Math 2210 § 4. **First Midterm Exam** Name: Practice Problems
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1. *Graph the curve. Is the curve closed? simple? Find* x,y, $\dfrac{dy}{dx}$ *and* $\dfrac{d^2\,y}{dx^2}$ *at* $t = 0$. *Obtain the Cartesian equation of the parameterized curve.*

$$x = t^3 - 2t; \qquad y = t^2 - 2t; \qquad -2 \le t \le 2.$$

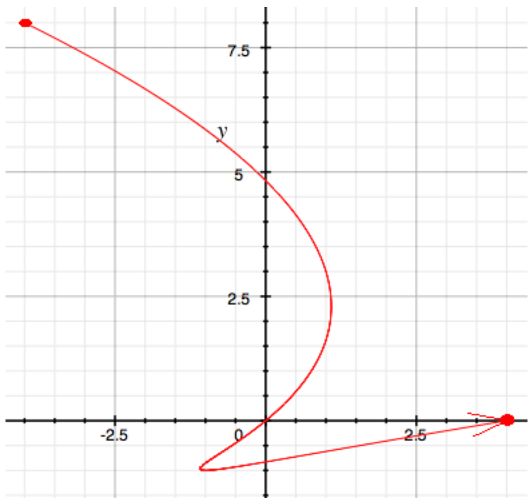

Figure 1: Problem 1.

Here is the output from my Grapher program on my laptop Macintosh. The curve starts at $(x(-2), y(-2)) = (-4, 8)$ and ends at $(x(2), y(2)) = (4, 0)$. Observe that $y = (t-1)^2 - 1$ so that $y = -1$ when $t = 1$ and $y \ge -1$ for all t. Solving for t in the second equation $t^2 - 2t - y = 0$ we find from the quadratic formula

$$t = \frac{2 \pm \sqrt{4 + 4y}}{2} = 1 \pm \sqrt{1 + y}$$

which makes sense for $y \ge -1$. There are two branches, the one corresponding to $-2 \le t \le 1$ when we take the "$-$" root and to $1 \le t \le 2$ when we take the "+" root.

Substituting into the other equation, $x = t(t^2 - 2)$ we see that for $1 \le t \le 2$ where we take the "+" root

$$\begin{aligned} x &= (1 + \sqrt{1+y})\left[(1 + \sqrt{1+y})^2 - 2\right] = (1 + \sqrt{1+y})\left[(1 + 2\sqrt{1+y} + 1 + y) - 2\right] \\ &= (1 + \sqrt{1+y})\left[y + 2\sqrt{1+y}\right] = [y + 2(1+y)] + [2+y]\sqrt{1+y} \\ &= [3y + 2] + [2+y]\sqrt{1+y} \end{aligned}$$

For $-3 \le t \le 1$ where we take the "$-$" root

$$\begin{aligned} x &= (1 - \sqrt{1+y})\left[(1 - \sqrt{1+y})^2 - 2\right] = (1 - \sqrt{1+y})\left[(1 - 2\sqrt{1+y} + 1 + y) - 2\right] \\ &= (1 - \sqrt{1+y})\left[y - 2\sqrt{1+y}\right] = [y + 2(1+y)] - [2+y]\sqrt{1+y} \\ &= [3y + 2] - [2+y]\sqrt{1+y} \end{aligned}$$

Notice that the two branches have x as a function of y and that the x for $1 \le t \le 2$ is strictly greater than the x for $-2 \le t < 1$ over the common height $-1 < y \le 0$. This means that the two curves only intersect when $t = 1$ or $(x, y) = (-1, -1)$. This means that the curve is simple (no self intersections and not closed (starting and ending points are different).

Another way to combine both equations into a single one is to isolate and square the radical

$$(x - 3y - 2)^2 = (2 + y)^2(1 + y).$$

Finally, let's compute the derivatives. At $t = 0$, $(x(0), y(0)) = (0, 0)$.

$$\frac{dx}{dt} = 3t^2 - 2; \qquad \frac{dy}{dt} = 2t - 2;$$

Since $\frac{dx}{dt}(0) = -2 \neq 0$ we may compute

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 2}{3t^2 - 2}$$

so

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{2t - 2}{3t^2 - 2}\right|_{t=0} = \frac{-2}{-2} = 1.$$

Note that by the quotient rule

$$\frac{dy'}{dt} = \frac{2(3t^2 - 2) - (2t - 2)6t}{(3t^2 - 2)^2} = \frac{-6t^2 + 12t - 4}{(3t^2 - 2)^2}$$

Also

$$\frac{d^2\,y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{-6t^2+12t-4}{(3t^2-2)^2}}{3t^2 - 2} = \frac{-6t^2 + 12t - 4}{(3t^2 - 2)^3}$$

so

$$\left.\frac{d^2\,y}{dx^2}\right|_{t=0} = \left.\frac{-6t^2 + 12t - 4}{(3t^2 - 2)^3}\right|_{t=0} = \frac{-4}{-8} = \frac{1}{2}.$$

2. *Find the center and radius of the sphere whose equation is*

$$x^2 + y^2 + z^2 + 8x - 22y - 10z + 98 = 0.$$

Completing the squares we find

$$\begin{aligned} 0 &= \left[(x + 4)^2 - 16\right] + \left[(y - 11)^2 - 121\right] + \left[(z - 5)^2 - 25\right] + 98 \\ &= (x + 4)^2 + (y - 11)^2 + (z - 5)^2 - 64 \end{aligned}$$

Thus the center is $C = \boxed{(-4, 11, 5)}$ and the radius is $r = \sqrt{64} = \boxed{8}$.

3. *Find the equation of the sphere whose center is $C = (7, 9, 11)$ and which is tangent to the plane $x + 2y + 3z = 100$.*

We need to compute the distance from the plane L to the point C. The normal vector to the plane is $N = (1, 2, 3)$. The dot product is $N \cdot C = 7 + 18 + 33 = 58$. If Q were a point on the plane then $N \cdot Q = D = 100$. Let $\theta = \angle(N, C - Q)$ be the angle between N and $C - Q$. Then this distance is

$$\begin{aligned} r &= \|\,\mathbf{proj}_N(C - Q)\| = \left|\cos\theta\,\|C - Q\|\right| = \frac{|N \cdot (C - Q)|}{\|N\|\,\|C - Q\|}\|C - Q\| = \frac{|N \cdot C - D|}{\|N\|} \\ &= \frac{|58 - 100|}{\sqrt{1 + 4 + 9}} = \frac{42}{\sqrt{14}} = 3\sqrt{14}. \end{aligned}$$

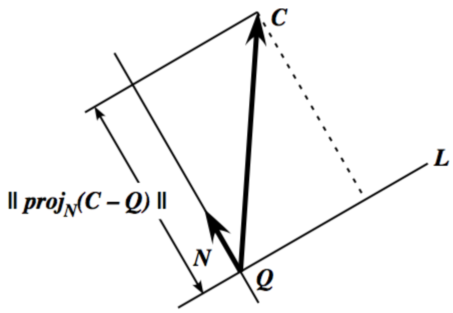

Figure 2: Problem 3.

Hence the equation of sphere with center C and radius r is

$$(x-7)^2 + (y-9)^2 + (z-11)^2 = 9 \cdot 14$$

or

$$x^2 + y^2 + z^2 - 14x - 18y - 22z + 125 = 0.$$

4. *Find the arclength of the given curve.*

$$x = t^2, \qquad y = \frac{4\sqrt{3}}{3} t^{3/2}, \qquad z = 3t, \qquad 1 \le t \le 9.$$

Plug into the arclength formula

$$L = \int_{t=0}^{3\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt.$$

Computing,

$$\frac{dx}{dt} = 2t; \qquad \frac{dy}{dt} = 2\sqrt{3}\, t^{1/2}; \qquad \frac{dx}{dt} = 3.$$

Substituting,

$$\begin{aligned} L &= \int_{t=1}^{3} \sqrt{4t^2 + 12t + 9}\, dt \\ &= \int_{t=1}^{3} 2t + 3\, dt \\ &= \left[t^2 + 3t\right]\Big|_{t=1}^{3} = \left[3^2 + 3 \cdot 3\right] - \left[1 + 3\right] = \boxed{14}. \end{aligned}$$

5. An objects position P changes so that its distance from $(-3, 2, 1)$ is always twice the distance from $(3, 2, 1)$. Show that P is on a sphere and find its center and radius.

Write the equation for $P = (x, y, z)$ geometrically and see what it is in coordinates. Let the center points be denoted $C = (-3, 2, 1)$ and $D = (3, 2, 1)$. P's distance to C_1 being twice its distance from C_2 becomes the equation

$$\|P - C_1\| = 2\|P - C_2\|.$$

Squaring we find

$$(x+3)^2 + (y-2)^2 + (z-1)^2 = \|P - C_1\|^2 = 4\|P - C_2\|^2 = 4\left((x-3)^2 + (y-2)^2 + (z-1)^2\right)$$

so

$$0 = 4(x-3)^2 - (x+3)^2 + 3(y-2)^2 + 3(z-1)^2.$$

The first two terms are after completing the square

$$\begin{aligned} 4(x-3)^2 - (x+3)^2 &= 4\left(x^2 - 6x + 9\right) - \left(x^2 + 6x + 9\right) \\ &= 3x^3 - 30x + 27 \\ &= 3\left(x^2 - 10x + 9\right) \\ &= 3\left((x-5)^2 - 16\right). \end{aligned}$$

Substituting into the equation for P,

$$0 = 3(x-5)^2 - 3 \cdot 16 + 3(y-2)^2 + 3(z-1)^2.$$

which simplifies to

$$(x-5)^2 + (y-2)^2 + (z-1)^2 = 16$$

so the center of the desired circle is $C = \boxed{(5,2,1)}$ and radius is $r = \boxed{4}$.

6. *Let n points be equally spaced on a circle and let $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots \mathbf{v}_n$, be the vectors from the center of the circle to these n points. Show that*

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \mathbf{0}.$$

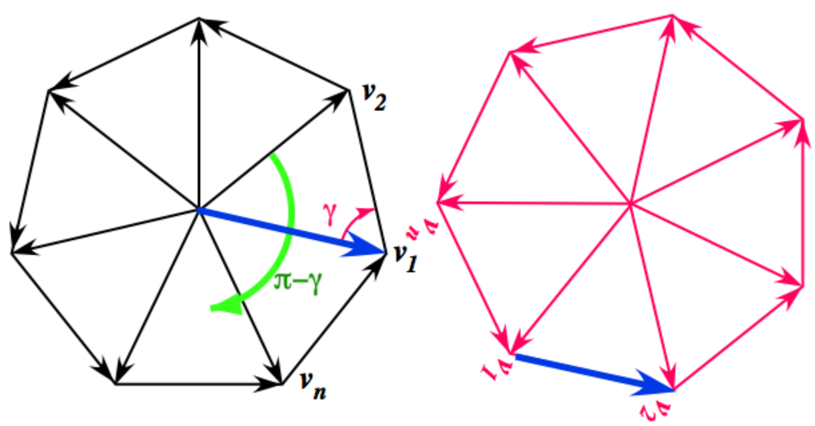

Figure 3: Unrotated and Rotated n-gon for Problem 6.

Consider the regular n-gon connecting the ends of the vectors from $\mathbf{v}_1$ to $\mathbf{v}_2$ all the way around to $\mathbf{v}_n$ to $\mathbf{v}_1$. The vectors along the rim are $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_2$, all the way around to $\mathbf{v}_n - \mathbf{v}_{n-1}$ to $\mathbf{v}_1 - \mathbf{v}_n$. Because the spokes are equally spaced around the circle, at each corner, the angle from spoke to rim vectors is the same

$$\gamma = \angle(\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1) = \angle(\mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2) = \cdots = \angle(\mathbf{v}_{n-1}, \mathbf{v}_n - \mathbf{v}_{n-1}) = \angle(\mathbf{v}_n, \mathbf{v}_1 - \mathbf{v}_n).$$

Thus if the original n-gon is rotated $\pi - \gamma$ radians to a new polygon, $\mathbf{v}'_2$, $\ldots \mathbf{v}'_n$ then the rim vectors of the rotated polygon point in the same direction as the original spoke vectors and their lengths are shortened by a scaling factor $s = \|\mathbf{v}_1\| / \|\mathbf{v}_2 - \mathbf{v}_1\|$.

Their sum must be zero because the polygon closes. Thus

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = s\left[(\mathbf{v}'_2 - \mathbf{v}'_1) + (\mathbf{v}'_3 - \mathbf{v}'_2) + \cdots + (\mathbf{v}'_1 - \mathbf{v}'_n)\right] = \mathbf{0}.$$

7. *Find the smallest angle between the main diagonals of a rectangular box 4 feet by 6 feet by 10 feet.*

Aligning the box with coordinate axes, we may take its vertices to be $(0,0,0)$,$(4,0,0)$, $(0,6,0)$, $(4,6,0)$, $(0,0,10)$, $(4,0,10)$, $(0,6,10)$ and $(4,6,10)$. The four main diagonals connect $(0,0,0)$ to $(4,6,10)$, $(4,0,0)$ to $(0,6,10)$, $(0,6,0)$ to $(4,0,10)$ and $(0,0,10)$ to $(4,6,0)$. Thus they have direction vectors $\mathbf{v}_1 = (4,6,10)$, $\mathbf{v}_2 = (-4,6,10)$, $\mathbf{v}_3 = (4,-6,10)$ and $\mathbf{v}_4 = (4,6,-10)$, respectively. All four diagonals have length squared $\|\mathbf{v}_i\|^2 = 16+36+100 = 152$. The cosines of angles between the six intersecting main diagonals angles

$$\begin{aligned}
\pm\cos\angle(\mathbf{v}_1,\mathbf{v}_2) = \pm\cos\angle(\mathbf{v}_3,\mathbf{v}_4) &= \frac{\mathbf{v}_1\cdot\mathbf{v}_2}{\|v_i\|^2} = \frac{-16+36+100}{152} = \frac{15}{19}\\
\pm\cos\angle(\mathbf{v}_1,\mathbf{v}_3) = \pm\cos\angle(\mathbf{v}_2,\mathbf{v}_4) &= \frac{\mathbf{v}_1\cdot\mathbf{v}_3}{\|v_i\|^2} = \frac{16-36+100}{152} = -\frac{10}{19}\\
\pm\cos\angle(\mathbf{v}_1,\mathbf{v}_4) = \pm\cos\angle(\mathbf{v}_2,\mathbf{v}_3) &= \frac{\mathbf{v}_1\cdot\mathbf{v}_4}{\|v_i\|^2} = \frac{16+36-100}{152} = -\frac{6}{19}
\end{aligned}$$

where minus cosine occurs if we reverse one of the diagonals. The smallest angle occurs between the largest cosine, which is between the $\mathbf{v}_1$ and $\mathbf{v}_2$ where $\cos\theta = 15/19$ or $\theta = \boxed{\arccos 15/19 = 0.661}$ radians.

8. *Find the distance between the parallel planes*

$$5x - 3y - 2z = 5, \qquad -5x + 3y + 2z = 7.$$

The normal vector to both is $N = (5,-3,-2)$ so $\|N\| = \sqrt{25+9+4} = \sqrt{38}$. Let P be a point on the first plane and Q on the second. Then the distance between the planes is the length of the projection of $Q-P$ onto the N direction

$$d = \|\,\mathbf{proj}_N(Q-P)\| = |\cos\theta|\,\|Q-P\|$$

where $\theta = \angle(N, Q-P)$ is the angle from the normal to $Q-P$. By inspection, the point $P = (1,0,0)$ is in the first plane and $Q = (-1,0,1)$ is in the second so $Q-P = (-2,0,1)$ and $N\cdot(Q-P) = -10+0-2 = -7$. Computing the length

$$d = |\cos\theta|\,\|Q-P\| = \left|\frac{N\cdot(Q-P)}{\|N\|\,\|Q-P\|}\right|\,\|Q-P\| = \frac{|N\cdot(Q-P)|}{\|N\|} = \frac{|-7|}{\sqrt{38}} = \boxed{\frac{7}{\sqrt{38}} = 1.136.}$$

9. *Let* $\mathbf{a} = (a_1,a_2,a_3)$ *and* $\mathbf{b} = (b_1,b_2,b_3)$ *be fixed vectors. Show that* $(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b}) = 0$ *is the equation of a sphere, and find its center and radius.*

Expand the vector equation and complete the square

$$\begin{aligned}
0 &= (\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{b})\\
&= \mathbf{x}\cdot\mathbf{x} - \mathbf{a}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b}\\
&= \mathbf{x}\cdot\mathbf{x} - (\mathbf{a}+\mathbf{b})\cdot\mathbf{x} + \mathbf{a}\cdot\mathbf{b}\\
&= \left(\mathbf{x} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot\left(\mathbf{x} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right) - \frac{1}{4}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + \mathbf{a}\cdot\mathbf{b}\\
&= \left\|\mathbf{x} - \frac{1}{2}(\mathbf{a}+\mathbf{b})\right\|^2 - \frac{1}{4}\left(\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a}\right) + \mathbf{a}\cdot\mathbf{b}
\end{aligned}$$

$$
\begin{aligned}
0 &= \left\| \mathbf{x} - \frac{1}{2}(\mathbf{a}+\mathbf{b}) \right\|^2 - \frac{1}{4}\left(\mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a}\right) \\
&= \left\| \mathbf{x} - \frac{1}{2}(\mathbf{a}+\mathbf{b}) \right\|^2 - \frac{1}{4}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
&= \left\| \mathbf{x} - \frac{1}{2}(\mathbf{a}+\mathbf{b}) \right\|^2 - \frac{1}{4}\|\mathbf{a}-\mathbf{b}\|^2
\end{aligned}
$$

thus the center and radius of the sphere are

$$
C = \boxed{\frac{1}{2}(\mathbf{a}+\mathbf{b})}, \qquad r = \boxed{\frac{1}{2}\|\mathbf{a}-\mathbf{b}\|}
$$

namely, the sphere centered at the midpoint of **a** and **b** which passes through both points.

10. *Find the area of the triangle with vertices* $(1,2,3)$, $(3,1,5)$ *and* $(5,4,6)$.

If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the vertices, the area is half the area of the parallelogram with sides $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ which in turn is given by the length of their crossproduct. Thus

$$
\begin{aligned}
A &= \frac{1}{2}\|(\mathbf{v}_2-\mathbf{v}_1)\times(\mathbf{v}_3-\mathbf{v}_1)\| = \frac{1}{2}\|(2,-1,2)\times(3,2,3)\| \\
&= \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix} \right\| = \frac{1}{2}\|\,(-1-4)\mathbf{i} + (6-2)\mathbf{j} + (4+3)\mathbf{k}\,\| \\
&= \frac{1}{2}\|\,-5\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\,\| = \frac{1}{2}\sqrt{25+16+49} = \frac{1}{2}\sqrt{90} = \boxed{\frac{3}{2}\sqrt{10}.}
\end{aligned}
$$

11. *Find the equation of a plane through the points*

$$
(2,3,4), \qquad (3,5,-7), \qquad (6,-8,9).
$$

If the vectors are called $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, then $\mathbf{v}_2 - \mathbf{v}_2$ and $\mathbf{v}_3 - \mathbf{v}_1$ are parallel to the plane and their cross product $N = (\mathbf{v}_2 - \mathbf{v}_2)\times(\mathbf{v}_3 - \mathbf{v}_1)$ is normal. Then the plane is given by the point-normal formula. $\mathbf{v}_2 - \mathbf{v}_2 = (1,2,-11)$ and $\mathbf{v}_3 - \mathbf{v}_1 = (4,-11,5)$ so

$$
\begin{aligned}
N &= (\mathbf{v}_2-\mathbf{v}_2)\times(\mathbf{v}_3-\mathbf{v}_1) = (1,2,-11)\times(4,-11,5) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -11 \\ 4 & -11 & 5 \end{vmatrix} = (10-121)\mathbf{i} + (-44-5)\mathbf{j} + (-11-8)\mathbf{k} \\
&= -111\mathbf{i} - 49\mathbf{j} - 19\mathbf{k}.
\end{aligned}
$$

The equation of the plane is thus

$$
0 = N\cdot(\mathbf{x}-\mathbf{v}_1) = N\cdot\mathbf{x} - N\cdot\mathbf{v}_1 = -111x - 49y + -19z - (-111\cdot 2 - 49\cdot 3 - 19\cdot 4)
$$

or

$$
\boxed{111x + 49y + 19z = 445.}
$$

12. *Prove Lagrange's identity without using* $\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\|\sin\angle(\mathbf{v},\mathbf{w})$.

$$\|\mathbf{v}\times\mathbf{w}\|^2 = \|\mathbf{v}\|^2\,\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2$$

Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$. One expands the left side and the right side and compares.

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2)\mathbf{i} + (v_3w_1 - v_1w_3)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k}.$$

Thus the left side is

$$\begin{aligned}\|\mathbf{v}\times\mathbf{w}\|^2 &= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \\ &= (v_2^2w_3^2 - 2v_2v_3w_2w_2 + v_3^2w_2^2) + (v_3^2w_1^2 - 2v_1v_3w_1w_3 + v_1^2w_3^2) \\ &\quad + (v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2).\end{aligned}$$

The right side is after cancelling

$$\begin{aligned}\|\mathbf{v}\|^2\,\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2 &= \left(v_1^2+v_2^2+v_3^2\right)\left(w_1^2+w_2^2+w_3^2\right) - (v_1w_1+v_2w_2+v_3w_3)^2 \\ &= \left(v_1^2w_1^2 + v_1^2w_2^2 + v_1^2w_3^2 + v_2^2w_1^2 + v_2^2w_2^2 + v_2^2w_3^2 + v_3^2w_1^2 + v_3^2w_2^2 + v_3^2w_3^2\right) \\ &\quad - \left(v_1^2w_1^2 + v_2^2w_2^2 + v_3^2w_3^2 + 2v_1v_2w_1w_2 + 2v_1v_3w_1w_3 + 2v_2v_3w_2w_3\right) \\ &= \left(v_1^2w_2^2 + v_1^2w_3^2 + v_2^2w_1^2 + v_2^2w_3^2 + v_3^2w_1^2 + v_3^2w_2^2\right) \\ &\quad - \left(2v_1v_2w_1w_2 + 2v_1v_3w_1w_3 + 2v_2v_3w_2w_3\right).\end{aligned}$$

This expression for $\|\mathbf{v}\times\mathbf{w}\|^2$ and the one for $\|\mathbf{v}\|^2\,\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2$ both have the same nine terms so they are equal.

13. *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{a}-\mathbf{b}$ *denote the three sides of a triangle whose lengths are* a, b *and* c, *respectively. Show* $2\,\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2$ *and use it to peove Heron's Formula for the area of a triangle,*

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where the semiperimeter is} \quad s = \frac{1}{2}(a+b+c).$$

Expanding the middle term yields the formula.

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2 - 2\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \left(\|\mathbf{a}\|^2 - 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2\right) - 2\mathbf{a}\cdot\mathbf{b} = 0.$$

Sixteen times the square of area of the triangle given by cross product is simplified using Lagrange's Formula and factoring the difference of squares.

$$\begin{aligned}16A^2 &= 4\|\mathbf{a}\times\mathbf{b}\|^2 \\ &= 4\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (2\,\mathbf{a}\cdot\mathbf{b})^2 \\ &= 4\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - \left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right)^2 \\ &= \left(2\|\mathbf{a}\|\,\|\mathbf{b}\| + \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right)\left(2\|\mathbf{a}\|\,\|\mathbf{b}\| - \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 + \|\mathbf{a}+\mathbf{b}\|^2\right) \\ &= \Big((\|\mathbf{a}\| + \|\mathbf{b}\|)^2 - \|\mathbf{a}-\mathbf{b}\|^2\Big)\Big(-(\|\mathbf{a}\| - \|\mathbf{b}\|)^2 + \|\mathbf{a}+\mathbf{b}\|^2\Big) \\ &= (\|\mathbf{a}\| + \|\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\|)(\|\mathbf{a}\| + \|\mathbf{b}\| - \|\mathbf{a}-\mathbf{b}\|) \\ &\qquad \cdot (\|\mathbf{a}\| - \|\mathbf{b}\| + \|\mathbf{a}+\mathbf{b}\|)(-\|\mathbf{a}\| + \|\mathbf{b}\| + \|\mathbf{a}+\mathbf{b}\|) \\ &= 2s(2s-2c)(2s-2b)(2s-2a)\end{aligned}$$

which is Heron's Formula.

14. *Find the velocity* $\mathbf{v}$*, acceleration* $\mathbf{a}$ *and speed* s *of* $\mathbf{r}(t)$ *at the time* $t = 2$*.*

$$\mathbf{r}(t) = t\sin\pi t\,\mathbf{i} + t\cos\pi t\,\mathbf{j} + e^{-t}\mathbf{k}$$

Velocity and acceleration are first and second derivatives

$$\begin{aligned}\mathbf{v} = \mathbf{r}' &= (\sin\pi t + \pi t\cos\pi t)\mathbf{i} + (\cos\pi t - \pi t\sin\pi t)\mathbf{j} - e^{-t}\mathbf{k}\\ \mathbf{a} = \mathbf{r}'' &= (2\pi\cos\pi t - \pi^2 t\sin\pi t)\mathbf{i} + (-2\pi\sin\pi t - \pi^2 t\cos\pi t)\mathbf{j} + e^{-t}\mathbf{k}\end{aligned}$$

At time $t = 2$,

$$\mathbf{v}(2) = (\sin 2\pi + 2\pi\cos 2\pi)\mathbf{i} + (\cos 2\pi - 2\pi\sin 2\pi)\mathbf{j} - e^{-2}\mathbf{k} = \boxed{2\pi\,\mathbf{i} + \mathbf{j} - e^{-2}\mathbf{k}},$$

$$\mathbf{a}(2) = (2\pi\cos 2\pi - 2\pi^2\sin 2\pi)\mathbf{i} + (-2\pi\sin 2\pi - 2\pi^2\cos 2\pi)\mathbf{j} + e^{-2}\mathbf{k} = \boxed{2\pi\,\mathbf{i} - 2\pi^2\mathbf{j} + e^{-2}\mathbf{k}}.$$

At $t = 2$, the speed is the length of the velocity

$$s = \|\mathbf{v}\| = \boxed{\sqrt{4\pi^2 + 1 + e^{-4}}}.$$

15. *Assume the planet* $\mathbf{r}(t)$ *travels in a gravitational force field which points to the origin with a magnitude proportional to the inverse square of the distance. Show that the particle satisfies Kepler's Second Law: the vector* $\mathbf{r}(t)$ *sweeps out equal areas in equal times.*

The assumption about the force field felt by the particle is

$$\mathbf{F} = -\frac{GmM\,\mathbf{r}}{\|\mathbf{r}\|^3}$$

where m and M are masses of the particle and sun at the origin and gravitational constant so that $\|\mathbf{F}\| \propto \|\mathbf{r}\|^{-2}$. By Newton's Law, mass times acceleration equals the force

$$m\mathbf{r}'' = \mathbf{F} = -\frac{GmM\,\mathbf{r}}{\|\mathbf{r}\|^3}.$$

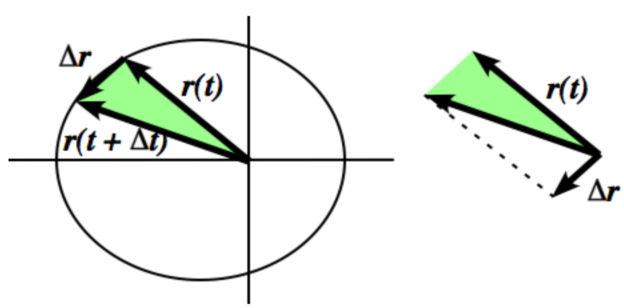

Figure 4: Kepler's Second Law in Problem 15.

The area swept out in a time Δt is approximately the area of the triangle with sides $\mathbf{r}(t)$ and $\Delta\mathbf{r}$ which is

$$\Delta A \approx \frac{1}{2}\|\mathbf{r}(t) \times \Delta\mathbf{r}\|.$$

Dividing by Δt this is

$$\frac{\Delta A}{\Delta t} \approx \frac{1}{2}\left\|\mathbf{r}(t) \times \frac{\Delta \mathbf{r}}{\Delta t}\right\|$$

which in the $\Delta t \to 0$ limit becomes

$$\frac{dA}{dt} = \frac{1}{2}\left\|\mathbf{r}(t) \times \mathbf{r}'\right\|.$$

Differentiating $\mathbf{r}(t) \times \mathbf{r}'$, and using Newton's Law above,

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{r}' \times \mathbf{r}' - \mathbf{r}(t) \times \frac{GM\,\mathbf{r}}{\|\mathbf{r}\|^3} = \mathbf{0} + \mathbf{0}$$

because $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. Thus $\mathbf{r} \times \mathbf{r}'$ is a constant vector with constant magnitude so $\dfrac{dA}{dt}$ is constant. Thus area swept grows proportionately to time.

16. *Find parametric and symmetric equations of the line of intersection of the given pair of planes*

$$x - 3y + z = -1; \qquad 6x - 5y + 4z = 9.$$

Eliminate a variable in the two equations in three variables. Solve for x in the first and substitute into the second.

$$x = 3y - z - 1, \qquad 6(3y - z - 1) - 5y + 4z = 9 \qquad \Longrightarrow \qquad 13y - 2z = 15.$$

Thus if we choose $z = t$ then $y = \frac{1}{13}(2t + 15)$ and $x = \frac{3}{13}(2t + 15) - t - 1$ so we get the parametric form of the line

$$x = -\frac{7}{13}t + \frac{17}{13}, \qquad y = \frac{2}{13}t + \frac{15}{13}, \qquad z = t \qquad \text{for } t \in \mathbb{R}.$$

Eliminating t gives the symmetric form of the line

$$t = \frac{x - \frac{17}{13}}{-\frac{7}{13}} = \frac{y - \frac{15}{13}}{\frac{2}{13}} = \frac{z - 0}{1}.$$

17. *Find the equation of the plane containing the point* $(1, -1, 5)$ *and the line*

$$x = 1 + 2t, \qquad y = -1 + 3t, \qquad z = 4 + t.$$

Let $P = (1, -1, 5)$. A vector in the plane is the tangent vector $T = (2, 3, 1)$ (the t derivative of the parametric line.) At $t = 0$ the line contains the point $Q = (1, -1, 4)$. Thus a second vector in the plane is $P - Q = (0, 0, 1)$ Thus a normal vector to the plane is

$$N = T \times (P - Q) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (3 - 0)\mathbf{i} + (0 - 2)\mathbf{j} + (0 - 0)\mathbf{k} = \mathbf{i} - 2\mathbf{j}.$$

Thus the point-normal equation of the plane is $N \cdot (X - P) = 0$ or

$$x - 2y = 1x - 2y + 0z = N \cdot X = N \cdot P = 1 \cdot 1 - 2 \cdot (-1) + 0 \cdot 5 = 3.$$

18. *Find the distance between the skew lines*

$$x = 1 + 2t, \quad y = -2 + 3t, \quad z = -4t; \qquad \text{and} \qquad \frac{x+4}{3} = \frac{y+5}{4} = \frac{z}{5}.$$

A point on the first line is when $t = 0$ or $P = (1, -2, 0)$. A point on the second line is when $z = 0$ so $Q = (-4, -5, 0)$. The direction of the first line is the time derivative of the parameter $V = (2, 3, -4)$. The direction of the second line is $W = (3, 4, 5)$ by symmetric form. You can see that if $X = (x, y, z)$ is on the line then so is $X + W$ since every term of the symmetric form is increased by one. The direction that is perpendicular to both lines is the cross product of tangents

$$N = V \times W = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 3 & 4 & 5 \end{vmatrix} = (15 + 16)\mathbf{i} + (-12 - 10)\mathbf{j} + (8 - 9)\mathbf{k} = 31\mathbf{i} - 22\mathbf{j} - \mathbf{k}.$$

A vector from the second line to the first is $P - Q = (5, 3, 0)$. The distance between the lines is the length of the projection of this vector to the normal line. Let the angle between N and $P - Q$ be $\theta = \angle(N, P - Q)$. Then

$$d = \| \mathbf{proj}_N(P - Q) \| = |\cos\theta| \, \|P - Q\| = \frac{|N \cdot (P - Q)|}{\|N\| \, \|P - Q\|} \|P - Q\|$$
$$= \frac{|N \cdot (P - Q)|}{\|N\|} = \frac{|31 \cdot 5 - 22 \cdot 3 - 1 \cdot 0|}{\sqrt{31^2 + 22^2 + 1}} = \boxed{\frac{89}{\sqrt{1446}} = 2.340.}$$

19. *Show that* $\mathbf{r}(t)$ *lies on a plane. Find the equation of the plane. Where does the tangent line at* $t = 2$ *intersect the* xy*-plane?*

$$\mathbf{r}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{j} + (1 - t^2)\,\mathbf{k}$$

The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2\mathbf{i} + 2t\,\mathbf{j} - 2t\,\mathbf{k}$$

At $t = 0$ this is $\mathbf{v}(0) = 2\,\mathbf{i}$. At $t = 2$ this is $\mathbf{v}(2) = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$. If the curve is to lie in a plane, then the normal vector should be a constant. Computing the cross product between two velocities we find a candidate for the normal vector.

$$N = \mathbf{v}(0) \times \mathbf{v}(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 2 & 4 & -4 \end{vmatrix} = (0 - 0)\mathbf{i} + (0 + 8)\mathbf{j} + (8 - 0)\mathbf{k} = 8\mathbf{j} + 8\mathbf{k}.$$

Let's check that the curve lies in the plane through $\mathbf{r}(0) = \mathbf{k}$ with normal N. Indeed

$$N \cdot (\mathbf{r}(t) - \mathbf{r}(0)) = 0 \cdot (2t - 0) + 8 \cdot (t^2 - 0) + 8(1 - t^2 - 1) = 0$$

for all t, so it is in this plane. By the formula $N \cdot (X - \mathbf{r}(0)) = 0$ we have the eqution of this plane

$$8y - 8z = N \cdot X = N \cdot \mathbf{r}(0) = 8.$$

Plugging $\mathbf{r}(t)$ into this equation we see again that the curve lies in the plane

$$8t^2 + 8(1 - t^2) = 8.$$

The tangent line at $t = 2$ passes through $\mathbf{r}(2) = (4, 4, -3)$ in the direction $\mathbf{v}(2)$. Thus the parametric equation of the tangent line is

$$x = 4 + 2t, \quad y = 4 + 4t, \quad z = -3 - 4t.$$

This line passes the xy-plane when $z = 0$ or when $t = -\frac{3}{4}$. At that time the xy coordinates are

$$x = 4 + 2\left(-\frac{3}{4}\right) = \frac{5}{2}, \qquad y = 4 + 4\left(-\frac{3}{4}\right) = 1.$$

20. *Find the unit tangent vector $T(t)$ and the curvature $\kappa(t)$ of the plane curve.*

$$x(t) = t\cos t, \qquad y(t) = t\sin t$$

The tangent vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$$

The speed is

$$\begin{aligned}
s(t) = \|\mathbf{v}(t)\| &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \\
&= \sqrt{\cos^2 t - 2t\cos t\,\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\,\sin t + t^2\cos^2 t} \\
&= \sqrt{1 + t^2}
\end{aligned}$$

Thus the unit tangent vector is

$$T(t) = \frac{1}{s(t)}\mathbf{v}(t) = \frac{\cos t - t\sin t}{\sqrt{1+t^2}}\mathbf{i} + \frac{\sin t + t\cos t}{\sqrt{1+t^2}}\mathbf{j}$$

Note that

$$x' = \cos t - t\sin t; \qquad x'' = -2\sin t - t\cos t; \qquad y' = \sin t + t\cos t; \qquad y'' = 2\cos t - t\sin t.$$

The curvature may be computed using the formula.

$$\begin{aligned}
\kappa &= \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}} \\
&= \frac{|(\cos t - t\sin t)(2\cos t - t\sin t) - (\sin t + t\cos t)(-2\sin t - t\cos t)|}{[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2]^{3/2}} \\
&= \frac{|(2\cos^2 t - 3t\cos t\,\sin t + t^2\sin^2 t) - (-2\sin^2 t - 3\cos t\,\sin t - t^2\cos^2 t)|}{\left[(\cos^2 t - 2t\cos t\,\sin t + t^2\sin^2 t) + (\sin^2 t + 2t\cos t\,\sin t + t^2\cos^2 t\right]^{3/2}} \\
&= \frac{2 + t^2}{[1 + t^2]^{3/2}}
\end{aligned}$$

21. *Find the curvature κ, the unit tangent vector* $\mathbf{T}$*, the unit normal vector* $\mathbf{N}$ *and the binormal vector* $\mathbf{B}$ *at t.*

$$\mathbf{r}(t) = e^{-2t}\mathbf{i} + e^{2t}\mathbf{j} + 2\sqrt{2}\,t\,\mathbf{k}$$

The velocity vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2e^{-2t}\mathbf{i} + 2e^{2t}\mathbf{j} + 2\sqrt{2}\,\mathbf{k}$$

so the speed is

$$s(t) = \|(t)\| = \sqrt{4e^{-4t} + 4e^{4t} + 8} = \sqrt{(2e^{-2t} + 2e^{2t})^2} = 2(e^{-2t} + e^{2t}) = 4\cosh 2t.$$

The derivative of speed is

$$s' = 4(-e^{-2t} + e^{2t}) = 8\sinh 2t.$$

The acceleration is

$$\mathbf{a} = \mathbf{v}' = 4e^{-2t}\mathbf{i} + 4e^{2t}\mathbf{j}.$$

We compute

$$\begin{aligned} -s'\mathbf{v} + s\mathbf{a} &= -4(-e^{-2t} + e^{2t})\left(-2e^{-2t}\mathbf{i} + 2e^{2t}\mathbf{j} + 2\sqrt{2}\,\mathbf{k}\right) + 2(e^{-2t} + e^{2t})\left(4e^{-2t}\mathbf{i} + 4e^{2t}\mathbf{j}\right) \\ &= 16\mathbf{i} + 16\mathbf{j} - 8\sqrt{2}\,(-e^{-2t} + e^{2t})\,\mathbf{k} \end{aligned}$$

Its norm is

$$\begin{aligned} \| - s'\mathbf{v} + s\mathbf{a}\|^2 &= \|16\mathbf{i} + 16\mathbf{j} - 8\sqrt{2}\,(-e^{-2t} + e^{2t})\,\mathbf{k}\|^2 \\ &= 265 + 256 + 128(-e^{-2t} + e^{2t})^2 \\ &= 128(2 + e^{-4t} + e^{4t}) = 128(e^{2t} + e^{-2t})^2 = 512\cosh^2 2t. \end{aligned}$$

Thus the unit tangent vector is

$$\mathbf{T} = \frac{1}{s}\mathbf{v} = -\frac{e^{-2t}}{e^{-2t} + e^{2t}}\mathbf{i} + \frac{e^{2t}}{e^{-2t} + e^{2t}}\mathbf{j} + \frac{\sqrt{2}}{e^{-2t} + e^{2t}}\,\mathbf{k}$$

The derivative of $\mathbf{T} = \frac{1}{s}\mathbf{v}$ is

$$\mathbf{T}' = -\frac{s'}{s^2}\mathbf{v} + \frac{1}{s}\mathbf{a} = \frac{-s'\mathbf{v} + s\mathbf{a}}{s^2}$$

so the normal vector is

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{-s'\mathbf{v} + s\mathbf{a}}{\| - s'\mathbf{v} + s\mathbf{a}\|} = \frac{16\mathbf{i} + 16\mathbf{j} - 8\sqrt{2}\,(-e^{-2t} + e^{2t})\,\mathbf{k}}{\|16\mathbf{i} + 16\mathbf{j} - 8\sqrt{2}\,(-e^{-2t} + e^{2t})\,\mathbf{k}\|} = \frac{\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - (-e^{-2t} + e^{2t})\,\mathbf{k}}{e^{2t} + e^{-2t}}$$

The binormal vector is

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{1}{(e^{2t} + e^{-2t})^2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^{-2t} & e^{2t} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & -(e^{2t} - e^{-2t}) \end{vmatrix} = \frac{[-e^{4t} - 1]\mathbf{i} + [3 - e^{-4t}]\mathbf{j} - \sqrt{2}(e^{2t} - e^{-2t})\mathbf{k}}{(e^{2t} + e^{-2t})^2}.$$

The curvature is

$$\kappa = \frac{1}{s}\|\mathbf{T}'\| = \frac{\| - s'\mathbf{v} + s\mathbf{a}\|}{s^3} = \frac{8\sqrt{2}\,(e^{2t} + e^{-2t})}{8(e^{2t} - e^{-2t})^3} = \frac{\sqrt{2}}{(e^{2t} - e^{-2t})^2}.$$

22. *Find the position and velocity of particle* $\mathbf{r}(t)$ *whose position and velocity at* $t = 0$ *are* $\mathbf{r}(0) = (2, 1, 5)$, $\mathbf{v}(0) = (1, 0, 0)$ *and whose acceleration at all times* $t > -1$ *is*

$$\mathbf{a}(t) = (t+1)^{3/2}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \cos(t)\,\mathbf{k}.$$

This curve satisfies $\mathbf{r}''(t) = \mathbf{a}(t)$. Thus we integrate once to get velocity

$$\begin{aligned}
\mathbf{r}'(t) - \mathbf{r}'(0) = \mathbf{v}(t) - \mathbf{v}(0) &= \int_0^t \Big[(t+1)^{3/2}\,\mathbf{i} + e^{-t}\,\mathbf{j} + \cos(t)\,\mathbf{k}\Big]\,dt \\
&= \left[\int_0^t (t+1)^{3/2}\,dt\right]\mathbf{i} + \left[\int_0^t e^{-t}\,dt\right]\mathbf{j} + \left[\int_0^t \cos(t)\,dt\right]\mathbf{k} \\
&= \left[\frac{2}{5}(t+1)^{5/2}\right]_0^t \mathbf{i} + \Big[-e^{-t}\Big]_0^t \mathbf{j} + \Big[\sin(t)\Big]_0^t \mathbf{k} \\
\mathbf{r}'(t) - \mathbf{i} &= \frac{2}{5}\Big[(t+1)^{5/2} - 1\Big]\mathbf{i} + \Big[1 - e^{-t}\Big]\mathbf{j} + \Big[\sin(t)\Big]\mathbf{k}
\end{aligned}$$

Integrating velocity we find

$$\begin{aligned}
\mathbf{r}(t) - \mathbf{r}(0) &= \int_0^t \left[\left(\frac{2}{5}(t+1)^{5/2} + \frac{3}{5}\right)\mathbf{i} + \left(1 - e^{-t}\right)\mathbf{j} + \sin(t)\,\mathbf{k}\right]dt \\
&= \left[\int_0^t \left(\frac{2}{5}(t+1)^{5/2} + \frac{3}{5}\right)dt\right]\mathbf{i} + \left[\int_0^t \left(1 - e^{-t}\right)dt\right]\mathbf{j} + \left[\int_0^t \sin(t)\,dt\right]\mathbf{k} \\
&= \left[\frac{4}{35}(t+1)^{7/2} + \frac{3}{5}t\right]_0^t \mathbf{i} + \Big[t + e^{-t}\Big]_0^t \mathbf{j} + \Big[-\cos t\Big]_0^t \mathbf{k} \\
\mathbf{r}(t) - \Big[2\mathbf{i} + \mathbf{j} + 5\mathbf{k}\Big] &= \left[\frac{4}{35}\Big((t+1)^{7/2} - 1\Big) + \frac{3}{5}t\right]\mathbf{i} + \Big[t + e^{-t} - 1\Big]\mathbf{j} + \Big[1 - \cos t\Big]\mathbf{k}
\end{aligned}$$

so for $t > -1$,

$$\mathbf{r}(t) = \left[\frac{4}{35}(t+1)^{7/2} + \frac{66}{35} + \frac{3}{5}t\right]\mathbf{i} + \Big[t + e^{-t}\Big]\mathbf{j} + \Big[6 - \cos t\Big]\mathbf{k}.$$

23. *Find the nomal and tangential components of acceleration for the curve*

$$x(t) = t\cos t, \qquad y(t) = t\sin t.$$

Computing the velocity we have

$$x'(t) = \cos t - t\sin t, \qquad y'(t) = \sin t + t\cos t.$$

so that the speed

$$\begin{aligned}
\frac{ds}{dt} &= \sqrt{(x')^2 + (y')^2} = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \\
&= \sqrt{\cos^2 t - 2t\cos t\,\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\,\sin t + t^2\cos^2 t} \\
&= \sqrt{1 + t^2}
\end{aligned}$$

The tangential acceleration is

$$a_T = \frac{d^2 s}{dt^2} = \frac{t}{\sqrt{1+t^2}}.$$

The accelerations are

$$x''(t) = -2\sin t - t\cos t, \qquad y''(t) = 2\cos t - t\sin t.$$

The curvature is

$$\begin{aligned}
\kappa &= \frac{|x'y'' - y'x''|}{\left((x')^2 + (y')^2\right)^{3/2}} \\
&= \frac{|(\cos t - t\sin t)(2\cos t - t\sin t) - (\sin t + t\cos t)(-2\sin t - t\cos t)|}{(1+t^2)^{3/2}} \\
&= \frac{\left|2\cos^2 t - 3t\cos t\,\sin t + t^2\sin^2 t + 2\sin^2 t + 3t\cos t\,\sin t + t^2\cos^2 t\right|}{(1+t^2)^{3/2}} \\
&= \frac{2+t^2}{(1+t^2)^{3/2}}.
\end{aligned}$$

Thus the normal acceleration is

$$a_n = \left(\frac{ds}{dt}\right)^2 \kappa = \left(\sqrt{1+t^2}\right)^2 \frac{2+t^2}{(1+t^2)^{3/2}} = \frac{2+t^2}{\sqrt{1+t^2}}$$

24. *Derive the polar coordinate curvature formula where the derivatives are with respect to θ.*

$$\kappa = \frac{|r^2 + 2\dot{r}^2 - r\ddot{r}|}{(r^2 + \dot{r}^2)^{3/2}}$$

For a curve in polar coordinates, the radius is a function of the central angle $r(\theta)$. Written in Cartesian coordinates

$$\mathbf{r}(t) = r(\theta)\cos\theta\,\mathbf{i} + r(\theta)\sin\theta\,\mathbf{j}.$$

One way to do the problem is to plug into the formula for curvature for plane curves, as in Problem 23. The other way is to derive the formula from the defnition of curvature, which we do here. The tangent vector is

$$\dot{\mathbf{r}}(t) = (\dot{r}\cos\theta - r\sin\theta)\mathbf{i} + (\dot{r}\sin\theta + r\cos\theta)\,\mathbf{j}.$$

Thus the speed is

$$\begin{aligned}
\frac{ds}{d\theta} &= \|\dot{\mathbf{r}}\| \\
&= \sqrt{(\dot{x})^2 + (\dot{y})^2} \\
&= \sqrt{(\dot{r}\cos\theta - r\sin\theta)^2 + (\dot{r}\sin\theta + r\cos\theta)^2} \\
&= \sqrt{\dot{r}^2\cos^2\theta - 2r\dot{r}\cos\theta\,\sin\theta + r^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2r\dot{r}\cos\theta\,\sin\theta + r^2\cos^2\theta} \\
&= \sqrt{\dot{r}^2 + r^2}
\end{aligned}$$

Thus the unit tangent vector is

$$\mathbf{T} = \frac{\dot{\mathbf{r}}}{\frac{ds}{d\theta}} = \frac{1}{\sqrt{r^2 + \dot{r}^2}}\left[(\dot{r}\cos\theta - r\sin\theta)\,\mathbf{i} + (\dot{r}\sin\theta + r\cos\theta)\,\mathbf{j}\right].$$

Its rate of change with respect to θ is using the product rule

$$\begin{aligned}
\dot{\mathbf{T}} &= -\frac{r\dot{r}+\dot{r}\ddot{r}}{\left(r^2+\dot{r}^2\right)^{3/2}}\left[\left(\dot{r}\cos\theta - r\sin\theta\right)\mathbf{i} + \left(\dot{r}\sin\theta + r\cos\theta\right)\mathbf{j}\right] + \\
&\quad + \frac{1}{\sqrt{r^2+\dot{r}^2}}\left[\left(\ddot{r}\cos\theta - 2\dot{r}\sin\theta - r\cos\theta\right)\mathbf{i} + \left(\ddot{r}\sin\theta + 2\dot{r}\cos\theta - r\sin\theta\right)\mathbf{j}\right] \\
&= \frac{1}{\left(r^2+\dot{r}^2\right)^{3/2}}\left[-\left(r\dot{r}+\dot{r}\ddot{r}\right)\left(\dot{r}\cos\theta - r\sin\theta\right)\mathbf{i} - \left(r\dot{r}+\dot{r}\ddot{r}\right)\left(\dot{r}\sin\theta + r\cos\theta\right)\mathbf{j}\right] + \\
&\quad + \frac{1}{\left(r^2+\dot{r}^2\right)^{3/2}}\left[\left(r^2+\dot{r}^2\right)\left(\ddot{r}\cos\theta - 2\dot{r}\sin\theta - r\cos\theta\right)\mathbf{i} + \left(r^2+\dot{r}^2\right)\left(\ddot{r}\sin\theta + 2\dot{r}\cos\theta - r\sin\theta\right)\mathbf{j}\right] \\
&= \frac{r\ddot{r} - 2\dot{r}^2 - r^2}{\left(r^2+\dot{r}^2\right)^{3/2}}\left[\left(r\cos\theta + \dot{r}\sin\theta\right)\mathbf{i} + \left(-\dot{r}\cos\theta + r\sin\theta\right)\mathbf{j}\right].
\end{aligned}$$

The curvature is thus

$$\begin{aligned}
\kappa &= \frac{1}{\frac{ds}{d\theta}}\|\dot{\mathbf{T}}\| \\
&= \frac{1}{\sqrt{r^2+\dot{r}^2}}\frac{\left|r\ddot{r} - 2\dot{r}^2 - r^2\right|}{\left(r^2+\dot{r}^2\right)^{3/2}}\sqrt{\left(r\cos\theta + \dot{r}\sin\theta\right)^2 + \left(-\dot{r}\cos\theta + r\sin\theta\right)^2} \\
&= \frac{\left|r\ddot{r} - 2\dot{r}^2 - r^2\right|}{\left(r^2+\dot{r}^2\right)^2}\sqrt{r^2\cos^2\theta + 2r\dot{r}\cos\theta\,\sin\theta + \dot{r}^2\sin^2\theta + \dot{r}^2\cos^2\theta - 2r\dot{r}\cos\theta\sin\theta + r^2\sin^2\theta} \\
&= \frac{\left|r\ddot{r} - 2\dot{r}^2 - r^2\right|}{\left(r^2+\dot{r}^2\right)^2}\sqrt{r^2+\dot{r}^2} \\
&= \frac{\left|r\ddot{r} - 2\dot{r}^2 - r^2\right|}{\left(r^2+\dot{r}^2\right)^{3/2}},
\end{aligned}$$

as promised.

25. *Name and sketch the graph of the following equation in three space.*

$$x^2 + y^2 - 4z^2 + 4 = 0.$$

Put the equation into standard form with one on the right side

$$-\frac{x^2}{4} - \frac{y^2}{4} + \frac{z^2}{1} = 1.$$

The trace in the xy-plane (where $z = 0$) is

$$-\frac{x^2}{4} - \frac{y^2}{4} = 1.$$

which is the empty set since nonpositive left side can't equal one on the right. The trace in the xz-plane (where $y = 0$) is

$$-\frac{x^2}{4} + \frac{z^2}{1} = 1.$$

which is hyperbola opening in the z-directions since there is no solution with $z = 0$. The trace in the xz-plane (where $y = 0$) is

$$-\frac{x^2}{4} - \frac{y^2}{4} = 1.$$

which is the empty set since nonpositive left side can't equal one on the right. The trace in the yz-plane (where $z = 0$) is

$$-\frac{y^2}{4} + \frac{z^2}{1} = 1.$$

which is hyperbola opening in the z-directions since there is no solution with $z = 0$. Thus the graph is a hyperbola of two sheets opening along the z-axis.

We plot it in the Grapher program in my Mac (Figure 5). Since points with $z = 0$ don't occur, we solve for z in the equation

$$z = \pm\sqrt{1 + \frac{x^2}{4} + \frac{y^2}{4}}$$

and superimpose the graphs with "+" and "−."

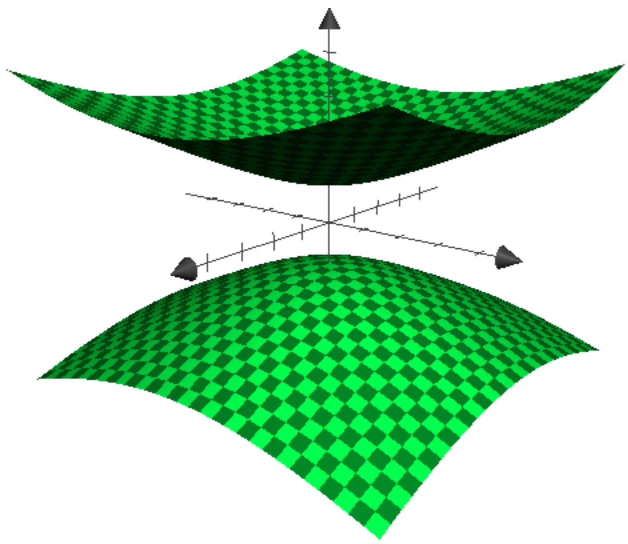

Figure 5: Problem 25.

26. *Find the equation of the surface that results when the given curve in the xy-plane is revolved about the x-axis,*

$$4x^2 - 3y^2 = 12$$

Revolving about the x axis means that the orbit is a circle in the $x =$ const. planes, where y^2, the squared radius in the y direction is replaced by the squared radius $r^2 = y^2 + z^2$ in the yz-direction. This results in the equation

$$4x^2 - 3(y^2 + z^2) = 12.$$

The standard form of this equation is

$$\frac{x^2}{3} - \frac{y^2}{4} - \frac{z^2}{4} = 1.$$

which is the equation of a hyperboloid of two sheets, opening in the $\pm x$-directions.

27. *Show that the projection in the xz-plane of the curve that is the intersection of of the given surfaces is an ellipse, and find its major and minor diameters.*

$$y = 4 - x^2, \qquad y = x^2 + z^2.$$

Eliminating the y variable from both equations (since on the curve of intersection the y's are the same) gives the desired curve. Thus equating y's yields the equation

$$4 - x^2 = x^2 + z^2$$

which in standard form is

$$\frac{x^2}{2} + \frac{z^2}{4} = 1,$$

the equation of an ellipse. Thus the minor and major radii are $a = \sqrt{2}$ and $b = 2$ so the diameters are $2\sqrt{2}$ and 4.

28. *Find the volume of solid bounded by the elliptical paraboloid and the xy-plane, where* $h > 0$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = h - z.$$

Use the method of slabs. Since the left side is nonnegative he have $0 \le h - z$ so $0 \le z \le h$. The intersection with the $z = z_0$ plane is the outer bounding curve

$$\frac{x^2}{(h - z_0)a^2} + \frac{y^2}{(h - z_0)b^2} = 1$$

which is an ellipse with minor and major radii $a\sqrt{h - z_0}$ and $b\sqrt{h - z_0}$. The area of such an ellipse is pi times the product of radii, or

$$A(z_0) = \pi ab(h - z_0).$$

Thus the volume of the solid is the integral of area times thickness of stacked slabs, or

$$V = \int_0^h A(z)\,dz = \int_0^h \pi ab(h - z)\,dz = \pi ab\left[hz - \frac{z^2}{2}\right]_0^h = \boxed{\frac{\pi abh^2}{2}}.$$