Math 5410 § 1. **First Midterm Exam** Name: Solutions

Treibergs Sept. 11, 2024

1. *Find the general solution. Find the Poincaré map for 2π periodic solutions. Is there a 2π-periodic solution? is it unique? Why?*

$$x' = [\sin(t) - 1]\, x + \cos t.$$

An integrating factor is

$$e^{-\int^t \sin(s)-1\,ds} = e^{\cos(t)+t}.$$

Multiplying the equation

$$\left(e^{\cos(t)+t}\, x\right)' = e^{\cos(t)+t}\left(x' - [\sin(t)-1]\, x\right) = e^{\cos(t)+t}\cos t$$

and integrating yields

$$e^{\cos(t)+t}\, x(t) - e^1 x(0) = \int_0^t e^{\cos(s)+s}\, \cos s\, ds.$$

Thus the solution starting from $x(0) = x_0$ is

$$x(t) = e^{-\cos(t)-t}\left(e x_0 + \int_0^t e^{\cos(s)+s}\, \cos s\, ds.\right)$$

The Poincaré map is gotten by evaluating the solution at one period $t = 2\pi$

$$\wp(x_0) = e^{-1-2\pi}\left(e x_0 + \int_0^{2\pi} e^{\cos(s)+s}\, \cos s\, ds.\right)$$

There are 2π-periodic solutions for every x_0 such that $x_0 = \wp(x_0)$. But this equation

$$x_0 = e^{-1-2\pi}\left(e x_0 + \int_0^{2\pi} e^{\cos(s)+s}\, \cos s\, ds\right)$$

is linear and has exactly one solution because $e^{-2\pi} \neq 1$, namely

$$x_0 = \frac{e^{-1-2\pi}}{1-e^{-2\pi}} \int_0^{2\pi} e^{\cos(s)+s}\, \cos s\, ds.$$

Thus the trajectory corresponding to this x_0 is the unique 2π-periodic solution.

2. *Determine the canonical form for this equation. Find the matrix T so that $Y = TX$ puts (1) in canonical form. Check that your matrix works.*

$$X' = \begin{pmatrix} -4 & -4 \\ 5 & 0 \end{pmatrix} X. \tag{1}$$

The characteristic equation with completed square is

$$0 = \det(A - \lambda I) = \begin{vmatrix} -4-\lambda & -4 \\ 5 & -\lambda \end{vmatrix} = (-4-\lambda)(-\lambda) - (-4)(5) = \lambda^2 + 4\lambda + 20 = (\lambda - 2)^2 + 16$$

Thus $(\lambda + 2)^2 = -16$ or $\lambda = -2 \pm 4i$. Thus the real canonical form is

$$M = \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix}$$

T consists of the real and imaginary parts of the eigenvector for $\lambda = -2 + 4i$.

$$0 = (A - \lambda I)\mathbf{w} = \begin{pmatrix} -2-4i & -4 \\ 5 & 2-4i \end{pmatrix} \begin{pmatrix} 4 \\ -2-4i \end{pmatrix}; \qquad \mathbf{w} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

so

$$T = \begin{pmatrix} 4 & 0 \\ -2 & -4 \end{pmatrix}.$$

To see that the transformation puts the equation into canonical form, it suffices to check if $AT = TM$. Computing, we see that T checks

$$AT = \begin{pmatrix} -4 & -4 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} -8 & 16 \\ 20 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} = TM.$$

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) STATEMENT: *Let A be a real 2×2 matrix such that $\det(A) \neq 0$. Then there are eigenvectors of A that form a basis for $\mathbf{R}^2$.*
   FALSE. The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has $\det A = 1 \neq 0$ but has only a one dimensional eigenspace. There is no second independent eigenvector so no basis of eigenvectors.

   (b) STATEMENT: *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix such that the vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent. Then for every $\mathbf{y} \in \mathbf{R}^2$ there is a unique solution of the equation $A\mathbf{x} = \mathbf{y}$.*
   TRUE. We know that if the columns are independent then $\det A \neq 0$ so A is nonsingular. This means that the equation $A\mathbf{x} = \mathbf{y}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{y}$ where $A^{-1} = (1/\det A)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(c) STATEMENT: *Let T and A be matrices such that T is invertible. If a solution $\mathbf{x}$ of $\dot{\mathbf{x}} = A\mathbf{x}$ is transformed by $\mathbf{x} = T\mathbf{y}$ then $\mathbf{y}$ satisfies $\dot{\mathbf{y}} = M\mathbf{y}$ where $M = T^{-1}AT$.*
TRUE. Since T is invertible, we may write $\mathbf{y} = T^{-1}\mathbf{x}$. Differentiating

$$\dot{\mathbf{y}} = \frac{d}{dt}(T^{-1}\mathbf{x}) = T^{-1}\dot{\mathbf{x}} = T^{-1}A\mathbf{x} = T^{-1}AT \cdot T^{-1}\mathbf{x} = M\mathbf{y}.$$

4. *Find the general solution. [Hint: first change variables to canonical form.]*

$$X' = \begin{pmatrix} -2 & -4 \\ 4 & 6 \end{pmatrix} X$$

The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -4 \\ 4 & 6-\lambda \end{vmatrix} = (-2-\lambda)(6-\lambda) - (-4)(4) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Thus $\lambda = 2, 2$. Thus the canonical form is

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

For $\lambda = 2$,

$$0 = (A - \lambda I)\mathbf{v} = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Choosing any vector independent of $\mathbf{v}$, let's take

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then

$$A\mathbf{w} = \begin{pmatrix} -2 & -4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = (-4) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The recipe gives $T = (\mathbf{v}\ \mathbf{u})$ so

$$\mathbf{u} = \frac{1}{-4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ 0 \end{pmatrix}. \qquad T = \begin{pmatrix} 1 & -\frac{1}{4} \\ -1 & 0 \end{pmatrix}.$$

To see that the transformation puts the equation into canonical form, it suffices to check if $AT = TM$. Computing, we see that T checks

$$AT = \begin{pmatrix} -2 & -4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{4} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{2} \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = TM.$$

Then the general solution of $\dot{\mathbf{y}} = M\mathbf{y}$ or

$$\begin{aligned} \dot{y}_1 &= 2y_1 + y_2 \\ \dot{y}_2 &= 2y_2 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \dot{y}_1 &= (c_1 + c_2 t)e^{2t} \\ \dot{y}_2 &= c_2 e^{2t} \end{aligned}$$

so that the general solution of the original equation

$$\mathbf{x} = T\mathbf{y} = \begin{pmatrix} 1 & -\frac{1}{4} \\ & \\ -1 & 0 \end{pmatrix} \begin{pmatrix} (c_1 + c_2 t)e^{2t} \\ \\ c_2 e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} c_1 + \frac{3}{4} c_2 t \\ \\ -c_1 - c_2 t \end{pmatrix}$$

We will encounter another method when dealing with cyclic vectors.

5. *Consider the family of differential equations depending on the parameter a*

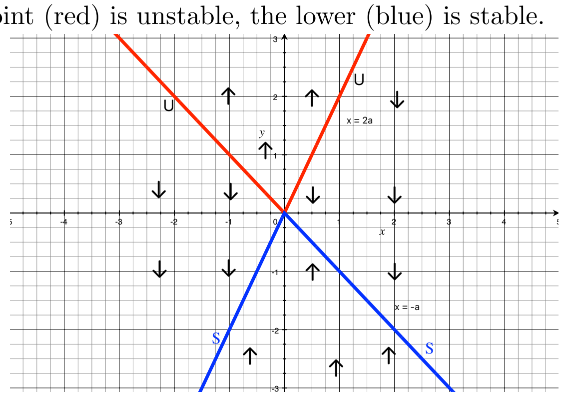
$$x' = x^2 - ax - 2a^2 = f(x, a)$$

Find the bifurcation points. Sketch the bifurcation diagram for this family of equations. Identify the rest points on the bifurcation diagram as sources, sinks or neither. Sketch the phase lines for values of a above and below the bifurcation values. [Hint: factor]

The rest points occur when $0 = f(x, a)$. Factoring we see

$$0 = f(x, a) = (x + a)(x - 2a)$$

so the rest points are $x = -a$ and $x = 2a$. $f > 0$ when x is very positive or negative. $f < 0$ when $|x|$ is small between the rest points, Thus $(0, 0)$ is a transcritical bifurcation point. The upper rest point (red) is unstable, the lower (blue) is stable.



For small a we plot the phase lines for $a > 0$ and $a < 0$.

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$$-a \qquad 0 \qquad\quad 2a$$
$$\longrightarrow\longrightarrow * \longleftarrow\longleftarrow\longleftarrow * \longrightarrow\longrightarrow\longrightarrow$$
$$2a \qquad\quad 0 \quad -a$$