance between x and 0 is less than 5. Therefore, x is between -5 and 5. Another way to write the previous sentence is $-5 < x < 5$.



Notice in the above paragraph that the precise number 5 wasn't really important for the problem. We could have replaced 5 with any positive number c to obtain the following translation.

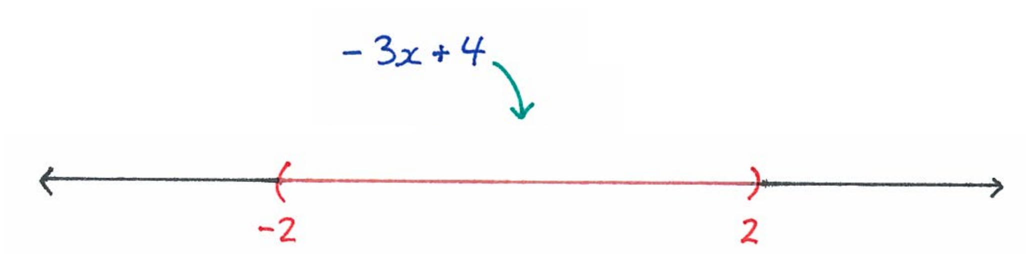
$$|x| < c \text{ means } -c < x < c$$

For example, writing $|x| < 2$ means the same thing as writing $-2 < x < 2$, and $|2x - 3| < \frac{1}{3}$ means the same as $-\frac{1}{3} < 2x - 3 < \frac{1}{3}$.

We can use the above rule to help us solve some inequalities that involve absolute values.

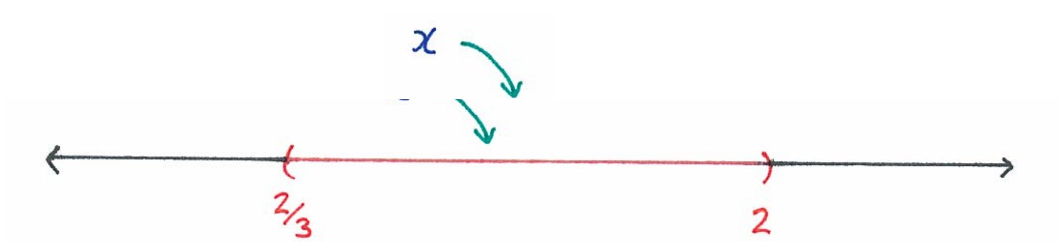
Problem. Solve for x if $|-3x + 4| < 2$.

Solution. We know from the explanation above that $-2 < -3x + 4 < 2$.



Subtracting 4 from all three of the quantities in the previous inequality yields $-2 - 4 < -3x < 2 - 4$, and that can be simplified as $-6 < -3x < -2$.

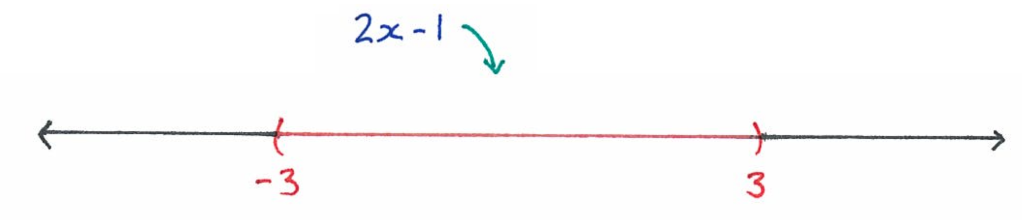
Next divide by -3, keeping in mind that dividing an inequality by a negative number "flips" the inequalities. The result will be $\frac{-6}{-3} > x > \frac{-2}{-3}$, which can be simplified as $2 > x > \frac{2}{3}$. That's the answer.



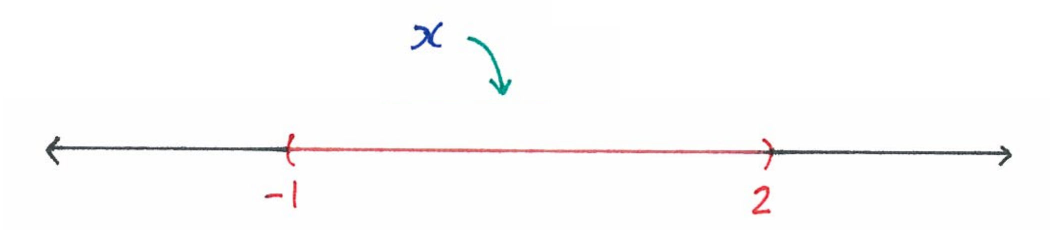
The inequality $2 > x > \frac{2}{3}$ could also be written as $\frac{2}{3} < x < 2$, or as $x \in (\frac{2}{3}, 2)$.

Problem. Solve for x if $|2x-1|<3$.

Solution. Write the inequality from the problem as $-3<2x-1<3$.



Add 1 to get $-2<2x<4$, and divide by 2 to get $-1<x<2$.



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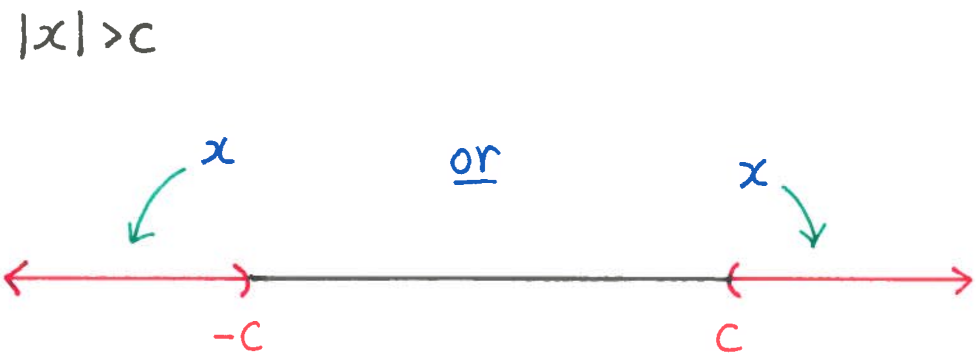
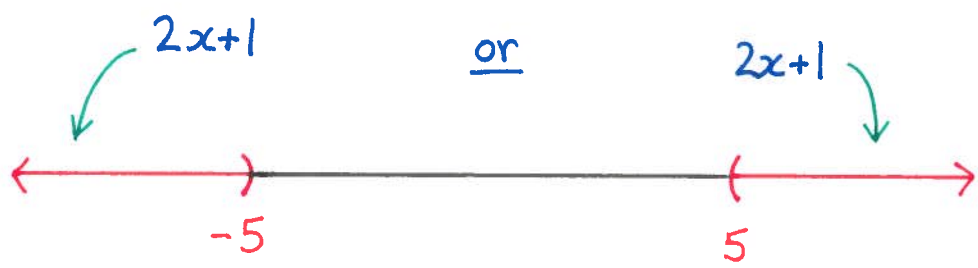
If c is a positive number, $|x|>c$ means that the distance between x and 0 is greater than c. There are two ways that the distance between x and 0 can be greater than c. Either $x<-c$ or $x>c$.



$$|x|>c \text{ means either } x<-c \text{ or } x>c$$

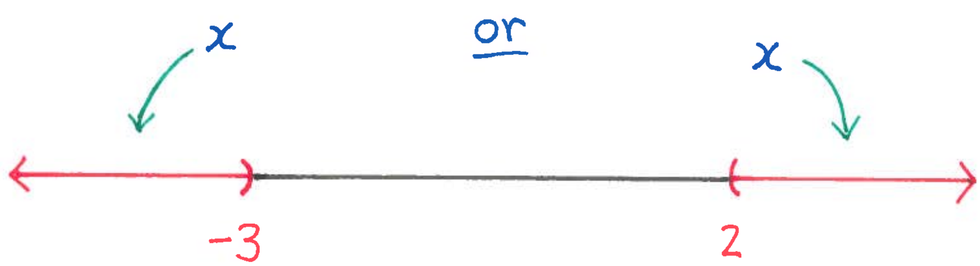
Problem. Solve for x if $|2x+1| > 5$.



Solution. $|2x+1| > 4$ means that either $2x+1 < -5$ or $2x+1 > 5$. This leaves us with two different inequalities to solve. Let's start with the inequality $2x+1 < -5$. Subtract 1, and divide by 2 to find that $x < -3$. That's one half of our answer.

For the second half of the answer, solve the second inequality: $2x+1 > 5$. Subtract 1, and divide by 2 to find that $x > 2$. That's the second half of our answer.

To summarize, if $|2x+1| > 4$, then either $x < -3$ or $x > 2$.



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Two important rules for absolute values

For the two rules below, $a, b, c \in \mathbb{R}$. Each rule is important for calculus. They'll be explained in class.

1. $|ab| = |a||b|$

2. $|a-c| \leq |a-b| + |b-c|$ (*triangle inequality*)

Exercises

1.) Suppose $f(x)$ is the piecewise defined function given by

$$f(x) = \begin{cases} x+1 & \text{if } x \in (-\infty, 2); \\ x+3 & \text{if } x \in [2, \infty). \end{cases}$$

What is $f(0)$? What is $f(10)$? What is $f(2)$?

2.) Suppose $g(x)$ is the piecewise defined function given by

$$g(x) = \begin{cases} 3 & \text{if } x \in [1, 5]; \\ 1 & \text{if } x \in (5, \infty). \end{cases}$$

What is $g(1)$? What is $g(100)$? What is $g(5)$?

3.) Suppose $h(x)$ is the piecewise defined function given by

$$h(x) = \begin{cases} 5 & \text{if } x \in (1, 3]; \\ x+2 & \text{if } x \in [3, 8). \end{cases}$$

What is $h(2)$? What is $h(7)$? What is $h(3)$?

4.) Suppose $f(x)$ is the piecewise defined function given by

$$f(x) = \begin{cases} 2 & \text{if } x \in [-3, 0); \\ e^x & \text{if } x \in [0, 2]; \\ 3x-2 & \text{if } x \in (2, \infty). \end{cases}$$

What is $f(-2)$? What is $f(0)$? What is $f(2)$? What is $f(15)$?

5.) Suppose $g(x)$ is the piecewise defined function given by

$$g(x) = \begin{cases} (x-1)^2 & \text{if } x \in (-\infty, 1]; \\ \log_e(x) & \text{if } x \in [1, 5]; \\ \log_e(5) & \text{if } x \in [5, \infty). \end{cases}$$

What is $g(0)$? What is $g(1)$? What is $g(5)$? What is $g(20)$?

6.) Suppose $h(x)$ is the piecewise defined function given by

$$h(x) = \begin{cases} e^x & \text{if } x \neq 2; \\ 1 & \text{if } x = 2. \end{cases}$$

What is $h(0)$? What is $h(2)$? What is $h\big(\log_e(17)\big)$?

7.) Write the following numbers as integers: $|8-5|$, $|-10-5|$, and $|5-5|$. The function $|x-5|$ measures the distance between x and which number?

8.) Write the following numbers as integers: $|1-2|$, $|3-2|$, and $|2-2|$. The function $|x-2|$ measures the distance between x and which number?

9.) Write the following numbers as integers: $|3+4|$, $|-1+4|$, $|-4+4|$. The function $|x+4|$ measures the distance between x and which number?

10.) The function $|x-y|$ measures the distance between x and which number?

11.) Solve for x if $|5x-2| < 7$.

12.) Solve for x if $|3x+4| < 1$.

13.) Solve for x if $|-2x+3| < 5$.

14.) Solve for x if $|x+3| > 2$.

15.) Solve for x if $|4x| > 12$.

16.) Solve for x if $|2x+4| > 8$.

Match the functions with their graphs.

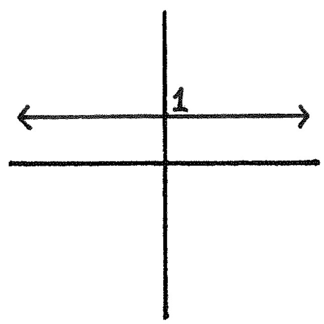
17.) $f(x) = 2x + 1$

18.) $g(x) = 1$

19.) $p(x) = \begin{cases} 2x+1 & \text{if } x \in (-\infty, 0); \\ 1 & \text{if } x \in [0, \infty). \end{cases}$

20.) $q(x) = \begin{cases} 1 & \text{if } x \in (-\infty, 0); \\ 2x+1 & \text{if } x \in [0, \infty). \end{cases}$

A.)



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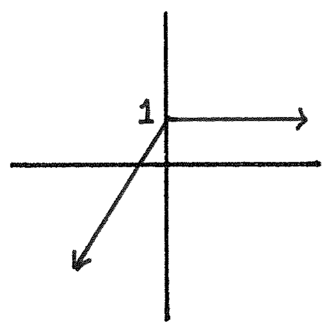
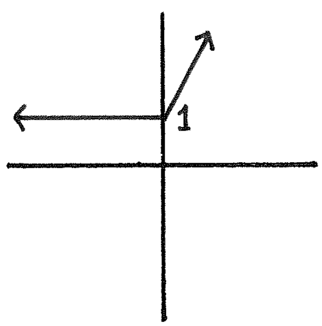
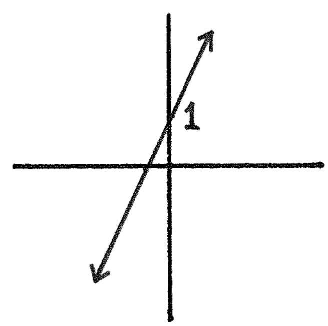


C.)



D.)



Match the functions with their graphs.

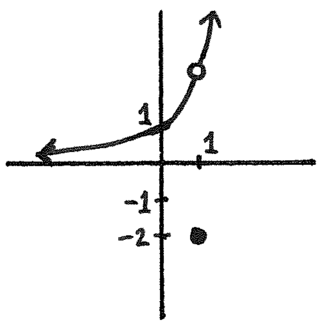
21.) $f(x) = e^x$

22.) $g(x) = -2$

23.) $p(x) = \begin{cases} e^x & \text{if } x \neq 1; \\ -2 & \text{if } x = 1. \end{cases}$

24.) $q(x) = \begin{cases} e^x & \text{if } x \neq -1; \\ -2 & \text{if } x = -1. \end{cases}$

A.)



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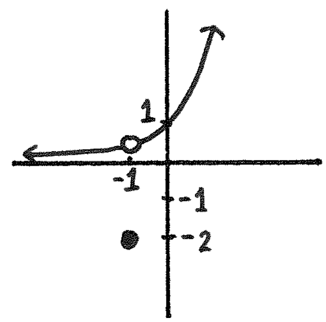

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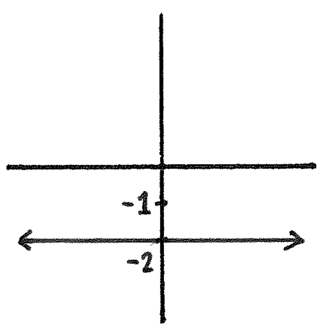
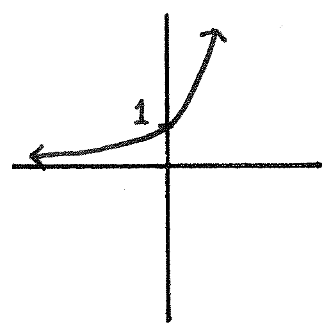


D.)



Match the functions with their graphs.

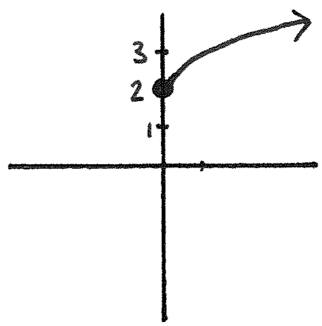
25.) $f(x) = \sqrt{x} + 2$

26.) $g(x) = 2$

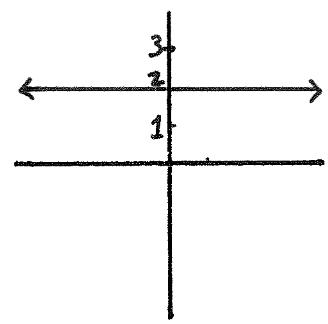
27.) $p(x) = \begin{cases} 2 & \text{if } x \in (-\infty, 0); \\ \sqrt{x} + 2 & \text{if } x \in [0, \infty). \end{cases}$

28.) $q(x) = \begin{cases} 2 & \text{if } x \in (-\infty, 1); \\ \sqrt{x} + 2 & \text{if } x \in [1, \infty). \end{cases}$

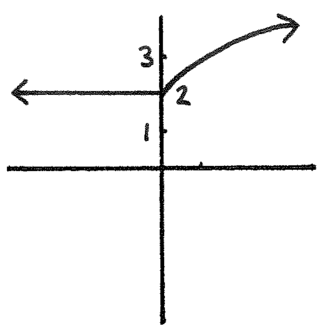
A.)



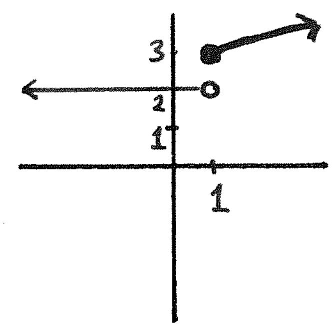
B.)



C.)



D.)



Simplify the expressions in #29-34.

29.) 3^4

30.) $\left(\frac{1}{4}\right)^{-2}$

31.) $\left(\frac{1}{e^{-2}}\right)^4$

32.) $8^{\frac{2}{3}}$

33.) $27^{-\frac{2}{3}}$

34.) $16^{\frac{3}{2}}$

Each of the numbers in #35-40 is an integer. Which integers are they?

35.) $\log_e(e^6)$

36.) $\log_e\left(\frac{1}{e^3}\right)$

37.) $\log_2(8)$

38.) $\log_2\big(32\big)$

39.) $\log_2(\frac{1}{4})$

40.) $\log_3(81)$