Helly's Theorem with Applications in Combinatorial Geometry

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`http://www.math.utah.edu/~treiberg/HellySlides.pdf`

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4. Outline.

- One-Dimensional Helly's Theorem
- Convex Sets, Convex Combinations, Convex Hull
  - Caratheodory's Theorem.
  - Radon's Theorem
- Helly's Theorem
- Applications of Helly's Theorem
  - Klee's Theorems on Covering by Translates
  - Rey-Pastór-Santaló Theorem on Common Transversal of Segments
  - Helly's Theorem on Approximation of Functions
  - Jung's Estimate of the Circumradius in Terms of Diameter
  - Kirschberger's Theorem on Separating Sets
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Intervals are real sets like $[1,2]$, $[1,2)$, $(-\infty,3]$, $(5,\infty)$, $(-\infty,\infty)$.

Baby Helly's Question. If J_1, J_2 and J_3 are intervals such that any two have a point in common, do all three have a point in common?

Answer: YES! We've assumed any two have a point in common so take $\alpha_{12} \in J_1 \cap J_2$, $\alpha_{13} \in J_1 \cap J_3$ and $\alpha_{23} \in J_2 \cap J_3$. Sort these numbers, say,

$$\alpha_{13} < \alpha_{12} < \alpha_{23}.$$

α_{12} is in both J_1 and J_2. Is it in J_3? Now $\alpha_{13} \in J_3$ so α_{12} is greater than a point in J_3. Also $\alpha_{23} \in J_3$ so α_{12} is less than a point in J_3. Hence α_{12} is between two points of J_3, which is an interval, so $\alpha_{12} \in J_3$ as well. $\square$

The one dimensional Helly's Theorem is the same assertion for arbitrary many intervals. The proof is similar too.

Theorem (One-Dimensional Helly's Theorem)

Suppose $J_i \subset \mathbf{R}$ for $i = 1, \ldots, k$ is a collection of intervals such that no two are disjoint. Then there is a point common to all k intervals.

Let $\alpha_{ij} = \alpha_{ji}$ be any point in $J_i \cap J_j$. Consider

$$\beta_1 = \max_{i=1,\ldots,k} \min_{j=1,\ldots,k} \alpha_{ij}; \qquad \beta_2 = \min_{i=1,\ldots,k} \max_{j=1,\ldots,k} \alpha_{ij}$$

First, $\beta_1 \le \beta_2$. If i_1 and i_2 are indices such that $\beta_1 = \min_{j=1,\ldots,k} \alpha_{i_1 j}$ and $\beta_2 = \max_{j=1,\ldots,k} \alpha_{i_2 j}$ it follows that $\beta_1 \le \alpha_{i_1 i_2} = \alpha_{i_2 i_1} \le \beta_2$.
Second, any $z \in [\beta_1, \beta_2]$ is in every J_i. As $z \le \max_{j=1,\ldots,k} \alpha_{ij} \in J_i$ for any i, it is less than an element of every J_i. Similarly $z \ge \min_{j=1,\ldots,k} \alpha_{ij} \in J_i$ for every i, it is also greater than a point of J_i. Thus z is in every J_i. □

The n-dimensional real vector space $\mathbb{E}^n$ with origin 0 has the scalar product $\langle \bullet, \bullet \rangle$ and induced norm $|\bullet|$. A vector $x \in \mathbb{E}^n$ is a linear combination if

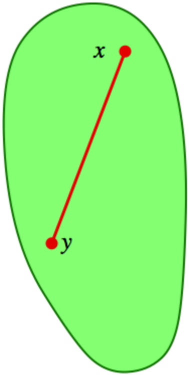
$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k$$

for vectors $x_1, \ldots x_k \in \mathbb{E}^n$ and suitable constants $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

If such λ_i exist with $\lambda_1 + \cdots + \lambda_k = 1$ we say x is an affine combination. Points $x_1, \ldots x_k \in \mathbb{E}^n$ are affinely independent if none of them is an affine combination of the others, *i.e.*,

$$\lambda_1 x_1 + \cdots + \lambda_k x_k = 0 \qquad \text{with} \qquad \lambda_1 + \cdots + \lambda_k = 0$$

implies $\lambda_1 = \cdots = \lambda_k = 0$. This is equivalent to the linear independence of the $k-1$ vectors $x_2 - x_1, \ldots, x_k - x_1$.



A set $A \subset \mathbf{E}^n$ is called convex if for any two points $x, y \in A$, the closed line segment from x to y is contained in A.

That is, $[x, y] \subset A$ where

$$[x, y] = \{(1-\lambda)x + \lambda y : 0 \leq \lambda \leq 1\}$$

So convex sets generalize segments to higher dimensions.
An example of a convex set is the open ball $B(z, \rho) = \{x \in \mathbb{E}^n : |z - x| < \rho\}$. Another is $B(z, \rho) \cup A$ where A is an arbitrary subset of the boundary of $B(z, \rho)$.

As a consequence of the definition, intersections of convex sets are convex. Images and preimages of affine transformations $T(x) = Ax + b$ of convex sets are convex too.

For subsets $A, B \subset \mathbb{E}^n$ and $\lambda \in \mathbb{R}$ we define Minkowski sum/product

$$A + B = \{a + b : a \in A,\ b \in B\}, \qquad \lambda A = \{\lambda a : a \in A\}$$

For $\lambda, \mu > 0$ we always have $\lambda A + \mu A \supset (\lambda + \mu)A$ but equality $\lambda A + \mu A = (\lambda + \mu)A$ holds for all $\lambda, \mu > 0$ if and only if A is convex.

A vector $x \in \mathbb{E}^n$ is a convex combination of the vectors $x_1, \ldots x_k \in \mathbb{E}^n$ if there are numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k, \qquad \lambda_i \geq 0 \text{ for all } i = 1, \ldots, k, \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

For $A \subset \mathbf{E}^n$, the convex hull, $\operatorname{conv} A$ is the set of all convex combinations of finitely many points in A.

Theorem

If $A \subset \mathbb{E}^n$ is convex then $A = \operatorname{conv} A$. For arbitrary sets $A \subset \mathbb{E}^n$, $\operatorname{conv} A$ is the intersection of all convex subsets of $\mathbf{E}^n$ containing A.
If $A, B \subset \mathbb{E}^n$ then $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$. If A is compact then so is $\operatorname{conv} A$.

Theorem (Caratheodory's Theorem)

If $A \subset \mathbb{E}^n$ and $x \in \operatorname{conv} A$ then x is a convex combination of affinely independent points in A. In particular, x is a combination of $n+1$ or fewer points of A.

Proof. A point in the convex hull is a convex combination of $k \in \mathbb{N}$ points

$$x = \sum_{i=1}^{k} \lambda_i x_i \qquad \text{with } x_i \in A, \text{ all } \lambda_i > 0 \text{ and } \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

where we may assume k is minimal.
For contradiction, assume $x_1, \ldots, x_k$ are affinely dependent. Then there would be numbers, not all zero, so that

$$\sum_{i=1}^{k} \alpha_i x_i = 0, \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_i = 0.$$

Choose an m such that $\dfrac{\lambda_m}{\alpha_m} > 0$ and is as small as possible. (All λ_i are already positive and one α_i must be positive.) Hence

$$x = \sum_{i=1}^{k} \left(\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) x_i$$

in another convex combination for x. All coefficients are non-negative because either α_i is negative, or $\lambda_i \geq \frac{\lambda_m}{\alpha_m} \alpha_i$ be choice of m.
Also, at least one (the mth) coefficient is zero, contradicting the minimality of k.
It follows that $x_1, \dots, x_k$ are affinely independent, which implies $k \leq n + 1$. □

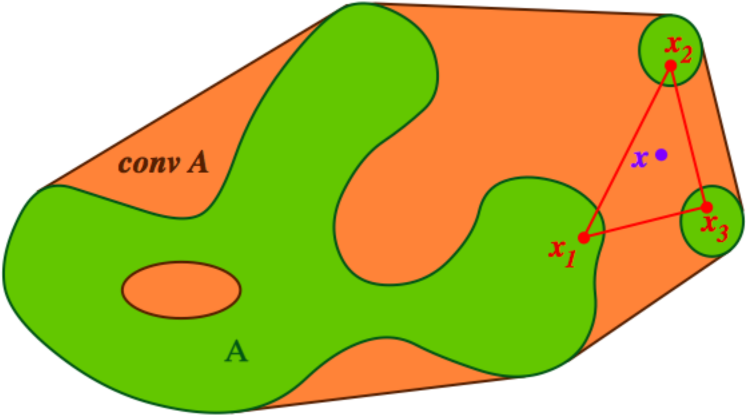


Figure: Convex hull $\operatorname{conv} A$ of a green set $A \subset \mathbb{E}^2$.

By Caratheodory's Theorem, for $A \subset \mathbb{E}^2$ any $x \in \operatorname{conv} A$ is the convex combination of at most three points, *i.e.*, in a simplex (triangle) with vertices $x_1, x_2, x_3 \in A$.

Theorem (Radon's Theorem)

Each finite set of affinely dependent points (in particular each set of at least $n+2$ points) can be expressed as the union of two disjoint sets whose convex hulls have a common point.

Proof. If $x_1, \ldots, x_k$ are affinely dependent, then there are numbers $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, not all zero, with

$$\sum_{i=1}^k \alpha_i x_i = 0, \qquad \text{and} \qquad \sum_{i=1}^k \alpha_i = 0.$$

We may assume, after renumbering that $\alpha_i > 0$ precisely for $i = 1, \ldots, j$ for $0 \le j < k$ (at least one $\alpha_i \neq 0$, but not all > 0 or all < 0). Put

$$\alpha = \alpha_1 + \cdots + \alpha_j = -(\alpha_{j+1} + \cdots + \alpha_k) > 0.$$

The weighted average of the positive points is

$$z = \sum_{i=1}^j \frac{\alpha_j}{\alpha} x_i = \sum_{i=j+1}^k \left(-\frac{\alpha_j}{\alpha}\right) x_i$$

Now $z \in \operatorname{conv}\{x_1, \ldots, x_j\} \cap \operatorname{conv}\{x_{j+1}, \ldots, x_k\}$ is the desired point. □

14. Johann Radon

Figure: Johann Radon (1887–1956)

Born in Bohemia, Austria, Radon earned his PhD in Vienna in 1910. He missed serving in WWI because of weak eyesight. He held several positions before returning to the University of Vienna. Radon developed this theorem especially to provide this nice proof of Helly's Theorem, published in 1922.

Radon is better known for he Radon-Nikodym Theorem of real analysis and the Radon Transform of X-ray tomography.

Theorem (Helly's Theorem)

Let $A_1, A_2, \dots, A_k \subset \mathbb{E}^n$ be convex sets. If any $n+1$ of these sets have a common point, then all sets have a common point.

Proof. We proceed by induction. There is nothing to prove if $k < n+1$ and the assertion is trivial if $k = n+1$. Thus we may suppose that $k > n+1$ and that the assertion is proved for $k-1$ convex sets. Thus for each $i \in \{1, \dots, k\}$ there is a point

$$x_i \in A_1 \cap \cdots \cap \widehat{A}_i \cap \cdots A_k$$

where $\widehat{A}_i$ indicates A_i is deleted. The $k \geq n+2$ points $x_1 \dots, x_k$ are affinely dependent (there are too many points.) By Radon's Theorem, after renumbering, we may infer that there is a point

$$z \in \mathrm{conv}\{x_1, \dots, x_j\} \cap \mathrm{conv}\{x_{j+1}, \dots, x_k\}$$

for some $j \in \{1, \dots, k-1\}$. Because $x_1, \dots, x_j \in A_{j+1}, \dots, A_k$ we have

$$z \in \mathrm{conv}\{x_1, \dots, x_j\} \subset A_{j+1} \cap \cdots \cap A_k.$$

Similarly $z \in \mathrm{conv}\{x_{j+1}, \dots, x_k\} \subset A_1 \cap \cdots \cap A_j$. □

Figure: Eduard Helly (1884–1943)

Helly was wounded in WWI and was prisoner of the Russians. He wrote about functional analysis from prison. Though discovered in 1913, the theorem in these notes wasn't published until 1921 when he was professor in Vienna. He fled the Nazi's to the US and worked at Monmouth College for a while, and joined the US Signal Corps in 1941 in Chicago, where he died.

17. Picture of Helly's Theorem in the Plane.

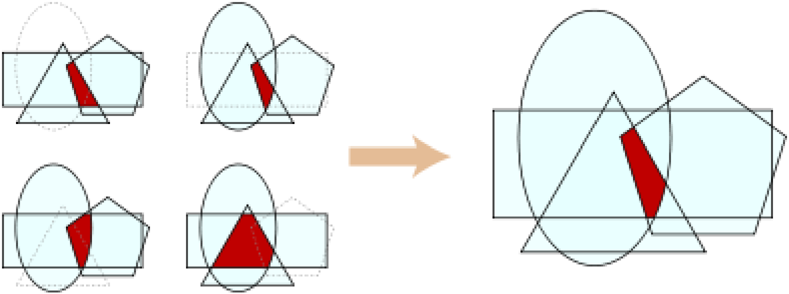


Figure: In a finite collection of planar convex sets, if every three have a point in common, then all have a pont in common.

Helly's Theorem can be generalized to infinite families of convex sets, provided some additional compactness is assumed.

Theorem (Helly's Theorem for Infinitely Many Sets)

Let $\mathcal{S}$ be a not necessarily finite family of convex sets in E^n. Assume that the intersection of any $n+1$ of these sets is compact and nonempty. Then all sets of $\mathcal{S}$ have a point in commom.

For example, to see that compactness is essential, consider the halfspaces

$$\{(x, y) \in \mathbb{E}^2 : y \geq n\}$$

for $n = 1, 2, 3 \ldots$. Their intersection is empty.

Theorem (Klee's Theorem)

Let K be a convex body (compact, convex set) and $\mathcal{S}$ a family of compact sets in $\mathbb{E}^n$. Assume that for any $n+1$ sets in $\mathcal{S}$ there is a translation $v \in \mathbb{E}^n$ such that $K+v$ covers the $n+1$ sets. Then there is a translation v_0 so that $K+v_0$ covers covers all sets of $\mathcal{S}$.

Proof. For any $S \in \mathcal{S}$ consider the compact set

$$T(S) = \{v \in \mathbb{E}^n : v \text{ is a translation such that } S \subset K+v.\}$$

Observe that $T(S)$ is convex: We have to show $S \subset K+v_1$ and $S \subset K+v_2$ implies $S \subset K+\lambda v_1+(1-\lambda)v_2$ for all $0 \leq \lambda \leq 1$. But a point $s \in S$ may be written $s = k_1+v_1 = k_2+v_2$ where $k_1, k_2 \in K$. Then $s = k+\lambda v_1+(1-\lambda)v_2$ where $k = \lambda k_1+(1-\lambda)k_2 \in K$.
It now follows from Helly's Theorem: the collection $\mathcal{S} = \{T(S) : S \in \mathcal{S}\}$ are convex sets with the property that for any $n+1$ of them $T(S_1), \ldots T(S_{n+1})$ there is a translate v such that $v \in T(S_j)$ for all $j = 1, \ldots, n+1$. Hence there is translation so v_0 is in each of $\mathcal{S}$. $\square$

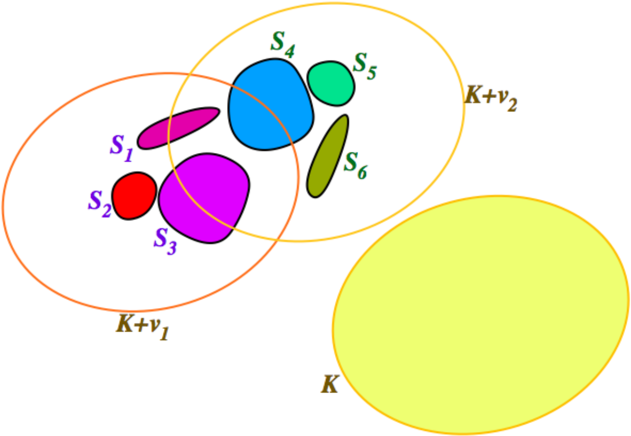


Figure: A translate of K covers S_1, S_2 and S_3. Another covers S_4, S_5 and S_6. Every three S_i's are covered by a translate, so all sets are covered by a translate.

Almost the same argument yields other versions of Klee's Theorem.

Theorem (More Klee's Theorem)

Let K be a convex body and $\mathcal{S}$ a family of convex bodies in $\mathbb{E}^n$. Assume that for any $n+1$ sets in $\mathcal{S}$ there is a translation $v \in \mathbf{E}^2$ such that $K+v$ intersects (is contained in) the $n+1$ sets. Then there is a translation v_0 so that $K+v_0$ intersects (is contained in) all sets of $\mathcal{S}$.

The argument considers alternate helper sets

$$T_2(S) = \{v : v \in \mathbb{E}^n \text{ is a translation such that } S \cap (K+v) \neq \emptyset.\}$$
$$T_3(S) = \{v : v \in \mathbb{E}^n \text{ is a translation such that } K+v \subset S.\}$$

Theorem (Rey, Pastór and Santaló)

Let $\mathcal{S}$ be a family of parallel segments in the plane. If any three segments of $\mathcal{S}$ have a common transversal, then all segments of $\mathcal{S}$ have a common transversal.

Proof. For simplicity, we assume that all segments are vertical with differing x_0 coordinates. Thus the segments have coordinates $\sigma = \{(x_0, y) : y_0 \leq y \leq y_1\}$. A transversal $y = ax + b$ intersects σ if

$$y_0 \leq ax_0 + b \leq y_1.$$

All possible transversals form a strip in the (a, b)-plane bounded by the lines

$$b = -x_0 a + y_0 \qquad \text{and} \qquad b = -x_0 a + y_1.$$

Different segments correspond to different slopes, thus the intersection of two strips is compact. Helly's Theorem implies all strips have a common point which corresponds to a common transversal. □

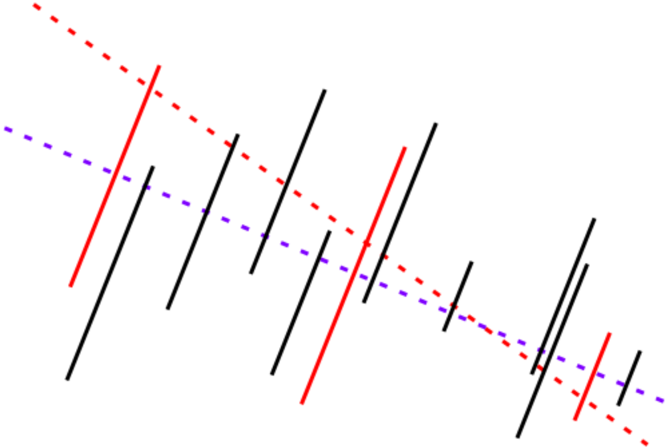


Figure: A family of parallel segments and a common transversal.

If every three of a family of parallel segments (such as the red ones) have a transversal (the red dashed line) then all segments have a common transversal (the blue dashed line).

The sup norm defines a distance on functions $f, g : [t_1, t_2] \to \mathbf{R}$ by

$$\|f - g\|_\infty = \sup_{t_0 \le t \le t_1} |f(t) - g(t)|$$

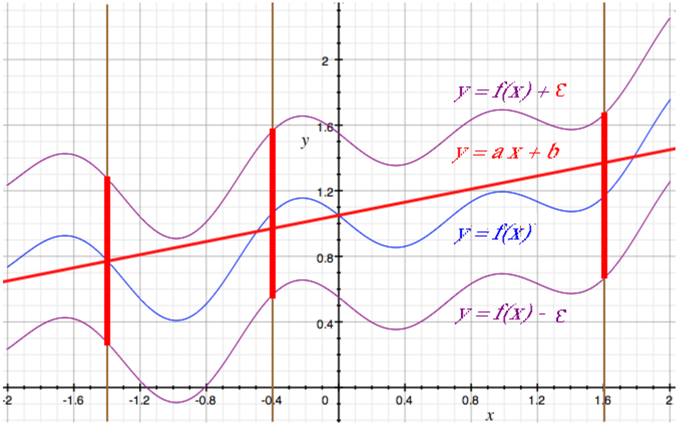
If $\|f - g\|_\infty \le \epsilon$ then for all $t_0 \le t \le t_1$

$$f(t) - \epsilon \le g(t) \le f(t) + \epsilon$$

The graph of g intersects the segments $x_0 = t$ and $y_0 = f(t) - \epsilon \le y \le f(t) + \epsilon = y_1$. If g is linear, then the Rey, Pastór, Santaló Theorem applies.

Proposition

A function $f : [t_1, t_1] \to \mathbf{R}$ may be approximated in the sup-norm by a linear function g to an error of $\le \epsilon$ on the interval $[t_1, t_1]$ if any three function values $f(t')$, $f(t'')$ and $f(t''')$ can be so approximated.



If for every three values $t', t'', t''' \in [t_1, t_2]$ there there are $a, b \in \mathbf{R}$ such that $f(t) - \epsilon \le at + b \le f(t) + \epsilon$ for each $t \in \{t', t'', t'''\}$ then there are $a, b \in \mathbf{R}$ such that $f(t) - \epsilon \le at + b \le f(t) + \epsilon$ for every $t \in [t_1, t_2]$. Here we take $t_1 = -2$, $t_2 = 2$, $\epsilon = .5$ and, for example, the three (red) vertical segments occur at $t' = -1.5$, $t'' = -.4$ and $t''' = 1.6$.

The diameter of a compact set $K \subset \mathbb{E}^n$ is the maximal distance between any two of its points

$$\operatorname{diam}(K) = \sup_{x,y \in K} |x - y|.$$

How big a ball is needed to cover a set with given diameter? The circumradius is the radius of the smallest ball that contains K.

Theorem (Jung's Theorem)

A set in $\mathbb{E}^n$ of diameter 1 is contained in a ball or radius $r_n = \sqrt{\frac{n}{2(n+1)}}$.

Proof. We show there is a point y such that every point of $x \in K$ is within r_n of y, *i.e.*, $|x - y| \leq r_n$. It suffices to show that all balls $x + r_n B$ intersect, where $x \in A$ and B is the closed unit ball. By Helly's Theorem, it suffices to show this for any $n + 1$ balls: given any $n + 1$ points $x_1, \ldots, x_{n+1} \in A$ there exists a point y whose distance from any of the x_i is at most r_n.

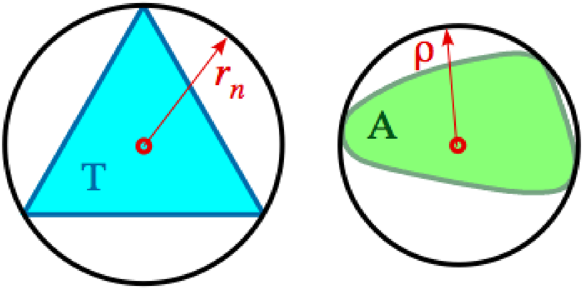


Figure: Among all sets $A \subset \mathbb{E}^2$ with unit diameter, the equilateral triangle T has largest circumradius: $r_n > \rho$.

Jung's theorem says that among all sets in $\mathbb{E}^n$ with the unit diameter, the regular simplex has the largest circumradius $r_n = \sqrt{\frac{n}{2(n+1)}}$. In the plane, the regular simplex is the equilateral triangle T with side length one.

Let $F = \{x_1, \ldots, x_{n+1}\} \subset A$ be any $n+1$ point subset. Let $\overline{B(c,r)}$ be a smallest ball containing F. $\overline{B(c,r)}$ is unique because if F were contained in two smallest balls, their intersection would contain F and be contained in an even smaller ball.

We may suppose that the center of this ball is the origin, $c = 0$. Let $F' \subset F$ be the points that intersect the boundary $\partial B(0,r)$. By renumbering, $F' = \{x_1 \ldots, x_k\}$ where $2 \le k \le n+1$. The smallest ball containing F' is the same as the smallest containing F. Note $|x_i| = r$ for all $i = 1, \ldots, k$.

We claim that the origin is in the convex hull $0 \in \operatorname{conv} F'$. If not, there is a closed halfspace $\overline{H} \subset \mathbb{E}^n$ such that $F' \subset \overline{H}$ but $0 \notin \overline{H}$. But this cannot be because $F' \subset \overline{H} \cap \overline{B(0,r)}$ whch is contained in a smaller ball.

Because $0 \in \operatorname{conv} F'$ there are numbers $\lambda_i \geq 0$ such that

$$0 = \sum_{i=1}^{k} \lambda_i x_i, \qquad 1 = \sum_{i=1}^{k} \lambda_i$$

for each j,

$$\begin{aligned} 1 - \lambda_j = \sum_{i \neq j} \lambda_i &\geq \sum_{i=1}^{k} \lambda_i |x_i - x_j|^2 \\ &= \sum_{i=1}^{k} \lambda_i \left(|x_i|^2 - 2x_i \bullet x_j + |x_j|^2 \right) \\ &= 2r^2 \sum_{i=1}^{k} \lambda_i - 2 \sum_{i=1}^{k} \lambda_i (x_i \bullet x_j) \\ &= 2r^2 - 2 \left(\sum_{i=1}^{k} \lambda_i x_i \right) \bullet x_j = 2r^2 \end{aligned}$$

Summing over j,

$$k - \sum_{j=1}^{k} \lambda_j = k - 1 \geq 2kr^2.$$

which implies

$$\frac{n}{2(n+1)} \geq \frac{k-1}{2k} \geq r^2. \quad \square$$

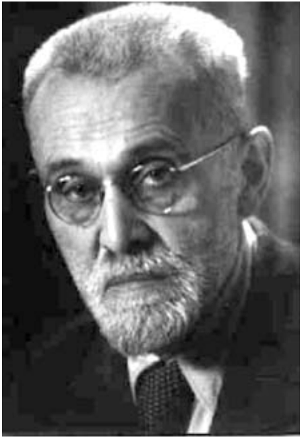


Figure: Heinrich Jung (1876–1953)

Jung's proved the theorem here in his 1899 Marberg thesis. His appointment at Kiel was interrupted because he had to serve in the army. After WWI, he held positions at Dorpat and at Halle after 1920. His main interests were theta functions and algebraic surfaces.

We say that two sets $A, B \subset \mathbb{E}^n$ can be strongly separated if there is a hyperplane $H = \{x \in \mathbb{E}^n : u \bullet x = \alpha\}$ for some nonzero $u \in \mathbb{E}^n$ and $\alpha \in \mathbf{R}$ such that A and B are on opposite sides of H and $\text{dist}(A, H)$ and $dist(B, H)$ are both positive.

For example, if $A, B \subset \mathbb{E}^n$ are compact, convex sets such that $A \cap B = \emptyset$ then A and B can be strongly separated.

Theorem

Let $A, B \subset \mathbb{E}^n$ be compact sets. For any subset $M \subset A \cup B$ with at most $n + 2$ points the sets $M \cap A$ and $M \cap B$ can be strongly separated then A and B can be strongly separated.

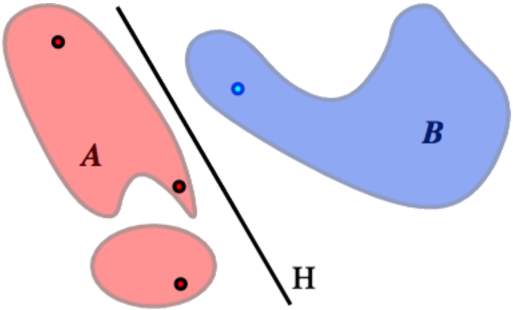


Figure: A, B are compact plane sets. If any four points in $A \cup B$ can be separated by a line H then A and B can be strongly separated by a line.

Kirschbergers's theorem says that for two compact sets $A, B \subset \mathbb{E}^n$, if any $n+2$ points of $A \cup B$ can be separated by a hyperplane H, then the sets can be strongly separated by a hyperplane.

Proof. Note that a halfspace is given by the set of points $x \in \mathbb{E}^n$ such that

$$v \bullet x + p \geq 0$$

where $v \neq 0$ is the inward pointing perpendicular vector and $p \in \mathbf{R}$.

First we assume A and B are finite sets. For $x \in \mathbb{E}^n$ define the sets of half-spaces that contain x

$$H_x^{\pm} = \{(v, p) \in \mathbb{E}^n \times \mathbf{R} : \pm(v \bullet x + p) > 0\}.$$

For card $M \leq n + 2$, by assumption there exists $u \in \mathbb{E}^n$ and $\alpha \in \mathbf{R}$ such that $u \bullet a > \alpha$ for $a \in M \cap A$ and $u \bullet b < \alpha$ for $b \in M \cap B$. Writing $p = -\alpha$ we see that $u \bullet a + p > 0$ and $u \bullet b + p < 0$ so that $(u, p) \in H_a^+$ for $a \in M \cap A$ and $(u, p) \in H_b^-$ for $b \in M \cap B$.

34. Kirschberger's Theorem Proof.

Thus the family $\{H_a^+ : a \in A\} \cup \{H_b^- : b \in B\}$ of finitely many convex sets in $\mathbb{E}^n \times \mathbf{R}$ has the property that any $n+2$ or fewer of the sets have nonempty intersection. By Helly's Theorem, the intersection of all sets of the family is nonempty. Since each set is open, the finite intersection is open too, so that we may assume that there is a point $(u, -\alpha)$ in the intersection that satisfies $u \neq 0$. Thus for every $a \in A$ we have $(v, -\alpha) \in H_a^+$ hence $v \bullet a > \alpha$ and for every $b \in B$, $(v, -\alpha) \in H_b^-$ hence $v \bullet b < \alpha$. Since $A \cup B$ is finite, they are strongly separated by the hyperplane $x \bullet u = \alpha$.

Now let A, B be compact sets satisfying the assumption. By compactness, separation implies strong separation. Suppose that A and B cannot be strongly separated. Then the compact sets conv A and conv B cannot be strongly separated. Hence there is $z \in \operatorname{conv} A \cap \operatorname{conv} B$. By Caratheodory's theorem, $z \in \operatorname{conv} A' \cap \operatorname{conv} B'$ where $A' \subset A$ and $B' \subset B$ are finite sets. Hence A' and B' cannot be strongly separated, which contradicts the result above. □

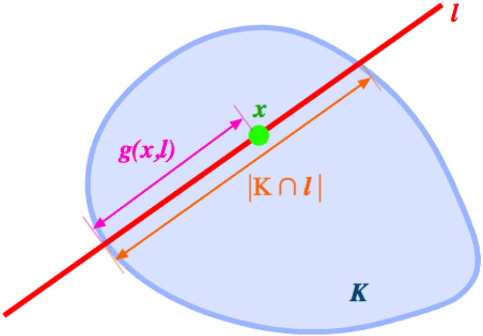


Figure: $x \in K$, ℓ is a line through x, $|K \cap \ell|$ is the length of the chord and $g(x, \ell)$ is the length of the larger subchord cut by x.

Theorem (Minkowski)

Let $K \subset \mathbb{E}^n$ be a convex body. Then $\min_{x \in K} \max_{\ell \ni x} \frac{g(x,\ell)}{|K \cap \ell|} \le \frac{n}{n=1}$.

Proof. The theorem asserts that there is a point where all chords are split in the ratio between $\frac{1}{n}$ and n. This point turns out to be the centroid. Define K_x to be the set K shrunk by factor $\frac{n}{n+1}$ about x. Then the result follows if

$$\bigcap_{x \in K} K_x \neq \emptyset.$$

To see it is sufficient, suppose that z is a point in the intersection and ℓ any line through it. Choose $x, y \in K \cap \ell$ on either side of z. All K_x and K_y are dilations and translations of K. So the length of the chords $w = |K_x \cap \ell| = |K_y \cap \ell| = \frac{n}{n+1}|K \cap \ell|$ are equal. It remains to show that the subchords $\ell_\pm$ of $K \cap \ell$ on either side of z are shorter than w. This follows if they are covered by $K_x \cap \ell$ and $K_y \cap \ell$ for suitable choices of x and y. For x equal to z, $z \in K_x$ but the end $\ell_+ \cap \partial K$ may not be in K_x. By moving x outward, eventually $\ell_+ \cap \partial K \in K_x$. By hypothesis $z \in K_x$ so $\ell_+ \subset K_x$. Similarly $\ell_- \subset K_y$ for suitable y.

The existence of a common point follows from Helly's Theorem if we could show

$$k_{x_1} \cap \cdots \cap K_{x_{n+1}} \neq \emptyset$$

for any points $x_1, \cdots, x_{n+1} \in K$. The centroid does the trick. Put

$$z = \frac{1}{n+1} \sum_{i=1}^{n+1} x_i.$$

z is a convex combination so is in K. To see that it is in K_{x_j} for every j,

$$z = x_j + \frac{n}{n+1} \left[\frac{1}{n} \sum_{i \neq j}^{n+1} (x_i - x_j) \right]$$

shows that z is the image under the homothety about x_j of the centroid of the points $x_1, \cdots, \widehat{x_j}, \ldots, x_{n+1} \in K$. □

38. Minkowski's Theorem.

Thanks!