Math 3160 § 1. **Final Exam** Name:________________________

Treibergs April 28, 2006

1. Find the Laurent Series centered at $z = 3i$ that converges at 3. In what region does your series converge?
$$f(z) = \frac{2z+7}{z^2+6z+5}.$$

2. Suppose that $f(z)$ is analytic in a domain $D \subset \mathbf{C}$. Suppose also that the disk with center z_0 and radius $R > 0$ is contained in D. (If $|z - z_0| \le R$ then $z \in D$.) Show that $f(z_0)$ is the average of f over the circle $|z - z_0| = R$.
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt.$$

3. (a) Find
$$\operatorname*{Res}_{z=0} \frac{1}{(\sin z)^3}$$
(b) Suppose $f(z)$ is analytic near z_0 and that z_0 is a zero of order n for f. Find
$$\operatorname*{Res}_{z=z_0} \left[\frac{f'(z)}{f(z)}\right]$$

4. Show that by choosing a branch of square root and by making some branch cuts in $\mathbf{C}$, we can find a domain $D \in \mathbf{C}$ on which
$$g(z) = \left(1 + z^2\right)^{\frac{1}{2}}$$
is single valued and analytic on D. Then find
$$\int_{-1-i}^{2+3i} \frac{z\,dz}{g(z)}$$
in the form $a + bi$, where the integral is taken along any contour from $-1-i$ to $2+3i$ in D.

5. (a) Show that $f(x + iy) = u + iv$ given by
$$u = e^x(x\cos y - y\sin y)$$
$$v = e^x(y\cos y + x\sin y)$$
is an entire function.

(b) Write $f(z)$ explicitly as a function of the single variable $z = x + iy$.

6. Let $\mathbf{c}$ denote the contour consisting of the square whose vertices are $0, i, i+1, 1$ traversed in the clockwise direction union the circle $|z| = 5$ taken in the counterclockwise direction. Find
$$\int_{\mathbf{c}} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2}$$

7. Find the improper integral using contour integration. Find a contour and formulate an expression for I involving a limit of an integral over the contour. Account for all pieces of your contour and explain why the "garbage terms" go to zero.

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5}$$

8. Find the improper integral using contour integration. Find a contour and formulate an expression for J involving as limit of an integral over the contour. Account for all pieces of your contour. You need not explain why the "garbage terms" go to zero.

$$J = \int_{-\infty}^{\infty} \frac{\cos 3x\, dx}{(x^2+1)^2}$$

Solutions.

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1. *Find the Laurent Series centered at $z = 3i$ that converges at 3. In what region does your series converge?*

$$f(z) = \frac{2z+7}{z^2+6z+5} = \frac{2z+7}{(z+1)(z+5)} = \frac{\frac{5}{4}}{z+1} + \frac{\frac{3}{4}}{z+5}$$

Laurent series converge in annular regions about the center $3i$ whose radii are distances to the poles $R_1 = |-1-3i| = \sqrt{10}$ and $R_2 = |-5-3i| = \sqrt{34}$. Since $|3-3i| = \sqrt{18}$ we will find an expansion that converges in the annulus $R_1 < |z-3i| < R_2$. Write the partial fractions in terms of $(z-3i)$.

$$f(z) = \frac{\frac{5}{4}}{(z-3i)+1+3i} + \frac{\frac{3}{4}}{5+3i-(z-3i)} = \frac{5}{4(z-3i)} \cdot \frac{1}{1+\dfrac{1+3i}{z-3i}} + \frac{\dfrac{3}{4(5+3i)}}{1+\dfrac{z-3i}{5+3i}}$$

Using the expansion that converges for $|w| < 1$

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots = \sum_{k=0}^{\infty} w^k$$

we obtain

$$f(z) = \frac{5}{4(z-3i)} \cdot \sum_{k=0}^{\infty} (-1)^k \left(\frac{1+3i}{z-3i}\right)^k + \frac{15-9i}{136} \sum_{k=0}^{\infty} (-1)^k \left(\frac{z-3i}{5+3i}\right)^k$$

The first sum converges if $\left|\dfrac{1+3i}{z-3i}\right| < 1$ or $|z-3i| > |1+3i| = \sqrt{10} = R_1$ and the second converges if $\left|\dfrac{z-3i}{5+3i}\right| < 1$ or $|z-3i| < |5+3i| = \sqrt{34} = R_2$. Combining, we find

$$f(z) = \sum_{k=0}^{\infty} \frac{15-9i}{136} \left(\frac{-5+3i}{34}\right)^k (z-3i)^k - \sum_{k=1}^{\infty} \frac{1-3i}{4} \frac{(-1-3i)^k}{(z-3i)^k}$$

which converges if $R_1 < |z-3i| < R_2$.

2. *Suppose that $f(z)$ is analytic in a domain $D \subset \mathbf{C}$. Suppose also that the disk with center z_0 and radius $R > 0$ is contained in D. (If $|z - z_0| \le R$ then $z \in D$.) Show that $f(z_0)$ is the average of f over the circle $|z - z_0| = R$.*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt.$$

We are given that $f(z)$ is analytic on the disk $|z - z_0| \le R$. Applying the Cauchy Integral Formula, for $|z - z_0| < R$ we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s - z}$$

For the particular value $z = z_0$, we parameterize the circle with $s = z_0 + Re^{i\theta}$ and $ds = iRe^{i\theta}\, d\theta$ to get

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s - z_0} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})\, iRe^{i\theta}\, d\theta}{z_0 + Re^{i\theta} - z_0} = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta.$$

3. (a) *Find* $\operatorname*{Res}_{z=0} \dfrac{1}{(\sin z)^3}$

We compute the Laurent expansion about zero.

$$\frac{1}{(\sin z)^3} = \frac{1}{z^3} \cdot \frac{1}{\left(1 - \frac{z^2}{6} + \cdots\right)^3} = \frac{1}{z^3} \cdot \left(1 + \frac{z^2}{6} + \cdots\right)^3 = \frac{1}{z^3} \cdot \left(1 + \frac{z^2}{2} + \cdots\right)$$

where we have used $(1 + w)^3 = 1 + 3w + \cdots$. Thus the residue is the z^{-1} coefficient $\operatorname*{Res}_{z=0} f(z) = \dfrac{1}{2}$.

(b) *Suppose $f(z)$ is analytic near z_0 and that z_0 is a zero of order n for f. Find* $\operatorname*{Res}_{z=z_0} \left[\dfrac{f'(z)}{f(z)}\right]$

Since $f(z)$ is analytic and has a zero of order n at z_0 it has a Taylor Series of the form

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$$

where $a_n \ne 0$. Then, dividing top and bottom by $a_n(z - z_0)^{n-1}$, the Laurent expansion near z_0 is

$$\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{na_n(z - z_0)^{n-1} + (n+1)a_{n+1}(z - z_0)^n + \cdots}{a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots} \\
&= \frac{n + \dfrac{(n+1)a_{n+1}}{a_n}(z - z_0) + \cdots}{(z - z_0)\left(1 + \dfrac{a_{n+1}}{a_n}(z - z_0) + \cdots\right)} \\
&= \frac{1}{z - z_0} \cdot \left(n + \frac{(n+1)a_{n+1}}{a_n}(z - z_0) + \cdots\right)\left(1 - \frac{a_{n+1}}{a_n}(z - z_0) + \cdots\right) \\
&= \frac{1}{z - z_0} \cdot \left(n + \frac{a_{n+1}}{a_n}(z - z_0) + \cdots\right) \\
&= \frac{n}{z - z_0} + \frac{a_{n+1}}{a_n} + \cdots
\end{aligned}$$

where we have used

$$\frac{1}{1+w} = 1 - w + w^2 - \cdots$$

Thus the residue is the $(z-z_0)^{-1}$ coefficient $\quad \operatorname*{Res}_{z=z_0}\left[\frac{f'(z)}{f(z)}\right] = n.$

4. *Show that by choosing a branch of square root and by making some branch cuts in* $\mathbf{C}$, *we can find a domain* $D \in \mathbf{C}$ *on which* $g(z) = \left(1+z^2\right)^{\frac{1}{2}}$ *is single valued and analytic on* D. *Then find*

$$\int_{-1-i}^{2+3i} \frac{z\,dz}{g(z)}$$

in the form $a+bi$, *where the integral is taken along any contour from* $-1-i$ *to* $2+3i$ *in* D.

Suppose that we consider the principal square root. That is, we take the branch $r > 0$ and $-\pi < \theta < \pi$ and

$$z^{\frac{1}{2}} = e^{(\ln r + i\theta)/2}.$$

Then the domain $D = \{z \in \mathbf{C} : \Re\mathrm{e}(z^2+1) > 0 \text{ or } \Im\mathrm{m}(z^2+1) \neq 0.\}$ where $g(z)$ misses the branch cut along the negative real axis. In terms of $z = x+iy$, $z \notin D$ if and only if

$$x^2 - y^2 + 1 \leq 0 \qquad \text{and} \qquad 2xy = 0.$$

The second equation says either $y = 0$ or $x = 0$. In case $y = 0$, the first condition says $x^2 + 1 \leq 0$ which is never true for real x. In case $x = 0$, the first condition says $y^2 \geq 1$ which is true if and only if $|y| \geq 1$, Thus the domain D is the complement of two slits along the imaginary axis for $y \geq 1$ and for $y \leq -1$. D is simply connected and $z/g(z)$ has an antiderivative in D, namely, $g(z)$.

$$g'(z) = \frac{d}{dz}(z^2+1)^{1/2} = \frac{z}{(z^2+1)^{1/2}} = \frac{z}{g(z)}.$$

Thus, the integral is independent of contour in D and gives

$$\begin{aligned}\int_{-1-i}^{2+3i} \frac{z\,dz}{g(z)} = g(-1-i) - g(2+3i) &= \left(1+(-1-i)^2\right)^{\frac{1}{2}} - \left(1+(2+3i)^2\right)^{\frac{1}{2}} \\ &= (1-2i)^{\frac{1}{2}} - (-4+12i)^{\frac{1}{2}}\end{aligned}$$

Now $1 - 2i = \sqrt{5}e^{i\alpha}$ where $\alpha = \operatorname{Atn} 2$ and $-4+12i = 4\sqrt{10}e^{i\beta}$ where $\beta = \pi - \operatorname{Atn} 3$. It follows that

$$\begin{aligned}(1-2i)^{\frac{1}{2}} &= \sqrt[4]{5}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right), \\ (-4+12i)^{\frac{1}{2}} &= 2\sqrt[4]{10}\left(\cos\frac{\beta}{2} + i\sin\frac{\beta}{2}\right),\end{aligned}$$

so

$$\int_{-1-i}^{2+3i} \frac{z\,dz}{g(z)} = \left(\sqrt[4]{5}\cos\frac{\alpha}{2} - 2\sqrt[4]{10}\cos\frac{\beta}{2}\right) + i\left(\sqrt[4]{5}\sin\frac{\alpha}{2} - 2\sqrt[4]{10}\sin\frac{\beta}{2}\right).$$

5. (a) *Show that $f(x+iy) = u + iv$ is an entire function, where*

$$u = e^x(x\cos y - y\sin y)$$
$$v = e^x(y\cos y + x\sin y)$$

For both functions u and v, both first order partial derivatives exist and are continuous at all points of the entire plane. Computing derivatives we find

$$u_x = e^x([x+1]\cos y - y\sin y) = v_y$$
$$u_y = e^x(-x\sin y - \sin y - y\cos y) = -v_x$$

Thus, $f(z)$ is entire since u and v satisfy the Cauchy Riemann equations at all points.

(b) *Write $f(z)$ explicitly as a function of the single variable $z = x + iy$.*
It seems that the function $e^z = e^x(\cos y + i\sin y)$ supplies some of the ingredients. Try multiplying

$$ze^z = (x+iy)e^x(\cos y + i\sin y) = e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y) = u + iy.$$

Yes, that's it!

6. *Let* $\mathbf{c}$ *denote the contour consisting of the square whose vertices are* $0,\ i,\ i+1,\ 1$ *traversed in the clockwise direction union the circle* $|z| = 5$ *taken in the counterclockwise direction. Find*

$$\int_{\mathbf{c}} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2}$$

The contour may be written $\mathbf{c} = C_5 - S$ where C_5 is the circle $|z| = 5$ oriented counterclockwise and S is the square oriented clockwise. The principal logarithm is defined away from the origin and the negative real axis. $\Re\mathrm{e}(z+6) = x + 6 > 0$ for all $z \in \mathbf{c}$ because $|x| \le |z| \le 5$ so the function $f(z) = \operatorname{Log}(z+6)$ is analytic in and on C_5. By the Cauchy Integral Formula for derivatives

$$\int_{C_5} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2} = 2\pi i f'(4i) = \frac{1}{6+4i} = \frac{(2+3i)\pi}{13}.$$

On the other hand since $4i$ is outside the square the quotient is analytic on and inside S. By the Cauchy Goursat Theorem,

$$\int_{S} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2} = 0.$$

Subtracting,

$$\int_{\mathbf{c}} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2} = \int_{\mathbf{C}_5} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2} - \int_{\mathbf{S}} \frac{\operatorname{Log}(z+6)\,dz}{(z-4i)^2} = \frac{(2+3i)\pi}{13}.$$

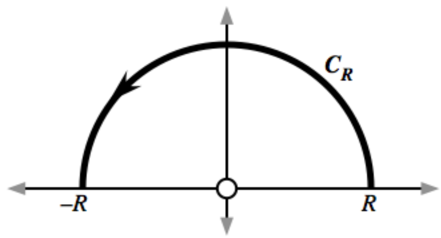
7. *Find the improper integral using contour integration. Find a contour and formulate an expression for I involving a limit of an integral over the contour. Account for all pieces of your contour and explain why the "garbage terms" go to zero.*

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5}$$

The integrand is continuous and decays like x^{-2} as $|x| \to \infty$. Thus the integral exists as an improper integral. It may be computed by

$$I = \lim_{R\to\infty} \int_{S_R} \frac{dz}{z^2 + 4z + 5}$$

where S_R is the line segment from $-R$ to R. Let C_R denote the semicircle $|z| = R$ from $\theta = 0$ to $\theta = \pi$.



We now show that the garbage term, the integral over C_R vanishes as $R \to \infty$. The length of C_R is $L_R = \pi R$. For $z \in C_R$ with $R > 6$, we have

$$\left| \frac{1}{z^2 + 4z + 5} \right| \le \frac{1}{|z^2 - (-4z - 5)|} \le \frac{1}{|z|^2 - 4|z| - 5} \le \frac{1}{R^2 - 4R - 5} = M_R$$

Thus by the contour integral estimate,

$$\left| \int_{C_R} \frac{dz}{z^2 + 4z + 5} \right| \le L_R M_R = \frac{\pi R}{R^2 - 4R - 5}$$

which tends to zero as $R \to \infty$.
The poles are at the zeros of $z^2 + 4z + 5 = (z + 2)^2 + 1$ which are $-2 \pm i$ of which only $z_0 = -2 + i$ is inside $C_R + S_R$ if $R > \sqrt{5}$. The residue

$$\operatorname*{Res}_{z=-2+i} \frac{1}{(z + 2 - i)(z + 2 + i)} = \frac{1}{(-2 + i) + 2 + i} = -\frac{i}{2}.$$

From the Residue theorem we have

$$\int_{C_R+S_R} \frac{dz}{z^2 + 4z + 5} = 2\pi i \operatorname*{Res}_{z=-2+i} \frac{1}{z^2 + 4z + 5} = 2\pi i \left(-\frac{i}{2} \right) = \pi.$$

Taking the limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5} = \pi.$$

8. *Find the improper integral using contour integration. Find a contour and formulate an expression for J involving as limit of an integral over the contour. Account for all pieces of your contour. You need not explain why the "garbage terms" go to zero.*

$$J = \int_{-\infty}^{\infty} \frac{\cos 3x \, dx}{(x^2 + 1)^2}$$

We observe that for real x, $\cos 3x = \Re\mathrm{e}(e^{3ix})$. The complex exponential

$$|e^{3iz}| = |e^{3ix-3y}| = e^{-3y} \le 1$$

if $y \ge 0$. Hence integrand is continuous and decays like x^{-4} as $|x| \to \infty$. Thus the integral exists as an improper integral. It may be computed by

$$J = \Re\mathrm{e}\left(\lim_{R\to\infty} \int_{S_R} \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} \right)$$

where S_R is the line segment from $-R$ to R. Let C_R denote the semicircle as in Problem (7).

We now show that the garbage term, the integral over C_R vanishes as $R \to \infty$. The length of C_R is $L_R = \pi R$. For $z \in C_R$ with $R > 6$, we have

$$\left| \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} \right| \le \frac{|e^{3iz}|}{|z^2+1|^2} \le \frac{1}{(|z|^2-1)^2} \le \frac{1}{(R^2-1)^2} = M_R$$

Thus by the contour integral estimate,

$$\left| \int_{C_R} \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} \right| \le L_R M_R = \frac{\pi R}{(R^2-1)^2}$$

which tends to zero as $R \to \infty$.

The poles are at the zeros of $(z^2+1)^2$ which are $\pm i$ of which only $z_0 = i$ is inside $C_R + S_R$ if $R > 1$. Factoring out the pole,

$$\frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} = \frac{1}{(z-i)^2} \cdot \frac{e^{3iz}}{(z+i)^2} = \frac{\phi(z)}{(z-i)^2}$$

where $\phi(z)$ is analytic and nonzero at $z = i$. The residue is the coefficient of the $(z-i)^{-1}$ term which corresponds to the $(z-i)$ coefficient of $\phi(z)$. Using Taylor's formula,

$$\operatorname*{Res}_{z=i} \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} = \phi'(i) = \left. \frac{(3iz-5)e^{3iz}}{\left(z+i\right)^3} \right|_{z=i} = -\frac{i}{e^3}$$

From the Residue theorem we have

$$\int_{C_R+S_R} \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} = 2\pi i \left(-\frac{i}{e^3} \right) = \frac{2\pi}{e^3}.$$

Taking the limit as $R \to \infty$,

$$J = \int_{-\infty}^{\infty} \frac{\cos 3x\,dx}{\left(x^2+1\right)^2} = \Re\mathrm{e}\left(\lim_{R\to\infty} \int_{S_R} \frac{e^{3iz}\,dz}{\left(z^2+1\right)^2} \right) = \frac{2\pi}{e^3}.$$