Math 2210 § 4. **First Midterm Exam** Name: Solutions

Treibergs February 6, 2019

1. *Let* $\mathbf{V} = 2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{W} = 5\mathbf{i} - 4\,\mathbf{k}$, $\mathbf{F}(t) = 6\,\mathbf{i} + 7t\,\mathbf{j} + 8t^2\,\mathbf{k}$.

   *For these vectors, compute or find and explain the following:*

   (a) $$\mathbf{V}\times\mathbf{W} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 5 & 0 & -4 \end{vmatrix} = [3(-4) - 4\cdot 0]\,\mathbf{i} + [4\cdot 5 - 2(-4)]\,\mathbf{j} + [2\cdot 0 - 3\cdot 5]\,\mathbf{k} = \boxed{-12\,\mathbf{i} + 28\,\mathbf{j} - 15\,\mathbf{k}}.$$

   (b) $\mathbf{V}\cdot(\mathbf{W}\times\mathbf{V}) = \boxed{0}$ because $\mathbf{W}\times\mathbf{V}$ is perpendicular to $\mathbf{V}$.

   (c) $$\left\|\mathbf{proj}_{\mathbf{V}}\,\mathbf{W}\right\| = \frac{|\mathbf{V}\cdot\mathbf{W}|}{\|\mathbf{V}\|} = \frac{|2\cdot 5 + 3\cdot 0 + 4(-4)|}{\sqrt{2^2+3^2+4^4}} = \frac{|-6|}{\sqrt{29}} = \boxed{\frac{6}{\sqrt{29}}}.$$

   (d) $$\mathbf{proj}_{\mathbf{V}}\,\mathbf{W} = \frac{(\mathbf{V}\cdot\mathbf{W})}{\|\mathbf{V}\|^2}\mathbf{V} = \frac{-6}{29}(2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}) = \boxed{-\frac{12}{29}\,\mathbf{i} - \frac{18}{29}\,\mathbf{j} - \frac{24}{29}\,\mathbf{k}}.$$

   (e) $\dfrac{d}{dt}\left\|\mathbf{F}(t)\times\mathbf{F}(t)\right\| = \boxed{0}$ because $\mathbf{F}(t)\times\mathbf{F}(t) = \mathbf{0}$ is a constant vector.

2. *Consider a line and a plane in three space given by the equations*

$$\mathbf{r}(t) = \mathbf{P} + t\mathbf{V},$$
$$\mathbf{N}\cdot(\mathbf{X} - \mathbf{Q}) = 0,$$

   *where* $\mathbf{P} = (4,5,6)$, $\mathbf{V} = (1,4,5)$, $\mathbf{N} = (3,-2,1)$, $\mathbf{Q} = (7,7,7)$ *and* $\mathbf{X} = (x,y,z)$ *is a variable point.*

   (a) *Show that the line and the plane are parallel.*

   The normal to the plane $\mathbf{N}$ has to be perpendicular to the direction $\mathbf{V}$ of the line for the line to be parallel to the plane. Indeed the vectors are perpendicular:

$$\mathbf{N}\cdot\mathbf{V} = 3\cdot 1 + (-2)4 + 1\cdot 5 = 0.$$

   (b) *Compute the distance between the line and the plane.*

   The distance between the line and the plane is the length of the projection of a vector from the plane to the line, $\mathbf{P} - \mathbf{Q} = (-3,-2,-1)$, onto the normal direction. Thus

$$d = \|\mathbf{proj}_{\mathbf{N}}\,(\mathbf{P}-\mathbf{Q})\| = \frac{|\mathbf{N}\cdot(\mathbf{P}-\mathbf{Q})|}{\|\mathbf{N}\|} = \frac{|3(-3) + (-2)(-2) + 1(-1)|}{\sqrt{3^2 + (-2)^2 + 1^2}} = \boxed{\frac{6}{\sqrt{14}}}$$

3. *Consider the line of intersection between the two planes*

$$x + 2y + 3z = 4$$
$$8x + 7y + 6z = 5$$

   (a) *Find a point* $\mathbf{P}$ *on the line.*

   (b) *Find a direction vector* $\mathbf{V}$ *of the line.*

   (c) *Find the parametric equations of the line.*

   (d) *Find the symmetric equations of the line.*

We first solve the two equations simultaneously to eliminate one of the unknowns. Multiply the first equation by eight

$$\begin{aligned} 8x + 16y + 24z &= 32 \\ 8x + 7y + 6z &= 5. \end{aligned}$$

Subtracting the second yields

$$9y + 18z = 27$$

which is the same as

$$y + 2z = 3.$$

Setting $z = t$ we find

$$y = 3 - 2t$$

and

$$x = 4 - 2y - 3z = 4 - 2(3 - 2t) - 3t = -2 + t.$$

Thus the parametic form of the line is

$$\boxed{\text{(c.)} \qquad x = -2 + t, \qquad y = 3 - 2t, \qquad z = t}.$$

Solvng for $t$ in each term gives the symmetric form of the line

$$\boxed{\text{(d.)} \qquad t = \frac{x+2}{1} = \frac{y-3}{-2} = \frac{z-0}{1}}.$$

A point on the line is when $t = 0$ or $\boxed{\text{(a.)} \qquad \mathbf{P} = (-2, 3, 0)}$. The direction of the line is the velocity of the parametric form, or the denominators in the symmetric form, namely, $\boxed{\text{(b.)} \qquad \mathbf{V} = (1, -2, 1)}$.

4. *Consider a sphere in space that passes through the distinct points* $\mathbf{V}$ *and* $\mathbf{W}$ *and whose center lies on the line from* $\mathbf{V}$ *to* $\mathbf{W}$.

   (a) *Find its center* $\mathbf{C}$.

   (b) *Find its radius* $r$.

   (c) *Find its equation in terms of* $\mathbf{V}$, $\mathbf{W}$ *and the variable point* $\mathbf{X} = (x, y, z)$.

The only way a sphere can pass through two points and have its center on the line through the two points is if the center is the midpoint of the two points

$$\boxed{\text{(a.)} \qquad \mathbf{C} = \frac{1}{2}(\mathbf{V} + \mathbf{W}).}$$

Then the radius is half the distance between the two ponts

$$\boxed{\text{(a.)} \qquad r = \frac{1}{2}\|\mathbf{V} - \mathbf{W}\|.}$$

Hence the equation of the sphere with this center and radius is

$$\boxed{\text{(c.)} \qquad \|\mathbf{X} - \mathbf{C}\|^2 = \frac{1}{4}\|\mathbf{V} - \mathbf{W}\|^2.}$$

5. *Suppose that* $\mathbf{r}(t)$ *is a curve in* $\mathbb{R}^3$. *You know its location* $\mathbf{P} = (3,1,0)$ *and velocity* $\mathbf{V} = (2,0,1)$ *at time* $t = 0$. *You also know its acceleration at all times* $\mathbf{a}(t) = (0, -\cos t, -\sin t)$. *Find its position* $\mathbf{r}(t)$ *at all times.*

   Because the velocity $\mathbf{v}' = \mathbf{a}$, we recover the change of velocity by integration

$$
\begin{aligned}
\mathbf{v}(t) - \mathbf{v}(0) &= \int_0^t \mathbf{a}(t)\,dt \\
\mathbf{v}(t) - (2,0,1) &= \left(\int_0^t 0\,dt, -\int_0^t \cos t\,dt, -\int_0^t \sin t\,dt\right) \\
&= (0,\ -\sin t,\ \cos t - 1)
\end{aligned}
$$

   Hence, the velocity vector for all time is

$$\mathbf{v}(t) = (2,\ -\sin t,\ \cos t)\,.$$

   Now the $\mathbf{r}' = \mathbf{v}$ so by integrating,

$$
\begin{aligned}
\mathbf{r}(t) - \mathbf{r}(0) &= \int_0^t \mathbf{v}(t)\,dt \\
\mathbf{r}(t) - (3,1,0) &= \left(\int_0^t 2\,dt, -\int_0^t \sin t\,dt, \int_0^t \cos t\,dt\right) \\
&= (2t,\ \cos t - 1,\ \sin t)\,.
\end{aligned}
$$

   Hence, the position vector for all time is

$$\boxed{\mathbf{r}(t) = (3 + 2t,\ \cos t,\ \sin t)\,.}$$

6. *Let the space curve be* $\quad \mathbf{r}(t) = 3(\sin t)\,\mathbf{i} + 5(\cos t)\,\mathbf{j} + 4(\sin t)\,\mathbf{k}.$

   *For each* $t$, *find the following:*

   (a) *The speed* $\dfrac{ds}{dt}$.
   (b) *The distance* $s$ *along the curve from* $\mathbf{r}(0)$.
   (c) *The unit tangent vector* $\mathbf{T}$.
   (d) *The curvature of the space curve* $\kappa$.
   (e) *The normal vector* $\mathbf{N}$.

   Differentiation we find

$$\mathbf{r}'(t) = 3(\cos t)\,\mathbf{i} - 5(\sin t)\,\mathbf{j} + 4(\cos t)\,\mathbf{k}.$$

   Hence the velocity is using $3^2 + 4^2 = 5^2$,

$$\frac{ds}{dt} = \|\mathbf{r}'\| = \sqrt{3^2\cos^2 t + 5^2\sin^2 t + 4^2\cos^2 t} = \sqrt{5^2\left(\cos^2 t + \sin^2 t\right)} = \boxed{5. \qquad \text{(a.)}}$$

   Thus the length along the curve from 0 to $t$ is

$$s(t) = \int_0^t \frac{ds}{dt}(t)\,dt = \int_0^t 5\,dt = \boxed{5t. \qquad \text{(b.)}}.$$

   The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\frac{ds}{dt}} = \boxed{\frac{3}{5}(\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + \frac{4}{5}(\cos t)\,\mathbf{k} \qquad \text{(c.)}}$$

The curvature is the norm of the rate of change of $\mathbf{T}$ with respect to $s$. Computing,

$$\mathbf{T}' = \frac{d}{dt}\mathbf{T}(t) = \frac{1}{5}\Big(-3(\sin t)\,\mathbf{i} - 5(\cos t)\,\mathbf{j} - 4(\sin t)\,\mathbf{k}\Big)$$

or

$$\kappa = \left\|\frac{d}{ds}\mathbf{T}\right\| = \frac{\|\mathbf{T}'\|}{\frac{ds}{dt}} = \frac{1}{25}\sqrt{3^2\sin^2 t + 5^2\cos^2 t + 4^2\sin^2 t} = \boxed{\frac{1}{5}. \qquad \text{(d.)}}$$

The normal vector is in the $\mathbf{T}'(t)$ direction, thus

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \boxed{-\frac{3}{5}(\sin t)\,\mathbf{i} - (\cos t)\,\mathbf{j} - \frac{4}{5}(\sin t)\,\mathbf{k} \qquad \text{(e.)}}$$

because $\|\mathbf{T}'\| = 1$. Indeed, we can also recover $\kappa$ and $\mathbf{N}$ from the formula

$$\kappa\mathbf{N} = \frac{d}{ds}\mathbf{T} = \frac{\mathbf{T}'}{\frac{ds}{dt}} = \frac{1}{5}\left(-\frac{3}{5}(\sin t)\,\mathbf{i} - (\cos t)\,\mathbf{j} - \frac{4}{5}(\sin t)\,\mathbf{k}\right).$$