Math 2210 § 4. **Second Midterm Exam** Name: Solutions
Treibergs March 20, 2019

1. *Let $f(x,y) = x^4 y$. Find all first and second partial derivatives of $f(x,y)$. Find the unit vector $\mathbf{u}$ in the direction in which f increases the fastest at the point $(1,2)$. What is the directional derivative $D_{\mathbf{u}} f(1,2)$? Find the second order Taylor polynomial $P_2(x,y)$ at the point $(1,2)$.*

   The first and second partial derivatives are

   $$f_x = 4x^3 y, \qquad f_y = x^4, \qquad f_{xx} = 12x^2 y, \qquad f_{xy} = 4x^3, \qquad f_{yy} = 0.$$

   $f(1,2) = 2$. At $(1,2)$ the derivatives equal

   $$f_x(1,2) = 8, \qquad f_y(1,2) = 1, \qquad f_{xx}(1,2) = 24, \qquad f_{xy}(1,2) = 4, \qquad f_{yy}(1,2) = 0.$$

   The vector in the direction of the fastest increase is the gradient.

   $$\nabla f = (f_x, f_y) = (4x^3 y, x^4), \qquad \mathbf{v} = \nabla f(1,2) = (8,1).$$

   The unit vector in the direction of fastest increase is

   $$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(8,1)}{\sqrt{8^2 + 1^2}} = \frac{1}{\sqrt{65}}(8,1).$$

   The directional derivative in that direction at $(1,2)$ is

   $$D_{\mathbf{u}} f(1,2) = \mathbf{u} \cdot \nabla f(1,2) = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{v} = \frac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|} = \|\mathbf{v}\| = \sqrt{65}.$$

   The second order Taylor Polynomial at $(x_0, y_0) = (1,2)$ is

   $$\begin{aligned} P_1(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + \\ &\quad + \frac{1}{2}\Big\{ f_{xx}(x_0,y_0)(x-x_0)^2 + 2 f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2 \Big\} \\ &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) + \\ &\quad + \frac{1}{2}\Big\{ f_{xx}(1,2)(x-1)^2 + 2 f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)(y-2)^2 \Big\} \\ &= 2 + 8(x-1) + (y-2) + \frac{1}{2}\Big\{ 24(x-1)^2 + 8(x-1)(y-2) \Big\}. \end{aligned}$$

2. (a) *Determine whether the limit exists. If it does, find the limit. If it doesn't exist, explain why not.*

   $$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4 + y^4}$$

   The limit does not exist. If one takes the path to the origin $(x,y) = (t,0)$ then

   $$\lim_{t\to 0} \frac{0}{t^4 + 0^4} = 0.$$

   If one takes the path to the origin $(x,y) = (t,t)$ then

   $$\lim_{t\to 0} \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

   Since the limit along the two paths disagree, there is no limit at the origin.

(b) *A surface satisfies the equation* $f(x,y,z) = \dfrac{x^2}{9} + \dfrac{y^2}{4} - z^2 = 1$. *Identify this surface. Find the equation of the tangent plane of this surface at* $P = (3,2,1)$.

Since for every horizontal plane $z = \text{const.}$ the slice is a nontrivial ellipsoid, the surface is a HYPERBOLOID OF ONE SHEET.

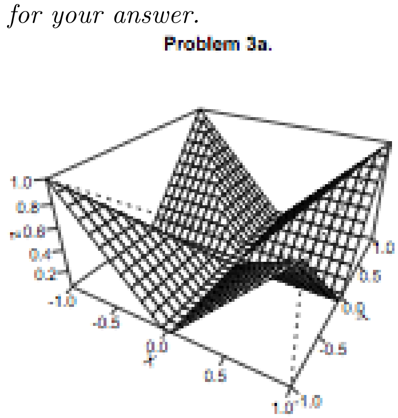
The normal to the surface is given by the gradient

$$\nabla f = (f_x, f_y, f_z) = \left(\frac{2}{9}x, \frac{1}{2}y, -2z\right), \qquad \mathbf{N} = \nabla f(3,2,1) = \left(\frac{2}{3}, 1, -2\right).$$

Using the point-normal form of the equation of the tangent plane, where $\mathbf{X} = (x,y,z)$ is an arbitrary point on the plane is

$$0 = \mathbf{N}\cdot(\mathbf{X} - \mathbf{P}) = \left(\frac{2}{3}, 1, -2\right)\cdot(x-3, y-2, z-1) = \frac{2}{3}(x-3) + (y-2) - 2(z-1).$$

3. (a) *Is the function* $f(x,y) = \min(|x|,|y|)$ *differentiable at* $(0,0)$*? Give a SHORT reason for your answer.*



THE FUNCTION IS NOT DIFFERENTIABLE AT $(0,0)$. The short reason is that the function is not almost linear at $(0,0)$, *i.e.*, the tangent plane $z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0$ does not well approximate the graph at the origin.

Here is a long reason (not needed for your answer.) Since $f(x,0) = f(0,x) = 0$ the partial derivatives exist at the origin and $f_x(0,0) = 0$ and $f_y(0,0) = 0$. Hence the tangent plane is dead zero

$$z = \lambda(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0.$$

Now $f(x,y)$ is differentiable at $(0,0)$ if the following limit exists and equals zero:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \lambda(x,y)}{\|(x-0, y-0)\|} = \lim_{(x,y)\to(0,0)} \frac{\min(|x|,|y|) - 0}{\sqrt{x^2+y^2}}$$

If one takes the path to the origin $(x,y) = (t,0)$ then

$$\lim_{t\to 0} \frac{0}{|t|} = 0.$$

If one takes the path to the origin $(x,y) = (t,t)$ then

$$\lim_{t\to 0} \frac{|t|}{\sqrt{2}|t|} = \frac{1}{\sqrt{2}}.$$

Since the limit along the two paths disagree, there is no limit at the origin so f is not differentiable there.

(b) *Find* $\frac{\partial f}{\partial v}$*. Express your answer in terms of* (u, v) *where*

$$f(x, y) = \sin(x + y)e^z, \qquad x = u + v^2, \qquad y = u^3, \qquad z = uv.$$

Using the chain rule, we find

$$\begin{aligned} \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} \\ &= \cos(x + y)e^z \cdot 2v + \cos(x + y)e^z \cdot 0 + \sin(x + y)e^z \cdot u \\ &= 2\cos(u + v^2 + u^3)e^{uv}v + \sin(u + v^2 + u^3)e^{uv}u \end{aligned}$$

4. (a) *Consider the function* $f(x, y) = x^3 + y^2 - 12x + 3y$ *Find the two critical points of* $f(x, y)$*. Find the discrimimant* $D = f_{xx}f_{yy} - f_{xy}^2$*. Use the discrimimant to determine whether each of the critical points in part (a) is a local minimum, local maximum, saddle or indeterminate.*

Critical points satisfy $\nabla f = (0, 0)$ or

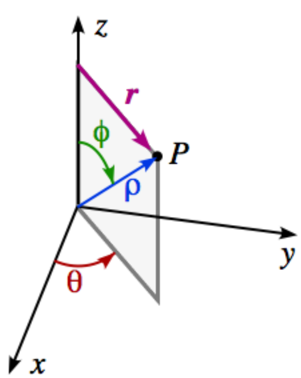
$$\begin{aligned} 0 &= f_x = 3x^2 - 12, \\ 0 &= f_y = 2y + 3 \end{aligned}$$

whose solution is $x = \pm 2$ and $y = -\frac{3}{2}$ hence the critical points are $(2, -\frac{3}{2})$ and $(-2, -\frac{3}{2})$. We have

$$f_{xx} = 6x, \qquad f_{xy} = 0, \qquad f_{yy} = 2, \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 12x.$$

At the first critical point $(2, -\frac{3}{2})$, $D = 24 > 0$ and $f_{xx} = 12 > 0$ so it is a local minimum. At the second critical point $(-2, -\frac{3}{2})$, $D = -24 < 0$ so it is a saddle.

(b) *Express the Cartesian coordinates* $P = (x, y, z)$ *of three space in terms of spherical coordinates* (ρ, ϕ, θ)*. Label* ρ*,* ϕ*,* θ *and* r *on the diagram. Change the spherical coordinates equation* $\rho \sin\phi = 1$ *to Cartesian coordinates.*



$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$r = \rho \sin\phi$$

The equation is $r = \rho \sin\phi = 1$, in other words, it is the cylinder surface whose equation is

$$\sqrt{x^2 + y^2} = 1 \qquad \text{or} \qquad x^2 + y^2 = 1.$$

5. *Find the maximum and minimum of the function $f(x, y) = xy$ on the circle $x^2 + y^2 = 4$.*

We use the method of Lagrange Multipliers. The objective function is $f(x, y) = xy$ and the constraint is

$$g(x, y) = x^2 + y^2 = 4.$$

The critical points subject to the constraint satisfy $\nabla f(x, y) = \lambda \nabla g(x, y)$ and the constraint equation. Componentwise these are

$$f_x = y = 2\lambda x = \lambda g_x \tag{1}$$

$$f_y = x = 2\lambda y = \lambda g_y \tag{2}$$

$$g = x^2 + y^2 = 4. \tag{3}$$

Substituting the second equation into the first yields

$$y = 4\lambda^2 y \qquad \text{or} \qquad (1 - 4\lambda^2)y = 0.$$

Either $y = 0$, so the second equation implies $x = 0$ and $x^2 + y^2 = 0$ doesn't satisfy the third equation, so this is impossible. Or $(1 - 4\lambda^2) = 0$ which implies that $\lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$. In case $\lambda = \frac{1}{2}$, the first equation says $x = y$ so that the third equation gives

$$x^2 + y^2 = 2x^2 = 4$$

so that $x = y = \sqrt{2}$ or $x = y = -\sqrt{2}$. For these critical points,

$$f\left(\sqrt{2}, \sqrt{2}\right) = f\left(-\sqrt{2}, -\sqrt{2}\right) = 2.$$

In case $\lambda = -\frac{1}{2}$, the first equation says $x = -y$ so that the third equation gives

$$x^2 + y^2 = x^2 + (-x)^2 = 2x^2 = 4$$

so that $x = -y = \sqrt{2}$ or $x = -y = -\sqrt{2}$. For these critical points,

$$f\left(\sqrt{2}, -\sqrt{2}\right) = f\left(-\sqrt{2}, \sqrt{2}\right) = -2.$$

Thus the minimum occurs at both critical points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ where $f = -2$. The maximum occurs at the two critical points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ where $f = 2$.

Alternative solution. Parameterize the circle

$$x = 2\cos t, \qquad y = 2\sin t.$$

Then we seek t where

$$h(t) = f(2\cos t, 2\sin t) = 4\cos t\,\sin t = 2\sin 2t$$

is maximum and minimum. Differentiating,

$$h'(t) = -4\sin^2 t + 4\cos^2 t = 4\cos 2t.$$

This is zero when $\cos 2t = 0$ or when

$$2t = \frac{\pi}{2} + 2\pi k \qquad \text{or} \qquad 2t = \frac{3\pi}{2} + 2\pi k$$

where k is an integer. In the range $0 \le t < 2\pi$ covering the whole circle, this occurs in the former case when $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$ for which

$$(x, y) = \left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right) = (\sqrt{2}, \sqrt{2}) \quad \text{or} \quad (x, y) = \left(2\cos\frac{5\pi}{4}, 2\sin\frac{5\pi}{4}\right) = (-\sqrt{2}, -\sqrt{2})$$

and $f(x,y) = 2$ in both cases. In the latter case when $t = \frac{3\pi}{4}$ or $t = \frac{7\pi}{4}$ we have

$$(x,y) = \left(2\cos\frac{3\pi}{4}, 2\sin\frac{3\pi}{4}\right) = (-\sqrt{2}, \sqrt{2}) \text{ or } (x,y) = \left(2\cos\frac{7\pi}{4}, 2\sin\frac{7\pi}{4}\right) = (\sqrt{2}, -\sqrt{2})$$

and $f(x,y) = -2$ in both cases. Thus the minimum occurs at both critical points $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ where $f = -2$. The maximum occurs at the two critical points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ where $f = 2$.