Math 2210 § 4. **Second Midterm Exam** Name: Practice Problems
Treibergs March 6, 2019

1. *Graph the surface and identify it.*

$$\frac{x^2}{9} - \frac{y^2}{4} = z$$

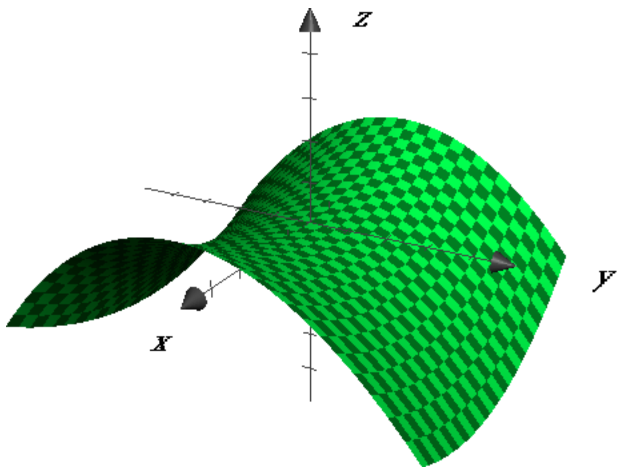

Figure 1: Problem 1.

The $y =$ const. slices are upward parabolas. The $x =$ const. slices are downward parabolas. The surface is a saddle. Here is the output from my Grapher program on my laptop Macintosh. It is a hyperbolic paraboloid.

2. *For the function in problem 1, find the equation of the tangent plane at $P = (2, 1)$.*

The partial derivatives of $z = f(x, y)$ at $(2, 1)$ where $f(2, 1) = \frac{4}{9} - \frac{1}{4} = \frac{7}{36}$ are

$$f_x(x, y) = \frac{2}{9}x, \qquad f(2, 1) = \frac{4}{9}; \qquad f_y(x, y) = -\frac{1}{2}y; \qquad f_y(2, 1) = -\frac{1}{2}.$$

The equation of the tangent plane at $P = (x_0, y_0)$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(x - x_0).$$

In our case, $(x_0, y_0) = (2, 1)$ so

$$\begin{aligned} z &= f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(x - 1) \\ &= \frac{7}{36} + \frac{4}{9}(x - 2) - \frac{1}{2}(y - 1) \\ &= \left(\frac{7}{36} - \frac{8}{9} + \frac{1}{2}\right) + \frac{4}{9}x - \frac{1}{2}y \\ &= -\frac{7}{36} + \frac{4}{9}x - \frac{1}{2}y. \end{aligned}$$

3. *Find the first, second and third order Taylor polynomials at* $(2,1)$ *for the function in Problem 1.*

The first order Taylor polynomial is the same as the equation for the tangent plane

$$P_1(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(x-x_0) = \frac{7}{36} + \frac{4}{9}(x-2) - \frac{1}{2}(y-1).$$

Continuing computing derivatives, we find

$$f_{xx}(x,y) = \frac{2}{9}, \qquad ; f_{xy}(x,y) = 0; \qquad f_{yy}(x,y) = -\frac{1}{2}.$$

The second order Taylor Polynomial is $P_2(x,y) =$

$$\begin{aligned}
&= P_1(x,y) + \frac{1}{2}\Big(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\Big)\\
&= \frac{7}{36} + \frac{4}{9}(x-2) - \frac{1}{2}(y-1) + \frac{1}{2}\Big(\frac{2}{9}(x-2)^2 - \frac{1}{2}(y-1)^2\Big)\\
&= \frac{x^2}{9} - \frac{y^2}{4}.
\end{aligned}$$

The second order Taylor polynomial of the function turns out to be the function itself. This is no surprise because the quadratic polynomial approximation of a quadratic polynomial should be itself! Continuing in this vein we see that the third derivatives all vanish

$$f_{xxx}(x,y) = f_{xxy}(x,y) = f_{xyy}(x,y) = f_{yyy}(x,y) = 0.$$

The third order Taylor Polynomial is

$$\begin{aligned}
P_3(x,y) = P_2(x,y) + \frac{1}{6}\Big(&f_{xxx}(x_0,y_0)(x-x_0)^3 + 3f_{xxy}(x_0,y_0)(x-x_0)^2(y-y_0)\\
&+3f_{xyy}(x_0,y_0)(x-x_0)(y-y_0)^2 + f_{yyy}(x_0,y_0)(y-y_0)^3\Big)\\
= P_2(x,y) + 0 = \frac{x^2}{9} - \frac{y^2}{4}&.
\end{aligned}$$

Thus since the second order polynomial already recovers the function on the nose, the third derivative correction is zero.

4. *Sketch the surface and indicate some level curves.*

$$z = \frac{5}{\sqrt{1+(x-2)^2+(y+3)^2}}$$

The level curves at $z = k$ are given by

$$\frac{25}{k^2} - 1 = (x-2)^2 + (y+3)^2$$

which are circles centered at $(2,-3)$ and radius $\sqrt{\frac{25}{k^2} - 1}$. A view from Macintosh's Grapher is given in Figure 2.

5. *Let* $A(x,y)$ *be the area of a nondegenerate rectangle of dimensions* x *and* y*, the rectangle being inside a circle of radius* 10*. Determine the domain and range of this function.*

To be nondegenerate we require $0 < x$ and $0 < y$. Every rectangle that fits inside the circle of radius $r \le 10$ may be moved so that the center of the rectangle coincides with the center

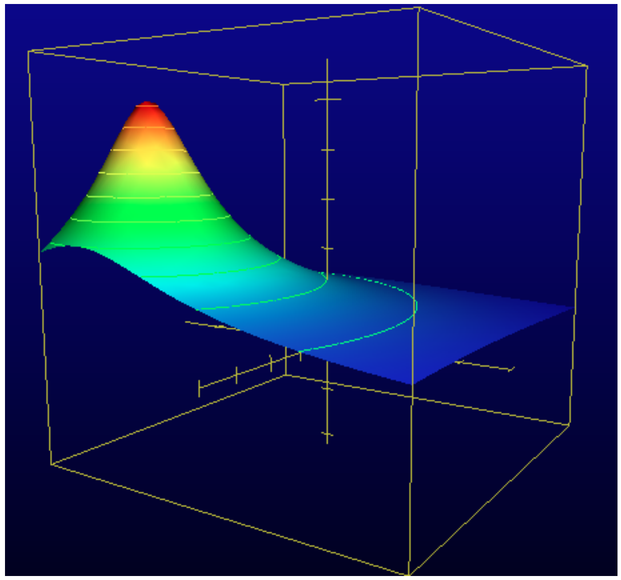

Figure 2: Problem 5.

of the circle. Thus the diagonal of the rectangle can be at most the diameter of the circle, namely $\sqrt{x^2+y^2} \le 20$. Thus the domain is the set shaped like a quarter sector

$$D = \{(x,y) \in \mathbf{R}^2 : 0 < x,\ 0 < y \text{ and } x^2+y^2 \le 400.\}$$

The range is the set of all possible values of the area $A(x,y)$ when $(x,y) \in D$. We see that for $(x,y) \in D$,

$$0 < A(x,y) = xy = \frac{1}{2}(x^2+y^2) - \frac{1}{2}(x-y)^2 \le 200 - 0.$$

In fact, the rectangle of sides $x = y = 10\sqrt{2}$ (the biggest possible square) has diagonal $\sqrt{x^2+y^2} = 20$ and area $A(x,y) = 200$. All intermediate areas occur for all smaller squares. Thus the range is $0 < A \le 200$.

6. *Show that the limit* $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist, where*

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & \text{if } (x,y) \ne (0,0); \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Along a straight line $(x,y) = (at,bt)$ where $(a,b) \ne (0,0)$ we have $f(at,bt) = \dfrac{a^2bt}{a^4t^2+b^2} \to 0$ as $t \to 0$. However, along the parabola $(x,y) = (t,t^2)$, $f(t,t^2) = \dfrac{1}{2} \to \dfrac{1}{2}$ as $t \to 0$. Since the limits along two approaches to the origin differ, there is no limit at the origin.

7. *Assume that the function $z = f(x, y)$ is differentiable and that*

$$f(3,4) = 2; \qquad \frac{\partial f}{\partial x}(3,4) = 5; \qquad \frac{\partial f}{\partial y}(3,4) = 6.$$

In what direction is the directional derivative increasing the fastest and what is tha directional derivative in that direction?

The direction of fastest increase is in the gradient direction

$$\nabla f(3,4) = \langle 5, 6 \rangle$$

The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 5, 6\rangle}{\sqrt{5^2 + 6^2}} = \frac{1}{\sqrt{61}}\langle 5, 6\rangle.$$

The directional derivative in this direction is

$$D_{\mathbf{u}} f(3,4) = \mathbf{u} \cdot \nabla f = \frac{\nabla f \cdot \nabla f}{\|\nabla f\|} = \|\nabla f(3,4)\| = \sqrt{61}.$$

8. *Find the equation of the plane tangent to the level surface $f(x, y, z)$ at the point $P = (2, 3, 4)$, where*

$$f(x, y, z) = x^3 + y^3 + z^3 - 6xyz.$$

The value of f on the level set is $c = f(2, 3, 4) = -45$. The gradient of f at P gives the normal vector.

$$\nabla f = \langle 3x^2 - 6yz, 3y^2 - 6xz, 3y^2 - 6xy\rangle$$

so that

$$N = \nabla f(P) = \langle -60, -21, -9\rangle.$$

Thus the point normal-form of a the equation of the tangent plane for the general point $X = \langle x, y, z\rangle$ is

$$0 = N \cdot (X - P)$$

or

$$0 = -60(x-2) - 21(y-3) - 9(z-4).$$

9. *Find all critical points, the global mininimum and the global maximum of the function $f(x, y)$. For each critical point, determine if the point is a local minimum, local maximum, saddle or indeterminate.*

$$f(x, y) = x^4 - 2x^2y + y^6.$$

The critical points occur if $\nabla f = \langle 0, 0\rangle$. The gradient is

$$\nabla f = \langle 4x^3 - 4xy, -2x^2 + 6y^5\rangle.$$

It vanishes when

$$4x(x^2 - y) = 0 \qquad \text{or} \qquad x^2 = 3y^5$$

The first equation tells us that either $x = 0$ in which case, from the second equation $y = 0$. Otherwise $y = x^2$ which is nonnegative, in which case the second equation becomes $0 = y(1-3y^4)$. Thus $y = 0$ so $x = 0$ from $y = x^2$ or $y = 3^{-1/4}$ (positive root) so $x = \pm 3^{-1/8}$. Hence there are only three critical points at $(0, 0)$, $(3^{-1/8}, 3^{-1/4})$ and $(-3^{-1/8}, 3^{-1/4})$. The value of the function at these points is

$$f(0,0) = 0; \qquad f(\pm 3^{-1/8}, 3^{-1/4}) = -2 \cdot 3^{-3/2} \approx -0.3849002.$$

Computing the second derivatives we find

$$f_{xx} = 12x^2 - 4y; \qquad f_{xy} = -4x; \qquad f_{yy} = 30y^4.$$

The discrimimant is

$$D = f_{xx}f_{yy} - f_{xy}^2 = 360x^2y^4 - 120y^5 - 16x^2$$

which is zero at $(0, 0)$ where the situation is indeterminate. At the other critical points where $y = x^2$,

$$D = 240x^{10} - 16x^2 = 16x^2(30x^8 - 1)$$

which is positive when $x = \pm 3^{-1/8}$. Since $f_{xx} = 8x^2 > 0$ also, the two critical points other than the origin are relative minima.

When $y = 0$ the $f(x, 0) \to \infty$ as $x \to \infty$ so f has no global maximum. On the other hand, $f(x, y) \geq -1$ for all (x, y). To see this, if $|y| \geq 1$ then

$$x^4 - 2x^2y + y^6 \geq x^4 - 2x^2y + y^2 = (x^2 - y)^2 \geq 0.$$

If $|y| \leq 1$ then

$$x^4 - 2x^2y + y^6 \geq x^4 - 2x^2$$

which is positive if $x^2 > 2$ and greater than -1 everywhere. Since the function takes negative values, it must take global minima in the region $-\sqrt{2} \leq x \leq \sqrt{2}$ and $-1 \leq y \leq 1$ where negative values are possible. Being differentiable, any global minima are critical points, thus are the two points with equal values which we found.

A plot using the **R**© package is given in Figure 3.

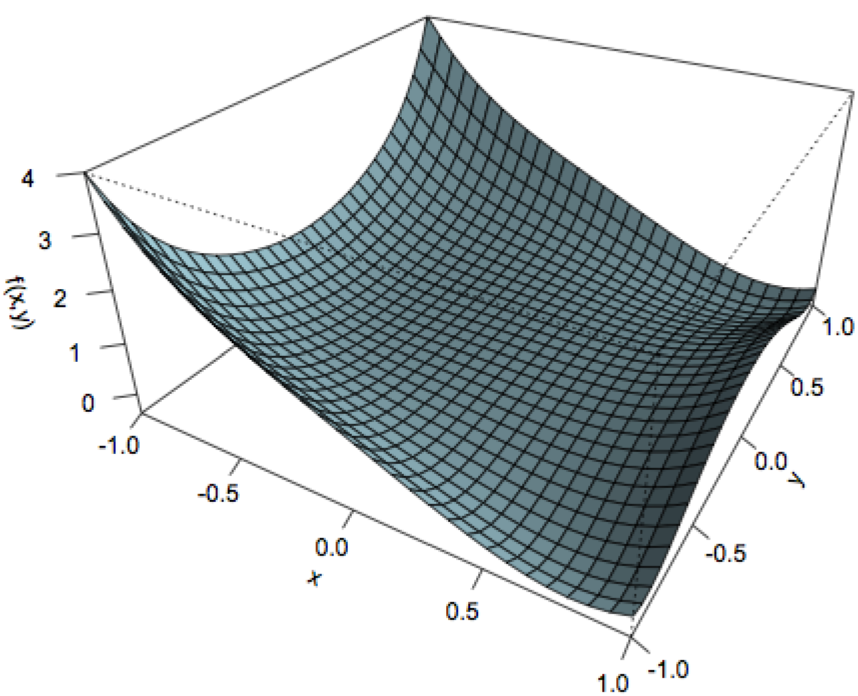

Figure 3: Problem 9.

10. *Find the limit* $L = \lim_{(x,y)\to(3,4)} f(x,y)$ *where*

$$f(x,y) = \begin{cases} \dfrac{(x-3)^2(y-4)^2}{(x-3)^2+(y-4)^2}, & \text{if } (x,y) \neq (3,4); \\ \\ 0, & \text{if } (x,y) = (3,4). \end{cases}$$

The limit is zero. To see it observe that

$$\begin{aligned} 0 &\le (x-3)^2(y-4)^2 \\ &\le \frac{1}{2}\left[2(x-3)^2(y-4)^2\right] \\ &\le \frac{1}{2}\left[(x-3)^4 + 2(x-3)^2(y-4)^2 + (y-4)^4\right] \\ &= \frac{1}{2}\left[(x-3)^2 + (y-4)^2\right]^2 \end{aligned}$$

so that

$$0 \le f(x,y) \le \frac{\left[(x-3)^2+(y-4)^2\right]^2}{2\left[(x-3)^2+(y-4)^2\right]} = \frac{1}{2}\left[(x-3)^2+(y-4)^2\right].$$

Since the function is squeezed between two functions that have a limit we conclude

$$0 = \lim_{(x,y)\to(3,4)} 0 \le \lim_{(x,y)\to(3,4)} f(x,y) \le \lim_{(x,y)\to(3,4)} \frac{1}{2}\left[(x-3)^2+(y-4)^2\right] = 0.$$

The last limit holds because it is the limit of a polynomial which is continuous at all points.

11. *Determine whether the function is continuous at* $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^4+y^4}, & \text{if } (x,y) \neq (0,0); \\ \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

The function is not continuous at $(0,0)$. To see it, consider two paths tending to the origin. For $(x,y) = (t,0)$, $f(t,0) = 0$ for all t so that

$$\lim_{t\to 0} f(t,0) = 0.$$

On the other hand if $(x,y) = (t,t)$ then $f(t,t) = \frac{1}{2}$ so that

$$\lim_{t\to 0} f(t,t) = \frac{1}{2}.$$

Because two paths yield different limits, there is no limit at $(0,0)$ so the function is not continuous there.

12. *Determine whether the function* $f(x,y) = x^3y^4$ *is differentiable at* $(1,2)$.

$f(1,2) = 16$. The partial derivatives are

$$f_x(x,y) = 3x^2y^4; \qquad f_x(1,2) = 48, \qquad f_y(x,y) = 4x^3y^3; \qquad f_y(1,2) = 32.$$

The tangent plane is

$$z = \lambda(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 16 + 48(x-2) + 32(y-2).$$

The function is differentiable at $(1,2)$ if it is almost linear there (well approximated by its tangent plane), namely,

$$\lim_{(h,k)\to(0,0)} \frac{|f(1+h,2+k)-\lambda(1+h,2+k)|}{\sqrt{h^2+k^2}} = 0.$$

To see if f is almost linear at $(1,2)$ we find

$$\begin{aligned}
f(1+h,2+k)-\lambda(1+h,2+k) &= (1+h)^3(2+k)^4 - [16+48h+32k] \\
&= \left[1+3h+3h^2+h^3\right]\left[16+32k+24k^2+8k^3+k^4\right] \\
&\qquad - [16+48h+32k] \\
&= \left[16+32k+24k^2+8k^3+k^4\right] \\
&\qquad + 3h\left[16+32k+24k^2+8k^3+k^4\right] \\
&\qquad + \left[3h^2+h^3\right]\left[16+32k+24k^2+8k^3+k^4\right] \\
&\qquad - [16+48h+32k] \\
&= \left[24k^2+8k^3+k^4\right] \\
&\qquad + 3h\left[32k+24k^2+8k^3+k^4\right] \\
&\qquad + \left[3h^2+h^3\right]\left[16+32k+24k^2+8k^3+k^4\right]
\end{aligned}$$

Thus the difference is quadratic in (h,k) and tends to zero faster than linear. To see it, let $r=\sqrt{h^2+k^2}$ so that $|h|\le r$ and $|k|\le r$. Using this estimate, we find

$$\begin{aligned}
|f(1+h,2+k)-\lambda(1+h,2+k)| &\le 24|k|^2+8|k|^3+|k|^4 \\
&\qquad + 3|h|\left[32|k|+24|k|^2+8|k|^3+|k|^4\right] \\
&\qquad + \left[3|h|^2+|h|^3\right]\left[16+32|k|+24|k|^2+8|k|^3+|k|^4\right] \\
&\le 24r^2+8r^3+r^4 \\
&\qquad + 3r\left[32r+24r^2+8r^3+r^4\right] \\
&\qquad + \left[3r^2+r^3\right]\left[16+32r+24r^2+8r^3+r^4\right] \\
&= 168r^2+192r^3+129r^4+51r^5+11r^6+r^7.
\end{aligned}$$

The difference quotient has the bounds

$$0 \le \frac{|f(1+h,2+k)-\lambda(1+h,2+k)|}{\sqrt{h^2+k^2}} \le 168r+192r^2+129r^3+51r^4+11r^5+r^6.$$

In the limit as $(h,k)\to(0,0)$ so $r\to 0$ we get

$$0 \le \lim_{(h,k)\to(0,0)} \frac{|f(1+h,2+k)-\lambda(1+h,2+k)|}{\sqrt{h^2+k^2}} \le 0.$$

In other words, $f(x,y)$ is almost linear (well approximated by a linear function) at $(1,2)$ which is the definition of being differentiable at $(1,2)$.

13. *Determine whether the function is differentiable at* $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & \text{if } (x,y)\ne(0,0); \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$$

Observe that $f(x,0) = f(0,y) = 0$ for all x, y so that both partial derivatives are defined at zero and equal zero $f_x(0,0) = f_y(0,0) = 0$. If $f(x,y)$ were differentiable, then it would be almost linear there and would be well approximated by its tangent plane

$$z = \lambda(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0.$$

Now compute the limit of the difference quotient

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - \lambda(0+h,0+k)}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{hk^2}{(h^2+k^2)^{3/2}}$$

which does not exist. The limit along the path $(h,k) = (t,0)$ yields zero whereas the limit along the path $(h,k) = (t,t)$ yields $2^{-3/2}$. Since the two limits along the two paths are inconsistent, there is no two-dimensional limit: the tangent plane does not well approximate the surface $z = f(x,y)$ so the function is not differentiable at $(0,0)$.

14. *Find* $\dfrac{\partial z}{\partial u}$. *Express your answer in terms of* (u,v).

$$z = x^3 e^{xy}, \qquad x = u^2 - v^2, \qquad y = u \sin v.$$

Using the chain rule

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left(3x^2 e^{xy} + x^3 y e^{xy}\right)(2u) + \left(x^4 e^{xy}\right)(\sin v) \\
&= e^{xy}\left[(6x^2 + 2x^3 y)u + x^4 \sin v\right] \\
&= e^{(u^2-v^2)u\sin v}\left[\big(6(u^2-v^2)^2 + 2(u^2-v^2)^3 u \sin v\big)u + (u^2-v^2)^4 \sin v\right]
\end{aligned}$$

15. *A function is homogeneous of degree one if* $f(tx,ty) = tf(x,y)$ *for all* $t > 0$. *Prove Euler's Theorem that such a function satisfies*

$$f(x,y) = x f_x(x,y) + y f_y(x,y).$$

Differentiating with respect to t using the chain rule we find the left side

$$\frac{\partial}{\partial t} f(tx,ty) = x f_x(tx,ty) + y f_y(tx,ty).$$

Differentiating the right side

$$\frac{\partial}{\partial t}(tf(x,y)) = f(x,y).$$

Since both are equal, for $t = 1$ we get Euler's Theorem follows.

16. *Use the total differential to approximate the change in* z *as* (x,y) *moves from* P *to* Q. *Use a calculator to find the exact change (up to the accuracy of your calculator).*

$$z = \text{Atn}(xy), \qquad P = (-2. - 0.5), \qquad Q = (-2.03, -0.51).$$

The total differential is

$$\begin{aligned}
dz &= f_x(x,y)\,dx + f_y(x,y)\,dy \\
&= \frac{y\,dx}{1+x^2y^2} + \frac{x\,dy}{1+x^2y^2}
\end{aligned}$$

At $(x, y) = (-2, -0.5)$, $dx = \Delta x = -.03$ and $dy = \Delta y = -0.01$. This gives the approximation

$$\Delta z \approx dz = \frac{y\,dx + x\,dy}{1 + x^2y^2} = \frac{(-0.5)(-.03) + (-2)(-.01)}{1 + (-0.5)^2(-2)^2} = \frac{.015 + .02}{1 + 1} = 0.0175.$$

The exact (up to calculator error) difference is

$$\Delta z = f(-2.03, -0.51) - f(-2, -0.5) = 0.01734214.$$

17. *Find the minimum distance between the point $(1, 2, 0)$ and the cone $z^2 = x^2 + y^2$.*

We illustrate two approaches to this problem. When we use polar coordinates, the problem is fairly difficult. When we represent the cone as a graph it is easier.

In the first method, we use polar coordinates for the cone, where $z = r$. Thus for $0 \le r$ and $0 \le \theta < 2\pi$, the points on the cone are given by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = \pm r.$$

Since the positive and negative nappes are equidistant from the point on the $z = 0$ plane, we can solve for the $z \ge 0$ closest point. The maximum distance occurs when the maximum squared distance occurs. The squared distance to the point $(1, 2, 0)$ is

$$\begin{aligned} f(r,\theta) &= (x-1)^2 + (y-2)^2 + (z-0)^2 \\ &= (r\cos\theta - 1)^2 + (r\sin\theta - 2)^2 + r^2 \\ &= r^2\cos^2 - 2r\cos\theta + 1 + r^2\sin^2\theta - 4r\sin\theta + 4 + r^2 \\ &= 2r^2 - 2r\cos\theta - 4r\sin\theta + 5. \end{aligned}$$

The partial derivatives are

$$f_r = 4r - 2\cos\theta - 4\sin\theta, \qquad f_\theta = 2r\sin\theta - 4r\cos\theta$$

Setting equal to zero

$$\begin{aligned} 2\cos\theta + 4\sin\theta &= 4r \\ (4\cos\theta - 2\sin\theta)r &= 0 \end{aligned}$$

If $r = 0$ then $f(0, \theta) = 5$. Otherwise $r > 0$ can be cancelled. Solving for $\sin\theta$ and $\cos\theta$ we find

$$\cos\theta = 0.4r, \qquad \sin\theta = 0.8r$$

But

$$1 = \cos^2\theta + \sin^2\theta = 0.16r^2 + 0.64r^2 = 0.8r^2$$

so

$$r = \frac{1}{2}\sqrt{5}, \qquad \cos\theta = \frac{1}{5}\sqrt{5}, \qquad \sin\theta = \frac{2}{5}\sqrt{5}.$$

Hence the closest points on the cone are

$$(x, y, z) = (r\cos\theta, r\sin\theta, \pm r) = \left(\frac{1}{2}, 1, \pm\frac{1}{2}\sqrt{5}\right)$$

whose distance to the point $(1, 2, 0)$ is

$$\sqrt{(x-1)^2 + (y-2)^2 + z^2} = \sqrt{\frac{1}{4} + 1 + \frac{5}{4}} = \sqrt{\frac{5}{2}}.$$

This is less than $\sqrt{5}$, the distance to the singular point (origin) so is the minimum distance to the cone.

For the second method, we determine the distance to the upper nappe. By reflection symmetry across the $z = 0$ plane, we get the same distance to the lower nappe.

Write the surface as $z = \sqrt{x^2 + y^2}$. Then the squared distance of (x, y, z) to $(1, 2, 0)$ is

$$\begin{aligned} g(x, y) &= (x - 1)^2 + (y - 2)^2 + (z - 0)^2 \\ &= x^2 - 2x + 1 + y^2 - 4y + 4 + x^2 + y^2 \\ &= 2x^2 - 2x + 2y^2 - 4y + 5. \end{aligned}$$

The partial derivatives vanish if

$$\begin{aligned} g_x &= 4x - 2 = 0 \\ g_y &= 4y - 4 = 0. \end{aligned}$$

Thus there is only one critical point at $x = \frac{1}{2}$ and $y = 1$ where $z = \sqrt{x^2 + y^2} = \frac{1}{2}\sqrt{5}$. We have

$$g_{xx} = 4, \qquad g_{xy} = 0, \qquad g_{yy} = 4, \qquad D = g_{xx}g_{yy} - g_{xy}^2 = 16.$$

Since both g_{xx} and D are positive at $(\frac{1}{2}, 1)$, this is a local minimum. Because the surface is an elliptic paraboloid, this is a global minimum. The distance to the cone is $\sqrt{g(\frac{1}{2}, 1)} = \sqrt{\frac{5}{2}}$.

18. *Using Lagrange Multipliers, show that for all x, y, z we have*

$$(x^3 + y^3 + z^3)^2 \le (x^2 + y^2 + z^2)^3.$$

We maximize and minimize the function $f(x, y, z) = x^3 + y^3 + z^3$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$. At the critical points, the four equations for x, y, z, λ hold

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \qquad g(z, y, z) = 1.$$

Computing,

$$\nabla f = (3x^2, 3y^2, 3z^2) = \lambda(2x, 2y, 2z) = \lambda \nabla g$$

which yields the equations

$$\begin{aligned} x(3x - 2\lambda) &= 0 \\ y(3y - 2\lambda) &= 0 \\ z(3z - 2\lambda) &= 0 \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

These imply that

$$\begin{aligned} x &= 0 \qquad \text{or} \qquad x = \frac{2\lambda}{3}, \\ y &= 0 \qquad \text{or} \qquad y = \frac{2\lambda}{3}, \\ z &= 0 \qquad \text{or} \qquad z = \frac{2\lambda}{3}. \end{aligned}$$

Of these eight possibilities, only $(x, y, z) = (0, 0, 0)$ does not occur since $x^2 + y^2 + z^2 = 1$. One possibility is that

$$x = y = z = \frac{2\lambda}{3}$$

so that

$$1 = x^2 + y^2 + z^2 = 3\left(\frac{2\lambda}{3}\right)^2 = \frac{4}{3}\lambda^2$$

implies

$$\lambda = \pm\frac{\sqrt{3}}{2}, \qquad \text{and} \qquad x = y = z = \pm\frac{1}{\sqrt{3}}.$$

For these values

$$f(x, y, z) = 3\left(\pm\frac{1}{\sqrt{3}}\right)^3 = \pm\frac{1}{\sqrt{3}}.$$

Three possibilities have one of the variables zero, say

$$x = 0, \qquad \text{and} \qquad y = z = \frac{2\lambda}{3}$$

so that

$$1 = x^2 + y^2 + z^2 = 2\left(\frac{2\lambda}{3}\right)^2 = \frac{8}{9}\lambda^2$$

implies

$$\lambda = \pm\frac{3}{2\sqrt{2}}, \qquad \text{and} \qquad y = z = \pm\frac{1}{\sqrt{2}}.$$

For these values

$$f(x, y, z) = 2\left(\pm\frac{1}{\sqrt{2}}\right)^3 = \pm\frac{1}{\sqrt{2}}.$$

The last three possibilities have two of the variables zero, say

$$x = y = 0, \qquad \text{and} \qquad z = \frac{2\lambda}{3}$$

so that

$$1 = x^2 + y^2 + z^2 = \left(\frac{2\lambda}{3}\right)^2 = \frac{4}{9}\lambda^2$$

implies

$$\lambda = \pm\frac{2}{3}, \qquad \text{and} \qquad z = \pm 1.$$

For these values

$$f(x, y, z) = (\pm 1)^3 = \pm 1.$$

It follows that $-1 \le f \le 1$ at all critical points. Since the sphere is closed and bounded, and f is smooth, the critical points include the global minimum and maximum. In other words

$$(x^3 + y^3 + z^3)^2 \le 1$$

for all (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$. Equality holds if (x, y, z) is on the coordinate axes.

To get the inequality for all triples (u, v, w) we assume $(u, v, w) \ne (0, 0, 0)$ and normalize

$$x = \frac{u}{\sqrt{u^2 + v^2 + w^2}}, \qquad y = \frac{v}{\sqrt{u^2 + v^2 + w^2}}, \qquad z = \frac{w}{\sqrt{u^2 + v^2 + w^2}}.$$

so that $u^2+v^2+w^2=1$. The inequality implies

$$\left\{\left(\frac{u}{\sqrt{u^2+v^2+w^2}}\right)^3+\left(\frac{v}{\sqrt{u^2+v^2+w^2}}\right)^3+\left(\frac{w}{\sqrt{u^2+v^2+w^2}}\right)^3\right\}^2 \le 1$$

which is equivalent to

$$(u^3+v^3+w^3)^2 \le (u^2+v^2+w^2)^3.$$

This inequality also holds for $u=v=w=0$, establishing the inequality for all (u,v,w).

19. *Find all critical points, local and global maximum and minimum points in $D=[-2,2]\times[-2,2]$ of*

$$f(x,y)=x^3+3xy-y^3.$$

The polynomial may have interior critical points or boundary extreme points, but no singular points.

The vanishing of partial derivatives yields

$$0=f_x(x,y)=3x^2+3y,$$
$$0=f_y(x,y)=3x-3y^2.$$

The first equation says $y=-x^2$ which is substituted into the second

$$0=x-(-x^2)^2=x(1-x^3)=x(1-x)(1+x+x^2)$$

whose roots are

$$x=0, \qquad 1, \qquad \frac{-1\pm\sqrt{1-4}}{2}.$$

The last two are complex, so the gradient vanishes at $(0,0)$ and $(1,-1)$ where the function equals $f(0,0)=0$ and $f(1,-1)=-1$. Computing second derivatives

$$f_{xx}=6x, \qquad f_{xy}=3, \qquad f_{yy}=-6y.$$

Thus

$$D=f_{xx}f_{yy}-f_{xy}^2=-36xy-9, \qquad D(0,0)=-9, \qquad D(1,-1)=27.$$

Thus $(0,0)$ is a saddle since $D(0,0)<0$ and $(1,-1)$ is local minimum since $D(1,-1)>0$ and $f_{xx}(1,-1)=6>0$.

At the boundary $x=-2$, $f(-2,y)=-8-6y-y^3$ is a decreasing function so there are no boundary critical points. At the corners $f(-2,-2)=12$ and $f(-2,2)=-28$. At the boundary $x=2$, $f(2,y)=8+6y-y^3$. $f_y(2,y)=6-3y^2$ is zero when $y=\pm\sqrt{2}$. $f_{yy}(2,y)=-6y$ so on the boundary line, $(2,-\sqrt{2})$ is local min since $f_{yy}(2,-\sqrt{2})>0$ and $(2,\sqrt{2})$ is local max since $f_{yy}(2,-\sqrt{2})<0$. There $f(2,-\sqrt{2})=8-6\sqrt{2}+2^{3/2}=8-2\sqrt{2}$ and $f(2,\sqrt{2})=8+6\sqrt{2}-2^{3/2}=8+2\sqrt{2}$. At the corners $f(2,-2)=4$ and $f(2,2)=12$.

At the boundary $y=-2$, $f(x,-2)=x^3-6x+8$ with $f_x(x,-2)=3x^2-6$ zero if $x=\pm\sqrt{2}$. $f_{xx}(x,-2)=6x$ so on the line, $x=-\sqrt{2}$ is local max and $x=\sqrt{2}$ is local min with values $f(-\sqrt{2},-2)=8-2\sqrt{2}$ and $f(\sqrt{2},-2)=8-2\sqrt{2}$. At the boundary $y=2$, $f(x,2)=x^3+6x-8$ which is increasing so without boundary critical points.

Looking at the values, the smallest among boundary points and interior critical points is at $(-2,2)$ where $f(-2,2)=-28$ is the global minimum. The global maximum is at $(-2,-2)$ and $(2,2)$ where $f(-2,-2)=f(2,2)=12$.