Math 2210 § 4. **Final Exam** Name: Practice Problems
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Half of the final exam will be comprehensive. The other half will focus on the material covered after the second midterm exam: multiple integrals and vector calculus. Here are some problems from the latter part of the course.

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1. *Let $R = \{(x, y) : 1 \le x \le 3,\ 4 \le y \le 5\}$. Approximate the integral $I = \iint_R \sqrt{x+y},\ dA$ by calculating the corresponding Riemann Sum using the partition of R into eight equal sized squares. Assume that the sample points (x_k, y_k) are the centers of the squares.*

   The squares are cut by the lines $y = 4$, $y = 4.5$, $y = 5$, $x = 1$, $x = 1.5$, $x = 2$, $x = 2.5$ and $x = 3$. Thus the first 0.5×0.5 square has center $(x_1, y_1) = (1.25, 4.25)$. Here is a little table of centers and function values at the centers generated by a program in ©**R**. We also compute their sum.

```
k     x_k   y_k    f(x_k,y_k)
--    ----  ----   ----------
 1    1.25  4.25      2.34521
 2    1.25  4.75      2.44949
 3    1.75  4.25      2.44949
 4    1.75  4.75      2.54951
 5    2.25  4.25      2.54951
 6    2.25  4.75      2.64575
 7    2.75  4.25      2.64575
 8    2.75  4.75      2.73861
                      -------
Total                20.37332
```

   Using $\Delta A_k = 0.5 \times 0.5 = .25$, the Riemann Sum is thus

$$I \approx \sum_{k=1}^{8} f(x_k, y_k)\,\Delta A_k = .25 \sum_{k=1}^{8} f(x_k, y_k) = .25 \cdot 20.37332 = 5.093331.$$

2. *Find the actual value of the integral I from Problem 1.*

   Using iterated integrals we have

$$\begin{aligned}
T &= \int_{x=1}^{3} \int_{y=4}^{5} (x+y)^{\frac{1}{2}}\,dy\,dx \\
&= \int_{x=1}^{3} \left[\frac{2}{3}(x+y)^{\frac{3}{2}}\right]_{y=4}^{5} dx \\
&= \frac{2}{3}\int_{x=1}^{3} (x+5)^{\frac{3}{2}} - (x+4)^{\frac{3}{2}}\,dx \\
&= \frac{2}{3}\left[\frac{2}{5}(x+5)^{\frac{5}{2}} - \frac{2}{5}(x+4)^{\frac{5}{2}}\right]_{x=1}^{3} \\
&= \frac{4}{15}\left[8^{\frac{5}{2}} - 6^{\frac{5}{2}} - 7^{\frac{5}{2}} + 5^{\frac{5}{2}}\right] \approx 5.092691.
\end{aligned}$$

   Thus the approximation in Problem 1 was fairly close.

3. *Find the volume of the solid in the first octant bounded by the coordinate planes and the planes $2x + y - 4 = 0$ and $8x + y - 4z = 0$.*

The lower surface is $z = 0$ and the upper surface from the last plane is

$$z = 2x + \frac{1}{4}y$$

which is positive in the first octant. The solid is bounded on the sides by the coordinate planes $x = 0$, $y = 0$ and second to last equation $y = 4 - 2x$ making the base of the solid a triangle. In $y = 4 - 2x$ we have $y = 0$ when $x = 2$ so $0 \le x \le 2$. Thus the volume is given by the iterated integral

$$\begin{aligned} V &= \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{2x+\frac{1}{4}y} dz\, dy\, dx \\ &= \int_{x=0}^{2} \int_{y=0}^{4-2x} 2x + \frac{1}{4}y\, dy\, dx \\ &= \int_{x=0}^{2} \left[2xy + \frac{1}{8}y^2 \right]_{y=0}^{4-2x} dx \\ &= \int_{x=0}^{2} 2x(4-2x) + \frac{1}{8}(4-2x)^2\, dx \\ &= \int_{x=0}^{2} [8x - 4x^2] + \left[2 - 2x + \frac{1}{2}x^2 \right] dx \\ &= \int_{x=0}^{2} 2 + 6x - \frac{7}{2}x^2\, dx \\ &= \left[2x + 3x^2 - \frac{7}{6}x^3 \right]_{x=0}^{2} \\ &= 4 + 12 - \frac{28}{3} = \frac{20}{3}. \end{aligned}$$

4. *Evaluate by using polar coordinates. Sketch the region of integration first.*

$$I = \int_{x=1}^{2} \int_{y=0}^{\sqrt{2x-x^2}} \left(x^2 + y^2\right)^{-\frac{1}{2}} dy\, dx.$$

The upper curve $y = \sqrt{2x - x^2}$ may be rewritten

$$y^2 = 2x - x^2$$

or

$$(x-1)^2 + y^2 = x^2 - 2x + 1 + y^2 = 1$$

which is a unit circle of radius one centered at the point $(1, 0)$. Thus the region of integration R is the quarter circle $1 \le x \le 2$ and $0 \le y \le \sqrt{2x - x^2}$, given in Figure 1.

Thus for polar angle θ when $x = r\cos\theta$ and $y = r\sin\theta$ satisfy the equation

$$1 = (r\cos\theta - 1)^2 + r^2\sin^2\theta = r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = r^2 - 2r\cos\theta + 1$$

so $r = 0$ or

$$r = 2\cos\theta.$$

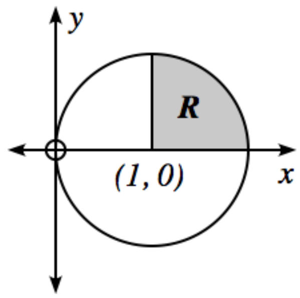

Figure 1: Region of integration R.

The curve $1 = x = r\cos\theta$ becomes

$$r = \frac{1}{\cos\theta} = \sec\theta.$$

The circle and $x = 1$ intersect at the point $(1, 1)$ where $\theta = \frac{\pi}{4}$. Thus as $0 \le \theta \le \frac{\pi}{4}$ we have $\sec\theta \le r \le 2\cos\theta$. Finally $x^2 + y^2 = r^2$. Thus the integral in polar coordinates is

$$\begin{aligned}
I &= \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=\sec\theta}^{2\cos\theta} \frac{1}{r} \cdot r\, dr\, d\theta \\
&= \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=\sec\theta}^{2\cos\theta} dr\, d\theta \\
&= \int_{\theta=0}^{\frac{\pi}{4}} 2\cos\theta - \sec\theta\, d\theta \\
&= \Big[2\sin\theta - \log|\sec\theta + \tan\theta|\Big]_{\theta=0}^{\frac{\pi}{4}} \\
&= \sqrt{2} - \log(\sqrt{2} + 1) \approx 0.53284.
\end{aligned}$$

5. *Find the mass and center of mass of the lamina of constant density k bounded by the cardoid $r = a(1 + \sin\theta)$ that is outside the circle $r = a$.*

Figure 2 gives the plot using my Macintosh's Grapher.

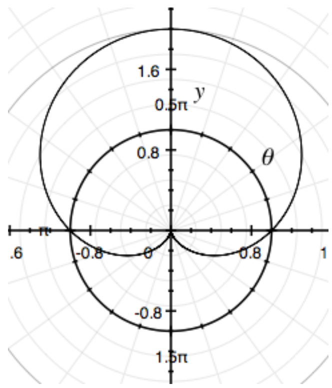

Figure 2: Cardoid and circle with $a = 1$.

The lamina satisfies $0 \le \theta \le \pi$ and $a \le r \le a(1 + \sin\theta)$. The mass is

$$\begin{aligned}
M &= \int_{\theta=0}^{\pi} \int_{r=a}^{a(1+\sin\theta)} k\, r\, dr\, d\theta \\
&= \int_{\theta=0}^{\pi} \left[\frac{k}{2} r^2\right]_{r=a}^{a(1+\sin\theta)} d\theta \\
&= \frac{k}{2} \int_{\theta=0}^{\pi} a^2(1+\sin\theta)^2 - a^2\, d\theta \\
&= \frac{a^2 k}{2} \int_{\theta=0}^{\pi} 2\sin\theta + \sin^2\theta\, d\theta \\
&= \frac{a^2 k}{2} \left(\Big[-2\cos\theta\Big]_{\theta=0}^{\pi} + \frac{\pi}{2} \right) \\
&= \frac{a^2 k}{2} \left(-2(-1) - (-2)(1) + \frac{\pi}{2} \right) \\
&= \left(2 + \frac{\pi}{4}\right) a^2 k.
\end{aligned}$$

The lamina is symmetric left/right, so it balances on the y-axis or $\bar{x} = 0$. To compute its moment about the x-axis, using the integral formula 113 on the back endpaper of the text,

$$\begin{aligned}
M_x &= \iint_R y\delta\, dA \\
&= \int_{\theta=0}^{\pi} \int_{r=a}^{a(1+\sin\theta)} kr\sin\theta\, r\, dr\, d\theta \\
&= k\int_{\theta=0}^{\pi} \left[\frac{1}{3}r^3\sin\theta\right]_{r=a}^{a(1+\sin\theta)} d\theta \\
&= \frac{k}{3}\int_{\theta=0}^{\pi} \left[a^3(1+\sin\theta)^3 - a^3\right]\sin\theta\, d\theta \\
&= \frac{ka^3}{3}\int_{\theta=0}^{\pi} 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta\, d\theta \\
&= \frac{ka^3}{3}\left(\frac{3\pi}{2} + 4 + \frac{3\pi}{8}\right) \\
&= \left(\frac{5\pi}{8} + \frac{4}{3}\right) ka^3.
\end{aligned}$$

It follows that the y coordinate of the center of mass is

$$\bar{y} = \frac{M_x}{M} = \frac{\left(\frac{5\pi}{8} + \frac{4}{3}\right) ka^3}{\left(2 + \frac{\pi}{4}\right) a^2 k} = \left(\frac{15\pi + 32}{6\pi + 48}\right) a \approx (1.183611)a.$$

6. *Find the area of the surface of the hemisphere* $x^2 + y^2 + z^2 = a^2$ *and* $z \ge 0$ *inside the elliptic cylinder* $b^2x^2 + a^2y^2 = a^2b^2$ *where* $0 < b \le a$.

Note that the ellipse $\mathcal{E}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has major and minor radii a and b so it lies within the equator of the sphere. Thus $-a \le x \le a$ and $-\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$. The upper surface is given by

$$z = f(x, y) = \left(a^2 - x^2 - y^2\right)^{\frac{1}{2}}$$

so

$$f_x = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \qquad f_y = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

and

$$1 + f_x^2 + f_y^2 = \frac{a^2 - x^2 - y^2}{a^2 - x^2 - y^2} + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} = \frac{a^2}{a^2 - x^2 - y^2}.$$

The area is thus

$$\begin{aligned}
A &= \iint_{\mathcal{E}} \sqrt{1 + f_x^2 + f_y^2}\, dA \\
&= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx
\end{aligned}$$

Substituting $y = \sqrt{a^2 - x^2}\sin u$ we get $dy = \sqrt{a^2 - x^2}\cos u\, du$ and $-\theta_0 \le u \le \theta_0$ where $\sin\theta_0 = \dfrac{b}{a}$. We get

$$
\begin{aligned}
A &= \int_{-a}^{a}\int_{-\theta_0}^{\theta_0} \frac{a\sqrt{a^2 - x^2}\ \cos u\, du}{\sqrt{a^2 - x^2}\sqrt{1 - \sin^2 u}}\, dx \\
&= a\int_{-a}^{a}\int_{-\theta_0}^{\theta_0} du\, dx \\
&= a\int_{-a}^{a} 2\theta_0\, dx = 4a^2\theta_0 = 4a^2\sin^{-1}\left(\frac{b}{a}\right).
\end{aligned}
$$

7. *Consider the part of the sphere $x^2 + y^2 + z^2 = a^2$ between the planes $z = h_1$ and $z = h_2$, where $0 \le h_1 < h_2 \le a$. Find the value h such that the plane $z = h$ cuts the surface area in half.*

   Answer: $h = \dfrac{h_1 + h_2}{2}$. This is because the area of the spherical region is proportional to the height. In fact it equals the area of the corresponding cylinder whose base is the equator of the sphere, namely

$$A = 2\pi a(h_2 - h_1)$$

   so that above the $z = h$ plane the area is

$$A_+ = 2\pi a(h_2 - h) = 2\pi a\left(h_2 - \frac{h_1 + h_2}{2}\right) = \pi a(h_2 - h_1) = \frac{1}{2}A.$$

8. *Find the center of mass of a solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 6$ if the density $\delta(x, y, z) = k(x^2 + y^2 + z^2)$.*

   It is natural to use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$ and z. Thus $\delta = k(r^2 + z^2)$ and the solid is $0 \le r \le 2$, $0 \le \theta \le 2\pi$ and $0 \le z \le 6$. The mass is

$$
\begin{aligned}
M &= \iiint_S \delta\, dV \\
&= \int_{z=0}^{6}\int_{\theta=0}^{2\pi}\int_{r=0}^{2} k(r^2 + z^2)\, r\, dr\, d\theta\, dz \\
&= k\int_{z=0}^{6}\int_{\theta=0}^{2\pi} \left[\frac{r^4}{4} + \frac{r^2 z^2}{2}\right]_{r=0}^{2} d\theta\, dz \\
&= k\int_{z=0}^{6}\int_{\theta=0}^{2\pi} 4 + 2z^2\, d\theta\, dz \\
&= 2\pi k\int_{z=0}^{6} 4 + 2z^2\, dz \\
&= 2\pi k\left(4 \cdot 6 + \frac{2}{3}\cdot 6^3\right) = 336\pi k.
\end{aligned}
$$

   The solid and density are rotationally symmetric about the z-axis so the center of mass is on the z-axis: $\bar{x} = \bar{y} = 0$. It remains to compute the moment about the xy-plane.

$$\begin{aligned}
M_{xy} &= \iiint_S \delta\, z\, dV \\
&= \int_{z=0}^{6} \int_{\theta=0}^{2\pi} \int_{r=0}^{2} k(r^2+z^2) z\, r\, dr\, d\theta\, dz \\
&= k \int_{z=0}^{6} \int_{\theta=0}^{2\pi} \left[\frac{r^4 z}{4} + \frac{r^2 z^3}{2} \right]_{r=0}^{2} d\theta\, dz \\
&= k \int_{z=0}^{6} \int_{\theta=0}^{2\pi} 4z + 2z^3\, d\theta\, dz \\
&= 2\pi k \int_{z=0}^{6} 4z + 2z^3\, dz \\
&= 2\pi k \left(2 \cdot 6^2 + \frac{1}{2} \cdot 6^4 \right) = 1440\pi k.
\end{aligned}$$

The z coordinate of the center of mass is

$$\bar{z} = \frac{M_{xy}}{M} = \frac{1440\pi k}{336\pi k} = \frac{30}{7} \approx 4.285714.$$

9. *Use spherical coordinates to find the the quantity*

$$I = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2-z^2}} \left(x^2+y^2+z^2\right)^{3/2}\, dy\, dz\, dx$$

The domain of integration is a sphere of radius $\rho = 3$ and $dV = dy\, dz\, dz$. Note that here the y-integration is taken on the inside but since the integrand and domain only depend only on the radius, the integration can be done in any order. Using $\rho^2 = x^2 + y^2 + z^2$ and the usual spherical coordinates,

$$\begin{aligned}
I &= \int_{\rho=0}^{3} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \rho^3 \cdot \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho \\
&= \int_{\rho=0}^{3} \int_{\theta=0}^{2\pi} \Big[-\rho^5 \cos\phi \Big]_{\phi=0}^{\pi} d\theta\, d\rho \\
&= \int_{\rho=0}^{3} \int_{\theta=0}^{2\pi} \rho^5 [-(-1) + (1)]\, d\theta\, d\rho \\
&= 2 \int_{\rho=0}^{3} \int_{\theta=0}^{2\pi} \rho^5\, d\theta\, d\rho \\
&= 4\pi \int_{\rho=0}^{3} \rho^5\, d\rho = 4\pi \cdot \frac{3^6}{6} = 486\pi.
\end{aligned}$$

10. *Let the region R in the first quadrant be bounded by the circles $x^2+y^2=2x$, $x^2+y^2=6x$ and the circles $x^2+y^2=2y$ and $x^2+y^2=8y$. Use a transformation to evaluate*

$$I=\iint_R \frac{1}{(x^2+y^2)^2}\,dx\,dy.$$

Rewritten, the bounding circle equations are $(x-1)^2+y^2=1$, $(x-3)^2+y^2=9$, $x^2+(y-1)^2=1$ and $x^2+(y-4)^2=16$, which are circles throught the origin whose centers are $(1,0)$, $(3,0)$, $(0,1)$ and $(0,4)$, resp. We sketch the region in Figure 3.

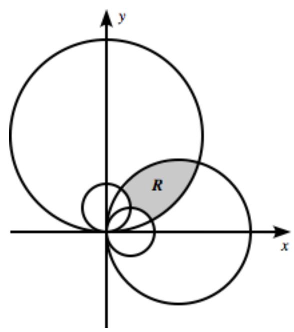

Figure 3: Region bounded by circles.

Making the change of variables

$$u=\frac{2x}{x^2+y^2},\qquad v=\frac{2y}{x^2+y^2}$$

we see that the corresponding region in the (u,v) plane is a rectangle T bounded by $\frac{1}{3}\le u\le 1$ and $\frac{1}{4}\le v\le 1$. Noting that

$$u^2+v^2=\frac{4(x^2+y^2)}{(x^2+y^2)^2}=\frac{4}{x^2+y^2}$$

we see that

$$u=\frac{1}{2}(u^2+v^2)x,\qquad v=\frac{1}{2}(u^2+v^2)y$$

so the transformation $G(u,v)=(x(u,v),y(u,v))$ is

$$x=\frac{2u}{u^2+v^2},\qquad y=\frac{2v}{u^2+v^2}.$$

It maps $G(T)=R$. Also, the integrand transforms to

$$f(x(u,v),y(u,v))=\frac{1}{[x(u,v)^2+y(u,v)^2]^2}=\frac{1}{16}(u^2+v^2)^2.$$

Computing the partial derivatives

$$\frac{\partial x}{\partial u}=\frac{2(v^2-u^2)}{(u^2+v^2)^2},\qquad \frac{\partial x}{\partial v}=-\frac{4uv}{(u^2+v^2)^2},\qquad \frac{\partial y}{\partial u}=-\frac{4uv}{(u^2+v^2)^2},\qquad \frac{\partial y}{\partial v}=\frac{2(u^2-v^2)}{(u^2+v^2)^2}.$$

The Jacobian determinant

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{2(v^2-u^2)}{(u^2+v^2)^2} & -\dfrac{4uv}{(u^2+v^2)^2} \\ -\dfrac{4uv}{(u^2+v^2)^2} & \dfrac{2(u^2-v^2)}{(u^2+v^2)^2} \end{vmatrix} = \frac{-4(u^2-v^2)^2 - 16u^2v^2}{(u^2+v^2)^4}$$

$$= \frac{-4u^4 + 8y^2v^2 - 4v^4 - 16u^2v^2}{(u^2+v^2)^4} = \frac{-4u^4 - 8y^2v^2 - 4v^4}{(u^2+v^2)^4} = \frac{-4(u^2+v^2)^2}{(u^2+v^2)^4} = \frac{-4}{(u^2+v^2)^2}.$$

Finally, transforming the integral

$$\begin{aligned} I &= \iint_{G(T)} f(x,y)\,dx\,dy \\ &= \iint_T f(x(u,v), y(u,v))\,|J(u,v)|\,du\,dv \\ &= \int_{u=\frac{1}{3}}^{1} \int_{v=\frac{1}{4}}^{1} \frac{1}{16}(u^2+v^2)^2 \cdot \frac{4}{(u^2+v^2)^2}\,dv\,du \\ &= \frac{1}{4}\int_{u=\frac{1}{3}}^{1} \int_{v=\frac{1}{4}}^{1} dv\,du = \frac{1}{4}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) = \frac{1}{8}. \end{aligned}$$

11. *Suppose* $c \neq 0$ *and* $m \neq 3$. *Show that* $\operatorname{div} \mathbf{F} \neq 0$ *but* $\operatorname{curl} \mathbf{F} = \mathbf{0}$, *where*

$$\mathbf{F}(x,y,z) = \frac{c\mathbf{r}}{\|\mathbf{r}\|^m}, \qquad \mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We have

$$\operatorname{div} \mathbf{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

and

$$\operatorname{curl} \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Let $f = c/\|\mathbf{r}\|^m$. We have

$$\begin{aligned} \nabla f &= \nabla\left(\frac{c}{\|\mathbf{r}\|^m}\right) \\ &= \nabla c\left(x^2+y^2+z^2\right)^{-\frac{m}{2}} \\ &= -\frac{cm}{2}\left(x^2+y^2+z^2\right)^{-\frac{m}{2}-1}(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \\ &= -\frac{cm\,\mathbf{r}}{\|\mathbf{r}\|^{m+2}} \end{aligned}$$

Using the formula $\operatorname{div}(f\mathbf{r}) = \nabla f \cdot \mathbf{r} + f \operatorname{div} \mathbf{r}$ with $m \neq 3$ and $c \neq 0$ we find that

$$\begin{aligned}
\operatorname{div} \mathbf{F} &= \operatorname{div}(f\mathbf{r}) \\
&= \nabla f \cdot \mathbf{r} + f \operatorname{div} \mathbf{r} \\
&= -\frac{cm\,\mathbf{r}}{\|\mathbf{r}\|^{m+2}} \cdot \mathbf{r} + \frac{3c}{\|\mathbf{r}\|^m} \\
&= -\frac{cm\,\|r\|^2}{\|\mathbf{r}\|^{m+2}} + \frac{3c}{\|\mathbf{r}\|^m} \\
&= \frac{(3-m)c}{\|\mathbf{r}\|^m} \neq 0
\end{aligned}$$

Using the formulae $\operatorname{curl}(f\mathbf{r}) = \nabla f \times \mathbf{r} + f \operatorname{curl} \mathbf{r}$ and $\mathbf{r} \times \mathbf{r} = \mathbf{0}$ we find that

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &= \operatorname{curl}(f\mathbf{r}) \\
&= \nabla f \times \mathbf{r} + f \operatorname{curl} \mathbf{r} \\
&= -\frac{cm\,\mathbf{r}}{\|\mathbf{r}\|^{m+2}} \times \mathbf{r} + \mathbf{0} = \mathbf{0}.
\end{aligned}$$

12. *Find the line integral I where C is the curve $x = 12\cos t$, $y = 12\sin t$, $z = 5t$, $0 \leq t \leq \pi$.*

$$I = \int_C (x^2 + y^2 + x^2)\, ds.$$

The element of arclength is

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = \sqrt{(-12\sin t)^2 + (12\cos t)^2 + 5^2}\, dt = \sqrt{12^2 + 5^2}\, dt = 13\, dt.$$

Thus

$$\begin{aligned}
I &= \int_0^\pi \left((12\cos t)^2 + (12\sin t)^2 + (5t)^2\right) 13\, dt \\
&= 13 \int_0^\pi \left(144 + 25t^2\right) dt \\
&= 13\left(144\pi + \frac{25}{3}\pi^3\right).
\end{aligned}$$

13. *Find the line integral J where C is the curve $x = e^t$, $y = e^{-t}$, $z = e^{2t}$, $0 \leq t \leq 1$.*

$$I = \int_C xz\, dx + (y+z)\, dy + x\, dz$$

Using $dx = \dot{x}\, dt = e^t\, dt$, *etc.*,

$$\begin{aligned}
I &= \int_0^1 e^t \cdot e^{2t} \cdot e^t\, dt + \left(e^{-t} + e^{2t}\right) \cdot (-e^{-t})\, dt + e^t \cdot 2e^{2t}\, dt \\
&= \int_0^1 e^{4t} - e^{-2t} - e^t + 2e^{3t}\, dt \\
&= \left[\frac{1}{4}e^{4t} - \frac{1}{2}e^{-2t} - e^t + \frac{2}{3}e^{3t}\right]_0^1 \\
&= \frac{1}{4}(e^4 - 1) - \frac{1}{2}(e^{-2} - 1) - e + 1 + \frac{2}{3}(e^3 - 1) \\
&= \frac{1}{4}e^4 - \frac{1}{2}e^{-2} - e + \frac{2}{3}e^3 + \frac{7}{12}.
\end{aligned}$$

14. *Find the work done by the force field* $\mathbf{F}(x, y, z) = (2x - y)\mathbf{i} + 2z\mathbf{j} + (y - z)\mathbf{k}$ *in moving the particle along a curve the line segment from* $(0, 0, 0)$ *to* $(1, 2, 3)$.

    The line segment C is given by $\gamma(t) = (x(t), y(t), z(t))$ where

    $$x = t, \qquad y = 2t, \qquad z = 3t, \qquad 0 \le t \le 1.$$

    The force field along the curve is

    $$\begin{aligned}\mathbf{F}(t) &= \mathbf{F}(x(t), y(t), z(t))\\ &= (2x(t) - y(t))\mathbf{i} + 2z(t)\mathbf{j} + (y(t) - z(t))\mathbf{k}\\ &= (2t - 2t)\mathbf{i} + 2 \cdot 3t\mathbf{j} + (2t - 3t)\mathbf{k}\\ &= 6t\mathbf{j} - t\mathbf{k}\end{aligned}$$

    The work is given

    $$\begin{aligned}W &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds\\ &= \int_C \mathbf{F} \cdot \frac{\dot\gamma}{\|\dot\gamma\|} \cdot \|\dot\gamma\|\, dt\\ &= \int_C \mathbf{F} \cdot \dot\gamma\, dt\\ &= \int_0^1 (6t\mathbf{j} - t\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})\, dt\\ &= \int_0^1 0 + 12t - 3t\, dt = \int_0^1 9t\, dt = \frac{9}{2}.\end{aligned}$$

15. *Use a line integral to compute the area of that part of the cylinder* $x^2 + y^2 = ay$ *inside the sphere* $x^2 + y^2 + z^2 = a^2$. *Hint: use polar coordinates.*

    If $x = r\cos\theta$ and $y = r\sin\theta$ then the cylinder equation is a curve C in the xy-plane given by

    $$r^2 = r^2\cos^2\theta + r^2\sin^2\theta = ar\sin\theta$$

    whose solution is $r = 0$ or $r = a\sin\theta$. The vertical distance is determined by the sphere. The half height from $z = 0$ to the sphere is

    $$z^2 = a^2 - x^2 - y^2 = a^2 - r^2\cos^2\theta - r^2\sin^2\theta = a^2 - r^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$$

    so $z = a|\cos\theta|$. We use absolute value since z is positive on both the $0 \le \theta \le \frac{\pi}{2}$ side and on the $\frac{\pi}{2} \le \theta \le \pi$ side. The arclength in polar coordinates is

    $$\begin{aligned}ds^2 &= dx^2 + dy^2\\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2\\ &= (\cos^2\theta + \sin^2\theta)\, dr^2 + (-r\cos\theta\sin\theta + r\sin\theta\cos\theta)\, dr\, d\theta + r^2(\sin^2\theta + \cos^2\theta)\, d\theta^2\\ &= dr^2 + r^2\, d\theta^2.\end{aligned}$$

    But $r = a\sin\theta$ so that

    $$dr = a\cos\theta\, d\theta$$

    and

    $$ds = \sqrt{dr^2 + r^2\, d\theta^2} = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}\, d\theta = a\, d\theta$$

The integral giving the area of the cylinder within the sphere is

$$\begin{aligned}
A &= \int_C 2z\,ds \\
&= \int_{\theta=0}^{\pi} 2a|\cos\theta| \cdot a d\theta \\
&= 4a^2 \int_{\theta=0}^{\frac{\pi}{2}} \cos\theta\,d\theta \\
&= 4a^2 \Big[\sin\theta\Big]_0^{\frac{\pi}{2}} = 4a^2.
\end{aligned}$$

16. *Determine whether the given vector field* $\mathbf{F}(x,y,z) = (2xy+z^2)\mathbf{i} + x^2\mathbf{j} + (2xz + \pi\cos\pi z)\mathbf{k}$ *is conservative. If so, find* f *so that* $\nabla f = \mathbf{F}$.

The field is conservative in $\mathbf{R}^3$ (which is simply connected) if it is continuously differentiable (it is because it consists of nice functions) and curl free. Computing the curl we fine

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy+z^2) & x^2 & 2xz + \pi\cos\pi z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(2xz+\pi\cos\pi z) - \frac{\partial}{\partial z}x^2\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(2xy+z^2) - \frac{\partial}{\partial x}(2xz+\pi\cos\pi z)\right)\mathbf{j} \\
&\quad + \left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy+z^2)\right)\mathbf{k} \\
&= 0\mathbf{i} + (2z-2z)\mathbf{j} + (2x-2x)\mathbf{k} = \mathbf{0}.
\end{aligned}$$

Thus $\mathbf{F}$ is curl free. The gradient satisfies

$$\nabla f = f_z\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} = (2xy+z^2)\mathbf{i} + x^2\mathbf{j} + (2xz+\pi\cos\pi z)\mathbf{k}$$

so

$$\begin{aligned}
f_x &= 2xy + z^2 \\
f_y &= x^2 \\
f_z &= 2xz + \pi\cos\pi z
\end{aligned}$$

Antidifferentiating the first equation with respect to x says

$$f(x,y,z) = x^2y + xz^2 + C(y,z)$$

where the "constant" of integration may depend on y and z. From the second gradient equation, the partial derivative of this with respect to y says

$$f_y = x^2 + C_y(y,z) = x^2$$

so $C_y = 0$ and $C(y,z) = D(z)$ which is a "constant" that may depend on z. Differentiating with respect to z and using the third gradient equation

$$f_z = 2xz + D_z = 2xz + \pi\cos\pi z$$

so that $D_z(z) = \pi \cos \pi z$ so that

$$D(z) = \sin \pi z + k$$

where k is constant. Finally

$$f(x, y, z) = x^2 y + xz^2 + \sin \pi z + k.$$

We check that its gradient is $\nabla f = \mathbf{F}$ as desired.

17. *Show that the integral is independent of path. Evaluate either by choosing a convenient path or by using a potential function.*

$$I = \int_{(0,0,0)}^{(1,1,1)} (6xy^3 + 2z^2)\, dx + 9x^2y^2\, dy + (4xz + 1)\, dz$$

If a potential function can be found, then the integral is independent of path. It satisfies

$$\nabla f = f_x\, dx + f_y\, dy + f_z\, dz = (6xy^3 + 2z^2)\, dx + 9x^2y^2\, dy + (4xz + 1)\, dz.$$

The first equation is

$$f_x = 6xy^3 + 2z^2.$$

Integrating with respect to x yields

$$f = 3x^2 + 2xz^2 + C(y, z).$$

Thus

$$f_y = C_y = 9x^2y^2$$

so $C(y, z) = 3x^2y^3 + D(z)$. Thus

$$f_z = 4xz + D_z = 4xz + 1$$

so $D_z = 1$ which means $D(z) = z + k$. Thus we have found our potential function.

$$f = 3x^2 + 2xz^2 + 3x^2y^3 + z + k$$

Note that this procedure would have failed if the integral was not independent of path. It follows that

$$I = f(1, 1, 1) - f(0, 0, 0) = (3 + 2 + 3 + k) - (k) = 8.$$

18. *Use Green's Theorem to evaluate the line integral over the curve C which is the rectangle with vertices* $(2, 1)$, $(6, 1)$, $(6, 4)$ *and* $(2, 4)$.

$$I = \oint_C (e^{3x} + 2y)\, dx + (x^2 + \sin y)\, dy$$

Green's Theorem is

$$\oint_C M\, dx + N\, dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

where S is th region bounded by C. We have $M(x, y) = e^{3x} + 2y$ and $N(x, y) = x^2 + \sin y$ so that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2$$

so

$$
\begin{aligned}
I &= \int_{x=2}^{6} \int_{y=1}^{4} 2x - 2\,dy\,dx \\
&= \int_{x=2}^{6} \Big[2xy - 2y\Big]_{y=1}^{4} dx \\
&= \int_{x=2}^{6} \Big[(8x - 2x) - 2(4-1)\Big]\,dx \\
&= \int_{x=2}^{6} 6x - 6\,dx \\
&= \Big[3x^2 - 6x\Big]_{x=2}^{6} = 3(6^2 - 2^2) - 6(6-2) = 72.
\end{aligned}
$$

19. *For* $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$*, use Green's Theorem to calculate the flux across* C *and the circulation along* C *where* C *is the boundary of the unit square with vertices* $(0,0)$*,* $(1,0)$*,* $(1,1)$*,* $(0,1)$*.*

Using the Divergence Theorem, the flux is

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_S \operatorname{div}\mathbf{F}\,dA = \int_{x=0}^{1}\int_{y=0}^{1} \frac{\partial y^2}{\partial x} + \frac{\partial x^2}{\partial y}\,dy\,dx = 0.$$

Here $M = y^2$ and $N = x^2$. Using Green's Theorem, the circulation is

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{T}\,ds &= \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA = \int_{x=0}^{1}\int_{y=0}^{1} \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\,dy\,dx \\
&= \int_{x=0}^{1}\int_{y=0}^{1} 2x - 2y\,dy\,dx = \int_{x=0}^{1} 2x - 1\,dx = 1 - 1 = 0.
\end{aligned}
$$

20. *Evaluate the integral over the surface* S *given by* $z = x^2 - y^2$*,* $0 \le x^2 + y^2 \le 1$*.*

$$I = \iint_S 2y^2 + z\,dS$$

The surface is the graph of the function $z = f(x,y) = x^2 - y^2$. The surface element is given by

$$dS = \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy = \sqrt{1 + (2x)^2 + (-2y)^2}\,dx\,dy = \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy.$$

Writing in cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

$$
\begin{aligned}
I &= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left(2r^2\sin^2\theta + r^2\cos^2\theta - r^2\sin^2\theta\right)\left(1 + 4r^2\right)^{\frac{1}{2}}\, r\,d\theta\,dr \\
&= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \left(r^2\sin^2\theta + r^2\cos^2\theta\right)\left(1 + 4r^2\right)^{\frac{1}{2}}\, r\,d\theta\,dr \\
&= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2\left(1 + 4r^2\right)^{\frac{1}{2}}\, r\,d\theta\,dr \\
&= \pi\int_{r=0}^{1} r^2\left(1 + 4r^2\right)^{\frac{1}{2}}\, 2r\,dr \\
&= \pi\int_{u=0}^{1} u\,(1 + 4u)^{\frac{1}{2}}\,du \\
&= \frac{\pi}{120}\Big[(12u - 2)\,(1 + 4u)^{\frac{3}{2}}\Big]_{u=0}^{1} = \frac{\pi}{60}\Big[5^{\frac{5}{2}} + 1\Big]
\end{aligned}
$$

where we substituted $u = r^2$ and used integration formula 96 in the back of the text.

21. *Calculate the flux with respect to upward surface normal of* $\mathbf{F} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ *across the surface* G *which is the part of the cone* $z = (x^2 + y^2)^{1/2}$, $z \geq 0$ *that is inside the cylinder* $x^2 + y^2 \leq 1$.

The surface is the graph of $f(x, y) = (x^2 + y^2)^{1/2}$. It has

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, \qquad f_y = \frac{y}{\sqrt{x^2 + y^2}},$$

The flux across the surface is given by

$$I = \iint_G \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and (away from the origin)

$$\mathbf{n} = \frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$$

which is upward since it has positive $\mathbf{k}$ component. Here $M = 2$, $N = 5$, $P = 3$ so that

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{n}\, dS &= (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot \frac{-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}} \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy \\
&= (-Mf_x - Nf_y + P)\, dx\, dy \\
&= \left(-\frac{2x}{\sqrt{x^2 + y^2}} - \frac{5y}{\sqrt{x^2 + y^2}} + 3 \right) dx\, dy
\end{aligned}$$

Thus, in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ so

$$\begin{aligned}
I &= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \left(-\frac{2r\cos\theta}{r} - \frac{5r\sin\theta}{r} + 3 \right) r\, d\theta\, dr \\
&= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (-2\cos\theta - 5\sin\theta + 3)\, r\, d\theta\, dr \\
&= 6\pi \int_{r=0}^{1} r\, dr = 3\pi.
\end{aligned}$$

It's no surprise. The sideways component of the field cancels and only the vertical component contributes to flux.

22. *Plot the parametric surface over the domain R:* $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, $0 \leq v \leq 2\pi$. *Find the surface area. Find the mass of the surface assuming that density is proportional to the distance to the z-axis.*

$$\mathbf{r}(u,v) = (2+\cos u)\cos v\,\mathbf{i} + (2+\cos u)\sin v\,\mathbf{j} + \sin u\,\mathbf{k}$$

The surface is the outside of a torus. Here is a picture using Grapher on my Macintosh.

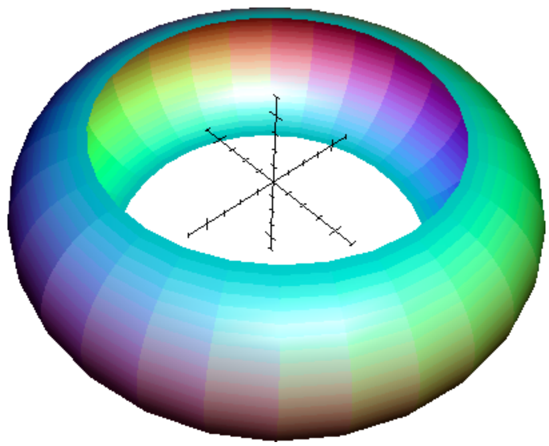

Figure 4: Image of R is the outside of a torus.

The cross product

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ -(2+\cos u)\sin v & (2+\cos u)\cos v & 0 \end{vmatrix} \\
&= -(2+\cos u)\cos u \cos v\,\mathbf{i} - (2+\cos u)\cos u \sin v\,\mathbf{j} \\
&\quad - (2+\cos u)(\sin u \cos^2 v + \sin u \sin^2 v)\,\mathbf{k} \\
&= -(2+\cos u)\big(\cos u \cos v\,\mathbf{i} + \cos u \sin v\,\mathbf{j} + \sin u\,\mathbf{k}\big)
\end{aligned}$$

so that

$$\|\mathbf{r}_u \times \mathbf{r}_v\|^2 = (2+\cos u)^2\big(\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \sin^2 u\big) = (2+\cos u)^2.$$

Thus the surface area is

$$\begin{aligned}
SA &= \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA \\
&= \int_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{v=0}^{2\pi} (2+\cos u)\,dv\,du \\
&= 2\pi \int_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} (2+\cos u)\,du \\
&= 2\pi \Big[2u + \sin u\Big]_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi(\pi+1)
\end{aligned}$$

If the density is proportional to the distance to the z-axis, it has the form

$$\begin{aligned}\delta(u,v) &= kr = k\sqrt{x^2+y^2}\\ &= k\sqrt{(2+\cos u)^2\cos^2 v + (2+\cos u)^2 \sin^2 v}\\ &= k(2+\cos u).\end{aligned}$$

Thus the mass is

$$\begin{aligned}M &= \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\|\, \delta\, dA\\ &= \int_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{v=0}^{2\pi} (2+\cos u)\cdot k(2+\cos u)\, dv\, du\\ &= 2\pi k \int_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 + 4\cos u + \cos^2 u\, du\\ &= 2\pi k \left[4u + 4\sin u + \frac{1}{2}u + \frac{1}{4}\sin 2u\right]_{u=-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi k(9\pi+16)\end{aligned}$$

using integral formula 21 from the back of the text.

23. *Use Gauss's Divergence Theorem to compute* $I = \iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\, dS$ *where* S *is the region* $x^2+z^2 \le y^2$, $x^2+y^2+z^2 \le 1$, $y \ge 0$ *and* $\mathbf{F} = (x^3+y)\,\mathbf{i} + (y^3+z)\,\mathbf{j} + (x+z^3)\,\mathbf{k}$.

S is the region inside the nappe of a cone in the positive y-axis direction cut by the unit sphere. It looks like a top. Gauss's Theorem says

$$I = \iint_{\partial S} \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_S \operatorname{div}\mathbf{F}\, dV$$

In this case,

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^3+y) + \frac{\partial}{\partial y}(y^3+z) + \frac{\partial}{\partial z}(x+z^3) = 3x^2+3y^2+3z^2$$

We use "spherical coordinates" where ϕ measures the angle form the y-axis and θ is polar angle in the yz-plane.

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\cos\phi, \qquad z = \rho\sin\phi\sin\theta.$$

The region is thus $0 \le \phi \le \frac{\pi}{4}$, $0 \le \theta \le 2\pi$ and $0 \le \rho \le 1$. The Jacobian is like spherical coordinates

$$\begin{aligned}J(\rho,\theta,\phi) &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta\\ \cos\phi & 0 & -\rho\sin\phi\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta\end{vmatrix}\\ &= \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\cos^2\theta + \rho^2\sin^3\phi\cos^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta\\ &= \rho^2(\sin^3\phi + \sin\phi\cos^2\phi) = \rho^2\sin\phi\end{aligned}$$

Thus

$$
\begin{aligned}
I &= \iiint_S \operatorname{div} \mathbf{F}\, dV \\
&= \int_{\rho=1}^{1} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \operatorname{div} \mathbf{F}\big(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)\big) |J(\rho,\theta,\phi)|\, d\phi\, d\theta\, d\rho \\
&= \int_{\rho=1}^{1} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} 3\rho^2 \cdot \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho \\
&= 3 \int_{\rho=1}^{1} \int_{\theta=0}^{2\pi} \Big[-\rho^4 \cos\phi \Big]_{\phi=0}^{\frac{\pi}{4}} d\theta\, d\rho \\
&= 3 \int_{\rho=1}^{1} \int_{\theta=0}^{2\pi} \rho^4 \left(1 - \frac{\sqrt{2}}{2}\right) d\theta\, d\rho \\
&= 6\pi \left(1 - \frac{\sqrt{2}}{2}\right) \int_{\rho=1}^{1} \rho^4\, d\rho = \frac{3\pi(2-\sqrt{2})}{5}.
\end{aligned}
$$

24. *Let $\mathbf{F} = (y-x)\,\mathbf{i} + (x-z)\,\mathbf{j} + (x-y)\,\mathbf{k}$. Use Stokes's Theorem to calculate the circulation $I = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ where C is the boundary of the plane $x + 2y + z = 2$ in the first octant, oriented clockwise as viewed from above.*

The plane is the level set $H(x,y,z) = 2$ of the function $H(x,y,z) = x + 2y + z$. It is a graph $z = f(x,y) = 2 - x - 2y$ so that $\nabla f = -\mathbf{i} - 2\mathbf{j}$. The piece of the plane in the first orthant is a triangle. Hence the area form

$$
dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy = \sqrt{1 + (-1)^2 + (-2)^2}\, dx\, dy = \sqrt{6}\, dx\, dy.
$$

The orientation means we follow the triangle in the order $(0,0,2)$ to $(2,0,0)$ to $(0,1,0)$ and back to the $(0,0,2)$. The normal is

$$
\mathbf{n} = \frac{\nabla H(x,y,z)}{\|\nabla H(x,y,z)\|} = \frac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{1}{\sqrt{6}} (\mathbf{i} + 2\mathbf{j} + \mathbf{k})
$$

It is consistent with the orientation because if your head is in the $\mathbf{n}$ dirction as you walk around the bounding curve of the triangle in the oriented direction, the triangle is on your left hand. The curl is

$$
\begin{aligned}
\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - x & x - z & x - y \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x-z) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(y-x) - \frac{\partial}{\partial x}(x-z) \right) \mathbf{j} \\
&\qquad + \left(\frac{\partial}{\partial x}(x-z) - \frac{\partial}{\partial y}(y-x) \right) \mathbf{k} \\
&= 0\mathbf{i} - \mathbf{j} + 0\mathbf{k} = -\mathbf{j}.
\end{aligned}
$$

By Stokes's Theorm,

$$
\begin{aligned}
I &= \oint_C \mathbf{F} \cdot \mathbf{T}\, ds \\
&= \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS \\
&= \int_{y=0}^{1} \int_{x=0}^{2-2y} -\mathbf{j} \cdot \frac{1}{\sqrt{6}} \left(\mathbf{i} + 2\mathbf{j} + \mathbf{k}\right) \sqrt{6}\, dx\, dy. \\
&= \int_{y=0}^{1} \int_{x=0}^{2-2y} -2\, dx\, dy \\
&= \int_{y=0}^{1} -4 + 4y\, dy = -4 + 2 = -2.
\end{aligned}
$$