Math 3160 § 1. **First Midterm Exam** Name: Solutions
Treibergs $\sigma\tau$ February 16, 2006

1.*Let* $f(x+iy) = e^{-y}\sin x - ie^{-y}\cos x$. *Is* f *an entire function? State what entire means.*

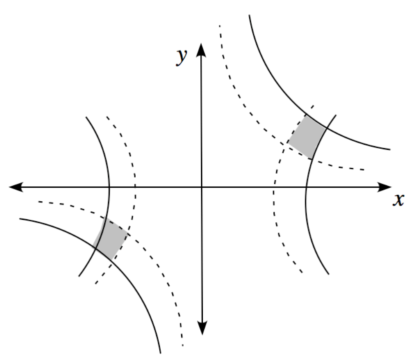
A function is *entire* if it is analytic on all of $\mathbf{C}$. A function is analytic on an open set S if the complex derivative of f exists at all points of S. According to the theorem that guarantees that a function has a complex derivative, we have to show, first, that the partial derivatives u_x, u_y, v_x and v_y are continuous at all points $x+iy \in \mathbf{C}$ and, second, that the Cauchy-Riemann Equations hold at all points, namely, $u_x = v_y$ and $u_y = -v_x$. In this case, $u(x,y) = e^{-y}\sin x$ so that $u_x = e^{-y}\cos x$, $u_y = -e^{-y}\sin x$, and $v(x,y) = -e^{-y}\cos x$ so $v_x = e^{-y}\sin x$ and $v_y = e^{-y}\cos x$. In the first place, as these functions are products of exponential, sine and cosine, which are continuous functions, the partial derivative functions are all continuous at all points. In the second place, we see that $u_x = v_y$ and $u_y = -v_x$ at all points. Thus the Cauchy - Riemann Equations are also satisfied everywhere.

2.*Let* $z = -2+2i$. *Find* $\dfrac{1}{z^2}$, $\operatorname{Arg} z$, $\arg z$, $|z|$, $\dfrac{z-\bar{z}}{2}$. *Find* $z^{\frac{1}{8}}$, $\sqrt[8]{z}$.

$z^2 = (-2+2i)^2 = 4-8i+4i^2 = -8i$. Thus $\frac{1}{z^2} = \frac{1}{-8i} = \frac{i}{8}$. In polar coordinates, $z = 2\sqrt{2}e^{\frac{3\pi i}{4}}$. Since $-\pi < \frac{3\pi i}{4} \le \pi$ so the principal argument is $\operatorname{Arg} z = \frac{3\pi i}{4}$. All arguments are the set $\arg z = \{2\sqrt{2}\exp(\frac{3\pi i}{4} + 2\pi k i) : k \in \mathbf{Z}\}$. $|z| = r = 2\sqrt{2}$. $\frac{z-\bar{z}}{2} = i\frac{z-\bar{z}}{2i} = i\,\Im\mathrm{m}\, z = 2i$. All eighth roots are the set $z^{\frac{1}{8}} = \{\sqrt[16]{8}\exp(\frac{3\pi i}{32} + \frac{\pi k i}{4}) : k \in \mathbf{Z}\}$. The principal eighth root is computed from the prinncipal argument, or $\sqrt[8]{z} = \sqrt[16]{8}\exp(\frac{3\pi i}{32})$.

3.*Let* $f(z) = z^2$. *Find set of all points* $z \in \mathbf{C}$ *such that* $1 < \Re\mathrm{e}(f(z)) \le 2$ *and* $8 < \Im\mathrm{m}(f(z)) \le 18$. *Sketch.*

$z^2 = x^2 - y^2 + 2xyi$, so that $1 < \Re\mathrm{e}(f(z)) = x^2 - y^2 \le 2$ is the region between the hyperbolas $1 = x^2 - y^2$ (not in the set) and the hyperbolas $x^2 - y^2 = 2$ (in the set). Also $8 < \Im\mathrm{m}(f(z)) = 2xy \le 18$ is the region between the hyperbolas $4 = xy$ (not in the set) and the hyperbolas $xy = 9$ (in the set). Thus the region consists of two curvilinear rectangles in the first and third quadrants.



4.*Does the complex limit exist? If it exists, find the limit.* $L = \displaystyle\lim_{z\to 0} \frac{(\Re\mathrm{e}\, z)\ (\Im\mathrm{m}\, z)}{z^2}$

Putting $z = x+iy$, $L = \displaystyle\lim_{x+iy\to 0} \frac{xy}{x^2 - y^2 + 2xyi}$. Taking the approach along the x-axis, $y = 0$, we have the limiting value $L = \displaystyle\lim_{x+0i\to 0} \frac{0}{x^2} = 0$. Taking the approach along the $x = y$ line, setting $x = y = t$, we have the limiting value $L = \displaystyle\lim_{t+ti\to 0} \frac{t^2}{2t^2 i} = -\frac{i}{2}$. As the two approaches yield inconsistent limiting values, there is no limit.

5.*Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is analytic in a domain $D \subset \mathbf{C}$. ($u(x,y)$ and $v(x,y)$ are real valued functions and x and y are real, as usual.) Show that $h(x,y)$ is a harmonic function on D, where $h(x,y) = (u(x,y))^2 - (v(x,y))^2$.*

Since we are given that f is analytic in D, we know that partial derivatives of all orders of u and v exist, and that they are continuous in D. Thus we may check harmonicity by computing $\Delta h = h_{xx} + h_{yy}$ and seeing if vanishes. $h_x = 2uu_x - 2vv_x$, $h_{xx} = 2u_x^2 - 2v_x^2 + 2uu_{xx} - 2vv_{xx}$. Similarly $h_y = 2uu_y - 2vv_y$, $h_{yy} = 2u_y^2 - 2v_y^2 + 2uu_{yy} - 2vv_{yy}$. Adding,

$$h_{xx} + h_{yy} = 2(u_x^2 - v_y^2) + 2(u_y^2 - v_x^2) + 2u(u_{xx} + u_{yy}) - 2v(v_{xx} + v_{yy}) = 2u \cdot 0 - 2v \cdot 0 + 0 - 0 = 0.$$

The last equation follows from the fact that f is analytic, which implies the Cauchy-Riemann Equations, $u_x = v_y$ which implies $u_x^2 - v_y^2 = u_x^2 - u_x^2 = 0$, and $u_y = -v_x$ which implies $u_y^2 - v_x^2 = u_y^2 - (-u_y)^2 = 0$. Analyticity also implies that both u and v are harmonic, so $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.