Contents

1 Normed Spaces. Banach Spaces.

1.1 Vector Space.

Definition 1.1.

1. *An arbitrary subset M of a vector space X is said to be* ***linearly independent*** *if every non-empty finite subset of M is linearly independent.*

2. *A vector space X is said to be* ***finite dimensional*** *if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of $n+1$ or more vectors of X is linearly dependent. n is called the* ***dimension*** *of X, written $n = \dim X$.*

3. *If X is any vector space, not necessarily finite dimensional, and B is a linearly independent subset of X which spans X, then B is called a* ***basis*** *(or* ***Hamel basis****) of X.*

   - *Hence if B is a basis for X, then every nonzero $x \in X$ has a unique representation as a linear combination of (**finitely many!**) elements of B with nonzero scalars as coefficients.*

Theorem 1.2. *Let X be an n-dimensional vector space. Then any proper subspace Y of X has dimension less than n.*

1. Show that the set of all real numbers, with the usual addition and multiplication, constitutes a one-dimensional real vector space, and the set of all complex numbers constitutes a one-dimensional complex vector space.

> **Solution:** The usual addition on $\mathbb{R}$ and $\mathbb{C}$ are commutative and associative, while scalar multiplication on $\mathbb{R}$ and $\mathbb{C}$ are also associative and distributive. For $\mathbb{R}$, the zero vector is $\mathbf{0}_R = 0 \in \mathbb{R}$, the identity scalar is $1_\mathbb{R} = 1 \in \mathbb{R}$, and the additive inverse is $-x$ for any $x \in \mathbb{R}$. For $\mathbb{C}$, the zero vector is $\mathbf{0}_\mathbb{C} = 0 + 0i \in \mathbb{C}$, the identity scalar is $1_\mathbb{C} = 1 + 0i \in \mathbb{C}$ and the additive inverse is $-z$ for all $z \in \mathbb{C}$.

2. Prove that $0x = \mathbf{0}$, $\alpha\mathbf{0} = \mathbf{0}$ and $(-1)x = -x$.

> **Solution:**
>
> $$\begin{aligned} 0x = (0+0)x = 0x + 0x \implies \mathbf{0} &= 0x + (-(0x)) \\ &= 0x + 0x + (-(0x)) \\ &= 0x + \mathbf{0} = 0x. \\ \alpha\mathbf{0} = \alpha(\mathbf{0}+\mathbf{0}) = \alpha\mathbf{0} + \alpha\mathbf{0} \implies \mathbf{0} &= \alpha\mathbf{0} + (-(\alpha\mathbf{0})) \\ &= \alpha\mathbf{0} + \alpha\mathbf{0} + (-(\alpha\mathbf{0})) \\ &= \alpha\mathbf{0} + \mathbf{0} = \alpha\mathbf{0}. \end{aligned}$$
>
> $$(-1)x = (-1(1))x = -1(1x) = -x.$$

3. Describe the span of $M = \{(1,1,1),(0,0,2)\}$ in $\mathbb{R}^3$.

Solution: The span of M is

$$\begin{aligned}\text{span } M &= \left\{\alpha \begin{bmatrix}1\\1\\1\end{bmatrix} + \beta \begin{bmatrix}0\\0\\2\end{bmatrix} : \alpha, \beta \in \mathbb{R}\right\}\\ &= \left\{\begin{bmatrix}\alpha\\ \alpha\\ \alpha+2\beta\end{bmatrix} : \alpha, \beta \in \mathbb{R}\right\}.\end{aligned}$$

We see that span M corresponds to the plane $x = y$ on $\mathbb{R}^3$.

4. Which of the following subsets of $\mathbb{R}^3$ constitute a subspace of $\mathbb{R}^3$? Here, $x = (\xi_1, \xi_2, \xi_3)$.

(a) All x with $\xi_1 = \xi_2$ and $\xi_3 = 0$.

Solution: For any $x, y \in W$ and any $\alpha, \beta \in \mathbb{R}$,

$$\alpha x + \beta y = \alpha \begin{bmatrix}\xi_1\\ \xi_2\\ 0\end{bmatrix} + \beta \begin{bmatrix}\eta_1\\ \eta_2\\ 0\end{bmatrix} = \begin{bmatrix}\alpha\xi_1 + \beta\eta_1\\ \alpha\xi_2 + \beta\eta_2\\ 0\end{bmatrix} \in W.$$

(b) All x with $\xi_1 = \xi_2 + 1$.

Solution: Choose $x_1 = \begin{bmatrix}2\\1\\0\end{bmatrix} \in W$, $x_2 = \begin{bmatrix}3\\2\\0\end{bmatrix} \in W$, then

$$x_1 + x_2 = \begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\2\\0\end{bmatrix} = \begin{bmatrix}5\\3\\0\end{bmatrix} \notin W$$

since $5 \neq 3 + 1$.

(c) All x with positive ξ_1, ξ_2, ξ_3.

Solution: Choose $\alpha = -1$, then for any $x \in W$, $\alpha x \notin W$.

(d) All x with $\xi_1 - \xi_2 + \xi_3 = k$.

Solution: For any $x, y \in W$,

$$x + y = \begin{bmatrix}\xi_1 + \eta_1\\ \xi_2 + \eta_2\\ \xi_3 + \eta_3\end{bmatrix}.$$

Since

$$\xi_1 + \eta_1 - (\xi_2 + \eta_2) + (\xi_3 + \eta_3) = (\xi_1 - \xi_2 + \xi_3) + (\eta_1 - \eta_2 + \eta_3) = 2k.$$

we see that W is a subspace of $\mathbb{R}^3$ if and only if $k = 0$.

5. Show that $\{x_1, \ldots, x_n\}$, where $x_j(t) = t^j$, is a linearly independent set in the space $C[a, b]$.

Solution: This is a simple consequence of **Fundamental Theorem of Algebra**. Fix a finite $n > 1$. Suppose that for all $t \in [a, b]$, we have

$$\sum_{j=1}^{n} \lambda_j x_j(t) = \sum_{j=1}^{n} \lambda_j t^j = 0.$$

Suppose $\lambda_n \neq 0$. Fundamental Theorem of Algebra states that any polynomials with degree n can have at most n real roots. Since the equation above is true for all $t \in [a, b]$, and the set of points in the interval $[a, b]$ is uncountable, $\sum_{j=1}^{n} \lambda_j t^j$ has to be the zero polynomial. Since $n \geq 1$ was arbitrary (but finite), this shows that any non-empty finite subset of $\{x_j\}_{j \in \Lambda}$, Λ a countable/uncountable indexing set, is linearly independent.

6. Show that in an n-dimensional vector space X, the representation of any x as a linear combination of a given basis vectors $e_1, \ldots, e_n$ is unique.

Solution: Let X is an n-dimensional vector space, with a basis $\{e_1, \ldots, e_n\}$. Suppose any $x \in X$ has a representation as a linear combination of the basis vectors, we claim that the representation is unique. Indeed, if $x \in X$ has two representations

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n = \beta_1 e_1 + \ldots + \beta_n e_n.$$

subtracting them gives

$$(\alpha_1 - \beta_1)e_1 + \ldots + (\alpha_n - \beta_n)e_n = \sum_{j=1}^{n} (\alpha_j - \beta_j)e_j = \mathbf{0}.$$

Since $\{e_1, \ldots, e_n\}$ is a basis of X, by definition it is linearly independent, which implies that $\alpha_j - \beta_j = 0$ for all $j = 1, \ldots, n$, i.e. the representation is unique.

7. Let $\{e_1, \ldots, e_n\}$ be a basis for a complex vector space X. Find a basis for X regarded as a real vector space. What is the dimension of X in either case?

Solution: A basis for X regarded as a real vector space is $\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$. The dimension of X is n as a complex vector space and $2n$ as a real vector space.

8. If M is a linearly dependent set in a complex vector space X, is M linearly dependent in X, regarded as a real vector space?

Solution: No. Let $X = \mathbb{C}^2$, with $K = \mathbb{C}$, and consider $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $y = \begin{bmatrix} i \\ -1 \end{bmatrix}$. $\{x, y\}$ is a linearly dependent set in X since $ix = y$. Now suppose $K = \mathbb{R}$, and

$$\alpha x + \beta y = \begin{bmatrix} \alpha + \beta i \\ \alpha i - \beta \end{bmatrix} = \mathbf{0} = \begin{bmatrix} 0 + 0i \\ 0 + 0i \end{bmatrix}.$$

Since α, β can only be real numbers, we see that $(\alpha, \beta) = (0, 0)$ is the only solution to the equation. Hence $\{x, y\}$ is a linearly independent set in $X = \mathbb{C}^2$ over $\mathbb{R}$.

9. On a fixed interval $[a, b] \subset \mathbb{R}$, consider the set X consisting of all polynomials with real coefficients and of degree not exceeding a given n, and the polynomial $x = 0$ (for which a degree is not defined in the usual discussion of degree).

   (a) Show that X, with the usual addition and the usual multiplication by real numbers, is a real vector space of dimension $n + 1$. Find a basis for X.

   Solution: Let X be the set given in the problem. It is clear that X is a real vector space. Indeed, for any $P, Q \in X$, with $\deg(P), \deg(Q) \leq n$, $\deg(P + Q) \leq n$ and $\deg(\alpha P) \leq n$ for any $\alpha \in \mathbb{R}$. A similar argument from Problem 5 shows that $\{1, t, t^2, \dots, t^n\}$ is a linearly independent set in X, and since $\{1, t, t^2, \dots, t^n\}$ spans X, it is a basis of X and X has dimension $n + 1$.

   (b) Show that we can obtain a complex vector space $\tilde{X}$ in a similar fashion if we let those coefficients be complex. Is X a subspace of $\tilde{X}$?

   Solution: No. Consider $P(t) = t \in X$, choose $\alpha = i$, then $\alpha P(t) = it \notin X$.

10. If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be one. Give examples.

Solution: Let Y and Z be subspaces of a vector space X. Take any $x, y \in Y \cap Z$, note that x, y are both elements of Y and Z. For any $\alpha, \beta \in K$, $\alpha x + \beta y \in Y$ (since Y is a subspace of X) and $\alpha x + \beta y \in Z$ (since Z is a subspace of X). Hence $\alpha x + \beta y \in Y \cap Z$.

For the second part, consider $Y = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$ and $Z = \left\{ \begin{bmatrix} 0 \\ \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$. It can be (easily) deduced that Y and Z are subspaces of $\mathbb{R}^2$, but $Y \cup Z$ is not a subspace of $\mathbb{R}^2$. To see this, choose $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x + y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin Y \cup Z$.

11. If $M \neq \varnothing$ is any subset of a vector space X, show that span M is a subspace of X.

Solution: This is immediate since a (scalar) field K is closed under addition and sums of two finite sums remain finite.

12. (a) Show that the set of all real two-rowed square matrices forms a vector space X. What is the zero vector in X?

Solution: This follows from Problem 1 and the definition of matrix addition and matrix scalar multiplication: we prove that $\mathbb{R}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$. The zero vector in X is $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Determine dim X. Find a basis for X.

Solution: We claim that dim $X = 4$. To prove this, consider the following four vectors in X

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose $\alpha_1 e_1 + \ldots + \alpha_4 e_4 = \mathbf{0} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$, we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, i.e. $\{e_1, e_2, e_3, e_4\}$ is a linearly independent set in X. However, any set of 5 or more vectors of X is linearly dependent, since any $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X$ can be written as a linear combination of $\{e_1, e_2, e_3, e_4\}$, i.e. $x = ae_1 + be_2 + ce_3 + de_4$. Hence, a basis for X is $\{e_1, e_2, e_3, e_4\}$.

(c) Give examples of subspaces of X.

Solution: An example is $W = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$.

(d) Do the symmetric matrices $x \in X$ form a subspace of X?

Solution: Yes. Consider any symmetric matrices $x = \begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \end{bmatrix}, y = \begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \end{bmatrix}$. For any $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$),

$$\alpha x + \beta y = \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha b_1 + \beta b_2 & \alpha d_1 + \beta d_2 \end{bmatrix}$$

which is a symmetric matrix.

(e) Do the singular matrices $x \in X$ form a subspace of X?

Solution: No. To see this, consider $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$; both x and y are singular matrices since they have zero determinant. However, $x+y = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$ is not a singular matrix since $\det(x+y) = 20 - 21 = -1 \neq 0$.

13. **(Product)** Show that the Cartesian product $X = X_1 \times X_2$ of two vector spaces over the same field becomes a vector space if we define the two algebraic operations by

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2), \\ \alpha(x_1, x_2) &= (\alpha x_1, \alpha x_2). \end{aligned}$$

Solution: This is a simple exercise. We first verify vector addition:

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2). \\ (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] &= \Big(x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)\Big) \\ &= \Big((x_1 + y_1) + z_1, (x_2 + y_2) + z_2\Big) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= \Big[(x_1, x_2) + (y_1, y_2)\Big] + (z_1, z_2). \\ (x_1, x_2) &= (x_1 + \mathbf{0}, x_2 + \mathbf{0}) \\ &= (x_1, x_2) + (\mathbf{0}, \mathbf{0}). \\ (\mathbf{0}, \mathbf{0}) &= (x_1 + (-x_1), y_1 + (-y_1)) \\ &= (x_1, y_1) + (-x_1, -y_1). \end{aligned}$$

Next, we verify scalar vector multiplication:

$$\begin{aligned} (x_1, x_2) &= (1_K x_1, 1_K x_2) \\ &= 1_K (x_1, x_2). \end{aligned}$$

$$
\begin{aligned}
\alpha\Big[\beta(x_1, x_2)\Big] &= \alpha(\beta x_1, \beta x_2) \\
&= \Big(\alpha(\beta x_1), \alpha(\beta x_2)\Big) \\
&= \Big((\alpha\beta)x_1, (\alpha\beta)x_2\Big) \\
&= (\alpha\beta)(x_1, x_2). \\
\alpha\Big[(x_1, x_2) + (y_1, y_2)\Big] &= \alpha(x_1 + y_1, x_2 + y_2) \\
&= \Big(\alpha(x_1 + y_1), \alpha(x_2 + y_2)\Big) \\
&= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\
&= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2) \\
&= \alpha(x_1, y_1) + \alpha(x_2, y_2). \\
(\alpha + \beta)(x_1, x_2) &= \Big((\alpha + \beta)x_1, (\alpha + \beta)x_2\Big) \\
&= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2) \\
&= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2) \\
&= \alpha(x_1, x_2) + \beta(x_1, x_2).
\end{aligned}
$$

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14. **(Quotient space, codimension)** Let Y be a subspace of a vector space X. The *coset* of an element $x \in X$ with respect to Y is denoted by $x + Y$ and is defined to be the set

$$x + Y = \{x \colon v = x + y, y \in Y\}.$$

(a) Show that the distinct cosets form a partition of X.

Solution:

(b) Show that under algebraic operations defined by

$$
\begin{aligned}
(w + Y) + (x + Y) &= (w + x) + Y \\
\alpha(x + Y) &= \alpha x + Y
\end{aligned}
$$

these cosets constitute the elements of a vector space. This space is called the *quotient space* (or sometimes *factor space*) of X by Y (or *modulo* Y) and is denoted by X/Y. Its dimension is called the *codimension* of Y and is denoted by codim Y, that is,

$$\textbf{codim } Y = \dim\,(X/Y).$$

Solution:

15. Let $X = \mathbb{R}^3$ and $Y = \{(\xi_1, 0, 0) \colon \xi_1 \in \mathbb{R}\}$. Find X/Y, X/X, $X/\{\mathbf{0}\}$.

Solution: First, $X/X = \{x + X : x \in X\}$; since $x + X \in X$ for any $x \in X$, we see that $X/X = \{\mathbf{0}\}$. Next, $X/\{\mathbf{0}\} = \{x + \mathbf{0} : x \in X\} = \{x : x \in X\} = X$. For X/Y, we are able to deduce (geometrically) that elements of X/Y are lines parallel to the ξ_1-axis. More precisely, by definition,$X/Y = \{x + Y : x \in X\}$; for a fixed $x_0 = (\xi_1^0, \xi_2^0, \xi_3^0)$,

$$\begin{aligned} x_0 + Y &= \{(\xi_1^0, \xi_2^0, \xi_3^0) + (0, 0, \xi_3) : \xi_3 \in \mathbb{R}\} \\ &= \{(\xi_1^0, \xi_2^0, \tilde{\xi_3}) : \tilde{\xi_3} \in \mathbb{R}\}. \end{aligned}$$

which corresponds to a line parallel to ξ_1-axis.

1.2 Normed Space. Banach Space.

Definition 1.3. *A **norm** on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties*

(N1) $\|x\| \geq 0$.

(N2) $\|x\| = 0 \Longleftrightarrow x = 0$.

(N3) $\|\alpha x\| = |\alpha|\|x\|$.

(N4) $\|x + y\| \leq \|x\| + \|y\|$. ***(Triangle inequality)***

Here, x and y are arbitrary vectors in X and α is any scalar.

- A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \qquad , x, y, \in X$$

  and is called the **metric induced by the norm**.

- The norm is continuous, that is, $x \mapsto \|x\|$ is a continuous mapping of $(X, \|\cdot\|)$ into $\mathbb{R}$.

Theorem 1.4. *A metric d induced by a norm on a normed space X satisfies*
(a) $d(x + a, y + a) = d(x, y)$.

(b) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.
for all $x, y, a \in X$ and every scalar α.

- This theorem illustrates an important fact: Every metric on a vector space might not necessarily be obtained from a norm.

- A counterexample is the space s consisting of all (bounded or unbounded) sequences of complex numbers with a metric d defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}.$$

1. Show that the norm $\|x\|$ of x is the distance from x to $\mathbf{0}$.

Solution: $\|x\| = \|x-y\|\Big|_{y=\mathbf{0}} = d(x,\mathbf{0})$, which is precisely the distance from x to $\mathbf{0}$.

2. Verify that the usual length of a vector in the plane or in three dimensional space has the properties (N1) to (N4) of a norm.

Solution: For all $x \in \mathbb{R}^3$, define $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. (N1) to (N3) are obvious. (N4) is an easy consequence of the **Cauchy-Schwarz inequality for sums**. More precisely, for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ we have:

$$\begin{aligned}\|x+y\|^2 &= (x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 \\ &= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) + 2(x_1y_1 + x_2y_2 + x_3y_3) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2.\end{aligned}$$

Taking square root on both sides yields (N4).

3. Prove (N4) implies $\Big|\|y\| - \|x\|\Big| \leq \|y - x\|$.

Solution: From triangle inequality of norm we have the following two inequalities:

$$\begin{aligned} \|x\| &= \|x-y+y\| \leq \|x-y\| + \|x\| \\ \implies \|x\| - \|y\| &\leq \|x-y\|. \\ \|y\| &= \|y-x+x\| \leq \|y-x\| + \|x\| \\ \implies \|y\| - \|x\| &\leq \|x-y\|. \end{aligned}$$

Combining these two yields the desired inequality.

4. Show that we may replace (N2) by $\|x\| = 0 \implies x = 0$ without altering the concept of a norm. Show that nonnegativity of a norm also follows from (N3) and (N4).

Solution: For any $x \in X$,

$$\begin{aligned} \|x\| = \|x+x-x\| \leq \|x+x\| + \|-x\| = 2\|x\| + \|x\| = 3\|x\| \\ \implies 0 \leq 2\|x\| \implies 0 \leq \|x\|. \end{aligned}$$

5. Show that $\|x\| = \left(\sum_{j=1}^n |\xi_j|^2\right)^{1/2} = \sqrt{|\xi_1|^2 + \ldots + |\xi_n|^2}$ defines a norm.

Solution: (N1) to (N3) are obvious. (N4) follows from the Cauchy-Schwarz inequality for sums, the proof is similar to that in Problem 3.

6. Let X be the vector space of all ordered pairs $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$, $\cdots$ of real numbers. Show that norms on X are defined by

$$\begin{aligned} \|x\|_1 &= |\xi_1| + |\xi_2| \\ \|x\|_2 &= (\xi_1^2 + \xi_2^2)^{1/2} \\ \|x\|_\infty &= \max\{|\xi_1|, |\xi_2|\}. \end{aligned}$$

Solution: (N1) to (N3) are obvious for each of them. To verify (N4), for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$,

$$\begin{aligned} \|x+y\|_1 &= |\xi_1 + \eta_1| + |\xi_2 + \eta_2| \\ &\leq |\xi_1| + |\eta_1| + |\xi_2| + |\eta_2| = \|x\|_1 + \|y\|_1. \end{aligned}$$

$$\begin{aligned} \|x+y\|_2^2 &= (\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2 \\ &= (\xi_1^2 + \xi_2^2) + (\eta_1^2 + \eta_2^2) + 2(\xi_1\eta_1 + \xi_2\eta_2) \\ &\leq \|x\|_2 + \|y\|_2 + 2\|x\|_2\|y\|_2 = (\|x\|_2 + \|y\|_2)^2 \\ \implies \|x+y\|_2 &\leq \|x\|_2 + \|y\|_2. \end{aligned}$$

$$\begin{aligned} \|x+y\|_\infty &= \max\{|\xi_1 + \eta_1|, |\xi_2 + \eta_2|\} \\ &\leq \max\{|\xi_1| + |\eta_1|, |\xi_2| + |\eta_2|\} \\ &\leq \max\{|\xi_1|, |\xi_2|\} + \max\{|\eta_1|, |\eta_2|\} = \|x\|_\infty + \|y\|_\infty. \end{aligned}$$

where we use the inequality $|a| \leq \max\{|a|, |b|\}$ for any $a, b \in \mathbb{R}$.

7. Verify that $\|x\| = \left(\sum_{j=1}^\infty |\xi_j|^p\right)^{1/p}$ satisfies (N1) to (N4).

Solution: (N1) to (N3) are obvious. (N4) follows from **Minkowski inequality for sums**. More precisely, for $x = (\xi_j)$ and $y = (\eta_j)$,

$$\|x+y\| = \left(\sum_{j=1}^\infty |\xi_j + \eta_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^\infty |\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{\frac{1}{p}}.$$

8. There are several norms of practical importance on the vector space of ordered n-tuples of numbers, notably those defined by

$$\|x\|_1 = |\xi_1| + |\xi_2| + \ldots + |\xi_n|$$

$$\|x\|_p = \Big(|\xi_1|^p + |\xi_2|^p + \ldots + |\xi_n|^p\Big)^{1/p} \qquad (1 < p < +\infty)$$
$$\|x\|_\infty = \max\{|\xi_1|, \ldots, |\xi_n|\}.$$

In each case, verify that (N1) to (N4) are satisfied.

Solution: This is a generalisation of Problem 6, with the proof being almost identicall. The only thing that differs is we use **Minkowski inequality for sums** to prove (N4) for $\|\cdot\|_p$.

9. Verify that $\|x\| = \max\limits_{t\in[a,b]} |x(t)|$ defines a norm on the space $C[a, b]$.

Solution: (N1) and (N2) are clear, as we readily see. For (N3), for any scalars α we have:
$$\|\alpha x\| = \max_{t\in[a,b]} |\alpha x(t)| = |\alpha| \max_{t\in[a,b]} |x(t)| = |\alpha|\|x\|.$$
Finally, for (N4),
$$\|x + y\| = \max_{t\in[a,b]} |x(t) + y(t)| \leq \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |y(t)| = \|x\| + \|y\|.$$

10. **(Unit Sphere)** The sphere
$$S_1(0) = \{x \in X : \|x\| = 1\}.$$
in a normed space X is called the *unit sphere*. Show that for the norms in Problem 6 and for the norm defined by $\|x\|_4 = (\xi_1^4 + \xi_2^4)^{1/4}$, the unit spheres look as shown in figure.

Solution: Refer to Kreyszig, page 65.

11. **(Convex set, segment)** A subset A of a vector space X is said to be *convex* if $x, y \in A$ implies
$$M = \{z \in X : z = \alpha x + (1 - \alpha)y, \quad 0 \leq \alpha \leq 1\} \subset A.$$
M is called a *closed segment* with *boundary points* x and y; any other $z \in M$ is called an *interior point* of M. Show that the *closed unit ball* in a normed space X is convex.

Solution: Choose any $x, y \in \tilde{B}_1(0)$, then for any $0 \leq \alpha \leq 1$,
$$\|\alpha x + (1 - \alpha)y\| \leq \alpha\|x\| + (1 - \alpha)\|y\| \leq \alpha + 1 - \alpha = 1.$$
This shows that the closed unit ball in X is convex.

12. Using Problem 11, show that

$$\psi(x) = \left(\sqrt{|\xi_1|} + \sqrt{|\xi_2|}\right)^2$$

does not define a norm on the vector space of all ordered pairs $x = (\xi_1, \xi_2)$ of real numbers. Sketch the curve $\psi(x) = 1$.

Solution: Problem 11 shows that if ψ is a norm, then the closed unit ball in a normed space $X = (X, \psi)$ is convex. Choose $x = (1, 0)$ and $y = (0, 1)$, x, y are elements of the closed unit ball in (X, ψ) since $\psi(x) = \psi(y) = 1$. However, if we choose $\alpha = 0.5$,

$$\psi(0.5x + 0.5y) = \left(\sqrt{|0.5|} + \sqrt{|0.5|}\right)^2 = (2\sqrt{0.5})^2 = 2 > 1.$$

This shows that for $0.5x + 0.5y$ is not an element of the closed unit ball in (X, ψ), and contrapositive of result from Problem 11 shows that $\psi(x)$ does not define a norm on X.

13. Show that the discrete metric on a vector space $X \neq \{\mathbf{0}\}$ cannot be obtained from a norm.

Solution: Consider a discrete metric space $X \neq \{\mathbf{0}\}$. Choose distinct $x, y \in X$, for $\alpha = 2$, $d(2x, 2y) = 1$ but $|2|d(x, y) = 2$. The statement then follows from theorem.

14. If d is a metric on a vector space $X \neq \{\mathbf{0}\}$ which is obtained from a norm, and $\tilde{d}$ is defined by

$$\tilde{d}(x, x) = 0, \qquad \tilde{d}(x, y) = d(x, y) + 1 \quad (x \neq y),$$

show that $\tilde{d}$ cannot be obtained from a norm.

Solution: Consider a metric space $X \neq \{\mathbf{0}\}$. Choose any $x \in X$, for $\alpha = 2$, $\tilde{d}(2x, 2x) = d(2x, 2x) + 1 = 1$ but $|2|\tilde{d}(x, x) = 2(d(x, x) + 1) = 2$. The statement then follows from theorem.

15. **(Bounded set)** Show that a subset M in a normed space X is bounded if and only if there is a positive number c such that $\|x\| \leq c$ for every $x \in M$.

Solution: Suppose a subset M in a normed space X is bounded. By definition, the diameter $\delta(M)$ of M is finite, i.e.

$$\delta(M) = \sup_{x,y \in M} d(x, y) = \sup_{x,y \in M} \|x - y\| < \infty.$$

Fix an $y \in M$, then for any $x \in M$,

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \leq \delta(M) + \|y\| < \infty.$$

Choosing $c = \delta(M) + \|y\|$ yields the desired result. Conversely, suppose there exists an $c > 0$ such that $\|x\| \leq c$ for all $x \in M$. Then for any $x, y \in M$,

$$d(x, y) = \|x - y\| \leq \|x\| + \|y\| \leq 2c.$$

Taking supremum over $x, y \in M$ on both sides, we obtain $\delta(M) \leq 2c < \infty$. This shows that M (in a normed space X) is bounded.

1.3 Further Properties of Normed Spaces.

Definition 1.5. *A* ***subspace*** *Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be* ***induced*** *by the norm on X.*

Theorem 1.6. *A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.*

Definition 1.7.

1. *If (x_k) is a sequence in a normed space X, we can associate with (x_k) the sequence (s_n) of* ***partial sums***
$$s_n = x_1 + x_2 + \ldots + x_n$$
where $n = 1, 2, \ldots$. If (s_n) is convergent, say, $s_n \longrightarrow s$ as $n \longrightarrow \infty$, then the ***infinite series*** *$\sum_{k=1}^{\infty} x_k$ is said to converge or to be* ***convergent****, s is called the* ***sum*** *of the infinite series and we write*
$$s = \sum_{k=1}^{\infty} x_k = x_1 + x_2 + \ldots .$$

2. *If $\|x_1\| + \|x_2\| + \ldots$ converges, the series $\sum_{k=1}^{\infty} x_k$ is said to be* ***absolutely convergent****.*

3. *If a normed space X contains a sequence (e_n) with the property that for every $x \in X$, there is a unique sequence of scalars (α_n) such that*
$$\|x - (\alpha_1 e_1 + \ldots + \alpha_n e_n)\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$
then (e_n) is called a ***Schauder basis*** *for X. The series $\sum_{j=1}^{\infty} \alpha_j e_j$ which has the sum x is called the* ***expansion*** *of x with respect to (e_n), and we write*
$$x = \sum_{j=1}^{\infty} \alpha_j e_j.$$

   - *If X has a Schauder basis, then it is separable. The converse is however not generally true.*

Theorem 1.8. *Let $X = (X, \|\cdot\|)$ be a normed space. There exists a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.*

1. Show that $c \subset l^\infty$ is a vector subspace of l^∞ and so is c_0, the space of all sequences of scalars converging to zero.

Solution: The space c consists of all convergent sequences $x = (\xi_j)$ of complex numbers. Choose any $x = (\xi_j), y = (\eta_j) \in c \subset l^\infty$, with limit $\xi, \eta \in \mathbb{C}$ respectively. For fixed scalars α, β, the result is trivial if they are zero, so suppose not. Given any $\varepsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|\xi_j - \xi| < \frac{\varepsilon}{2|\alpha|} \qquad \text{for all } j > N_1.$$
$$|\eta_j - \eta| < \frac{\varepsilon}{2|\beta|} \qquad \text{for all } j > N_2.$$

Choose $N = \max\{N_1, N_2\}$, then for all $j > N$ we have that

$$\begin{aligned} |\alpha\xi_j + \beta\eta_j - \alpha\xi - \beta\eta| &= |\alpha(\xi_j - \xi) + \beta(\eta_j - \eta)| \\ &\leq |\alpha||\xi_j - \xi| + |\beta||\eta_j - \eta|| \\ &< \cancel{|\alpha|}\frac{\varepsilon}{2\cancel{|\alpha|}} + \cancel{|\beta|}\frac{\varepsilon}{2\cancel{|\beta|}} = \varepsilon. \end{aligned}$$

This shows that the sequence $\alpha x + \beta y = (\alpha\xi_j + \beta\eta_j)$ is convergent, hence $x \in c$. Since α, β were arbitrary scalar, this proves that c is a subspace of l^∞. By replacing $\xi = \eta = 0$ as limit, the same argument also shows that c_0 is a subspace of l^∞.

2. Show that c_0 in Problem 1 is a closed subspace of l^∞, so that c_0 is complete.

Solution: Consider any $x = (\xi_j) \in \bar{c}_0$, the closure of c. There exists $x_n = (\xi_j^n) \in c_0$ such that $x_n \longrightarrow x$ in l^∞. Hence, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all j we have

$$|\xi_j^n - \xi_j| \leq \|x_n - x\| < \frac{\varepsilon}{2}.$$

in particular, for $n = N$ and all j. Since $x_N \in c_0$, its terms ξ_j^N form a convergent sequence with limit 0. Thus there exists an $N_1 \in \mathbb{N}$ such that for all $j \geq N_1$ we have

$$|\xi_j^N| < \frac{\varepsilon}{2}.$$

The triangle inequality now yields for all $j \geq N_1$ the following inequality:

$$|\xi_j| \leq |\xi_j - \xi_j^N| + |\xi_j^N| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the sequence $x = (\xi_j)$ is convergent with limit 0. Hence, $x \in c_0$. Since $x \in \bar{c}_0$ was arbitrary, this proves closedness of c_0 in l^∞.

3. In l^∞, let Y be the subset of all sequences with only finitely many nonzero terms. Show that Y is a subspace of l^∞ but not a closed subspace.

Solution: Consider any $x = (\xi_j), y = (\eta_j) \in Y \subset l^\infty$, there exists $N_x, N_y \in \mathbb{N}$ such that $\xi_j = 0$ for all $j > N_x$ and $\eta_j = 0$ for all $j > N_y$. Thus for any scalars α, β, $\alpha\xi_j + \beta\eta_j = 0$ for all $j > N = \max\{N_x, N_y\}$, and $\alpha x + \beta y \in Y$. This shows that Y is a subspace of l^∞. However, Y is not a closed subspace. Indeed, consider a sequence $x_n = (\xi_j^n) \in Y$ defined by

$$\xi_j^n = \begin{cases} \dfrac{1}{j} & \text{if } j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

Let $x = (\xi_j) = \left(\dfrac{1}{j}\right)$, then $x_n \longrightarrow x$ in l^∞ since

$$\|x_n - x\|_{l^\infty} = \sup_{j>n} |\xi_j| = \frac{1}{n+1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

but $x \notin Y$ since x has infinitely many nonzero terms.

4. **(Continuity of vector space operations)** Show that in a normed space X, vector addition and multiplication by scalars are continuous operation with respect to the norm; that is, the mappings defined by $(x, y) \mapsto x+y$ and $(\alpha, x) \mapsto \alpha x$ are continuous.

Solution: Consider any pair of points $(x_0, y_0) \in X \times X$. Given any $\varepsilon > 0$, choose $\delta_1 = \delta_2 = \dfrac{\varepsilon}{2} > 0$. Then for all x satisfying $\|x - x_0\| < \delta_1$ and all y satisfying $\|y - y_0\| < \delta_2$,

$$\|x + y - (x_0 + y_0)\| \leq \|x - x_0\| + \|y - y_0\| < \delta_1 + \delta_2 = \varepsilon.$$

Since $(x_0, y_0) \in X \times X$ was arbitrary, the mapping defined by $(x, y) \mapsto x + y$ is continuous with respect to the norm.

Choose any scalar α_0. Consider any nonzero $x_0 \in X$. Given any $\varepsilon > 0$, choose $\delta_1 = \dfrac{\varepsilon}{2\|x_0\|} > 0$ and $\delta_2 > 0$ such that $(\delta_1 + |\alpha_0|)\delta_2 = \dfrac{\varepsilon}{2}$. Then for all α satisfying $\|\alpha - \alpha_0\| < \delta_1$ and all x satisfying $\|x - x_0\| < \delta_2$,

$$\begin{aligned} \|\alpha x - \alpha_0 x_0\| &= \|\alpha x - \alpha x_0 + \alpha x_0 - \alpha_0 x_0\| \\ &\leq |\alpha|\|x - x_0\| + |\alpha - \alpha_0|\|x_0\| \\ &\leq \Big(|\alpha - \alpha_0| + |\alpha_0|\Big)\|x - x_0\| + |\alpha - \alpha_0|\|x_0\| \\ &< (\delta_1 + |\alpha_0|)\delta_2 + \delta_1\|x_0\| = \varepsilon. \end{aligned}$$

If $x_0 = \mathbf{0} \in X$, choose $\delta_1 = 1 > 0$ and $\delta_2 = \dfrac{\varepsilon}{1+|\alpha_0|} > 0$. Then for all α satisfying $|\alpha - \alpha_0| < \delta_1$ and all x satisfying $\|x\| < \delta_2$,

$$\begin{aligned}\|\alpha x\| = |\alpha|\|x\| &\leq \Big(|\alpha - \alpha_0| + |\alpha_0|\Big)\|x\| \\ &< \Big(\delta_1 + |\alpha_0|\Big)\delta_2 \\ &= \cancel{\Big(1 + |\alpha_0|\Big)}\frac{\varepsilon}{\cancel{1+|\alpha_0|}} = \varepsilon.\end{aligned}$$

Since α_0 and x_0 were arbitrary scalars and vectors in K and X, the mapping defined by $(\alpha, x) \mapsto \alpha x$ is continuous with respect to the norm.

5. Show that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ implies $x_n + y_n \longrightarrow x + y$. Show that $\alpha_n \longrightarrow \alpha$ and $x_n \longrightarrow x$ implies $\alpha_n x_n \longrightarrow \alpha x$.

Solution: If $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then

$$\|x_n + y_n - x - y\| \leq \|x_n - x\| + \|y_n - y\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

If $\alpha_n \longrightarrow \alpha$ and $x_n \longrightarrow x$, then

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \\ &= \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\| \\ &\leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\| \\ &\leq C\underbrace{\|x_n - x\|}_{\longrightarrow 0} + \underbrace{|\alpha_n - \alpha|}_{\longrightarrow 0}\|x\|\end{aligned}$$

where we use the fact that convergent sequences are bounded for the last inequality.

6. Show that the closure $\bar{Y}$ of a subspace Y of a normed space X is again a vector subspace.

Solution: If $x, y \in \bar{Y}$, there exists sequences $x_n, y_n \in Y$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Then for any scalars α, β,

$$\|\alpha x_n + \beta y_n - (\alpha x + \beta y)\| \leq |\alpha|\|x_n - x\| + |\beta|\|y_n - y\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

This shows that the sequence $(\alpha x_n + \beta y_n) \in Y$ converges to $\alpha x + \beta y$, which implies that $\alpha x + \beta y \in \bar{Y}$.

7. **(Absolute convergence)** Show that convergence of $\|y_1\| + \|y_2\| + \|y_3\| + \dots$ may not imply convergence of $y_1 + y_2 + y_3 + \dots$.

> **Solution:** Let Y be the set of all sequences in l^∞ with only finitely many nonzero terms, which is a normed space. Consider the sequence $(y_n) = (\eta_j^n) \in Y$ defined by
>
> $$\eta_j^n = \begin{cases} \dfrac{1}{j^2} & \text{if } j = n, \\ 0 & \text{if } j \neq n. \end{cases}$$
>
> Then $\|y_n\| = \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. However, $y_1 + y_2 + y_3 + \dots \longrightarrow y$, where $y = \left(\dfrac{1}{n^2}\right)$, since
>
> $$\left\|\sum_{j=1}^{n} y_j - y\right\| = \frac{1}{(n+1)^2} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$
>
> but $y \notin Y$.

8. If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

> **Solution:** Choose (x_n) be any Cauchy sequence in X. Given any $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that for all $m, n \geq N_k$, $\|x_m - x_n\| < \dfrac{1}{2^k}$; by construction, (N_k) is an increasing sequence. Consider the sequence (y_k) defined by $y_k = x_{N_{k+1}} - x_{N_k}$. Then
>
> $$\sum_{k=1}^{\infty} \|y_k\| = \sum_{k=1}^{\infty} \|x_{N_{k+1}} - x_{N_k}\| < \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$
>
> This shows that the series $\displaystyle\sum_{k=1}^{\infty} y_k$ is absolute convergent, which is also convergent by assumption. Thus,
>
> $$\begin{aligned} \sum_{k=1}^{\infty} y_k &= \lim_{n\to\infty} \sum_{k=1}^{n} y_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} x_{N_{k+1}} - x_{N_k} \\ &= \lim_{n\to\infty} x_{N_{n+1}} - x_{N_1} < \infty. \end{aligned}$$
>
> Hence, $(x_{N_{n+1}})$ is a convergent subsequence of (x_n), and since (x_n) is a Cauchy sequence, (x_n) is convergent. Since (x_n) was an arbitrary Cauchy sequence, X is complete.

9. Show that in a Banach space, an absolutely convergent series is convergent.

Solution: Let $\sum_{k=1}^{\infty} \|x_k\|$ be any absolutely convergent series in a Banach space X. Since a Banach space is a complete normed space, it suffices to show that the sequence (s_n) of partial sums $s_n = x_1 + x_2 + \ldots + x_n$ is Cauchy. Given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} \|x_k\| < \varepsilon.$$

For any $m > n > N$,

$$\begin{aligned} \|s_m - s_n\| &= \|x_{n+1} + x_{n+2} + \ldots + x_m\| \\ &\leq \|x_{n+1}\| + \|x_{n+2}\| + \ldots + \|x_m\| \\ &= \sum_{k=n+1}^{m} \|x_k\| \\ &\leq \sum_{k=n+1}^{\infty} \|x_k\| \\ &\leq \sum_{k=N+1}^{\infty} \|x_k\| < \varepsilon. \end{aligned}$$

This shows that (s_n) is Cauchy and the desired result follows from completeness of X.

10. **(Schauder basis)** Show that if a normed space has a Schauder basis, it is separable.

Solution: Suppose X has a Schauder basis (e_n). Given any $x \in X$, there exists a unique sequence of scalars $(\lambda_n) \in K$ such that

$$\|x - (\lambda_1 e_1 + \ldots + \lambda_n e_n)\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Consider the sequence $(f_n) \subset X$ defined by $f_n = \dfrac{e_n}{\|e_n\|}$. Note that $\|f_n\| = 1$ for all $n \geq 1$ and (f_n) is a Schauder basis for X; indeed, if we choose $\mu_j = \lambda_j \|e_j\| \in K$, then

$$\left\| x - \sum_{j=1}^{n} \mu_j f_j \right\| = \left\| x - \sum_{j=1}^{n} \lambda_j e_j \right\| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

In particular, for any $x \in X$, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left\| x - \sum_{j=1}^{n} \mu_j f_j \right\| < \frac{\varepsilon}{2} \qquad \text{for all } n > N.$$

Define M to be the set

$$M = \left\{ \sum_{j=1}^{n} \theta_j f_j \colon \theta_j \in \tilde{K}, n \in \mathbb{N} \right\}.$$

where $\tilde{K}$ is a countable dense subset of K. Since $\mu_j \in K$, given any $\varepsilon > 0$, there exists an $\theta_j \in \tilde{K}$ such that $|\mu_j - \theta_j| < \dfrac{\varepsilon}{2n}$ for all $j = 1, \ldots, n$. Then

$$\begin{aligned}
\left\| x - \sum_{j=1}^{n} \theta_j f_j \right\| &\leq \left\| x - \sum_{j=1}^{n} \mu_j f_j \right\| + \left\| \sum_{j=1}^{n} \mu_j f_j - \sum_{j=1}^{n} \theta_j f_j \right\| \\
&< \frac{\varepsilon}{2} + \sum_{j=1}^{n} |\mu_j - \theta_j| \|f_j\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\cancel{n}} \cancel{\sum_{j=1}^{n}} 1 \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This shows that there exists an $y \in M$ in any ε-neighbourhood of x. Since $x \in X$ was arbitrary, M is a countable dense subset of X and X is separable.

11. Show that (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for l^p, where $1 \leq p < +\infty$.

Solution: Let $x = (\xi_j)$ be any sequence in l^p, we have

$$\left(\sum_{j=n+1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Now choose a sequence of scalars $(\lambda_n) \in \mathbb{C}$ defined by $\lambda_j = \xi_j$,

$$\|x - (\lambda_1 e_1 + \ldots + \lambda_n e_n)\| = \left(\sum_{j=n+1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

This shows that $(e_n) = (\delta_{nj})$ is a Schauder basis for l^p. Uniqueness?

12. **(Seminorm)** A *seminorm* on a vector space X is a mapping $p \colon X \longrightarrow \mathbb{R}$ satisfying (N1), (N3), (N4). (Some authors call this a *pseudonorm*.) Show that

$$p(0) = 0, \qquad |p(y) - p(x)| \leq p(y - x).$$

(Hence if $p(x) = 0 \implies x = \mathbf{0}$, then p is a norm.)

Solution: Using (N3),

$$p(\mathbf{0}) = p(0x) = 0p(x) = 0.$$

Using (N4), for any $x, y \in X$,

$$\begin{aligned}
&p(y) \leq p(y-x) + p(x). \\
&p(x) \leq p(x-y) + p(y) = p(y-x) + p(y). \\
&\implies |p(y) - p(x)| \leq p(y-x).
\end{aligned}$$

13. Show that in Problem 12, the elements $x \in X$ such that $p(x) = 0$ form a subspace N of X and a norm on X/N is defined by $\|\hat{x}\|_0 = p(x)$, where $x \in \hat{x}$ and $\hat{x} \in X/N$.

Solution: Let N be the set consisting of all elements $x \in X$ such that $p(x) = 0$. For any $x, y \in N$ and scalars α, β,

$$0 \leq p(\alpha x + \beta y) \leq p(\alpha x) + p(\beta y) = |\alpha|\cancel{p(x)} + |\beta|\cancel{p(y)} = 0.$$

This shows that N is a subspace of X. Consider $\|\hat{x}\|_0 = p(x)$, where $x \in \hat{x}$ and $\hat{x} \in X/N$. We start by showing $\|\cdot\|_0$ is well-defined. Indeed, for any $u, v \in \hat{x}$, there exists $n_u, n_v \in N$ such that $u = x + n_u$ and $v = x + n_v$. Since N is a subspace of X,

$$0 \leq |p(u) - p(v)| \leq |p(u-v)| = |p(n_u - n_v)| = 0.$$

- $\|\hat{x}\|_0 = p(x) \geq 0$.
- Suppose $\hat{x} = \hat{\mathbf{0}} = N$, then $\|\hat{x}\|_0 = p(x) = 0$. Now suppose $\|\hat{x}\|_0 = 0$, then $p(x) = 0 \implies x \in N \implies \hat{x} = \hat{\mathbf{0}}$. Thus, (N2) is satisfied.
- For any nonzero scalars α, any $y \in \alpha\hat{x}$ can be written $y = \alpha x + n$ for some $n \in N$. Thus,

  $$\begin{aligned}
  \|\alpha\hat{x}\|_0 = p(\alpha x + n) &= |\alpha| p\left(x + \frac{n}{\alpha}\right) \\
  &= |\alpha|\|\hat{x}\|_0.
  \end{aligned}$$

  If $\alpha = 0$, then $0\hat{x} = N$ and $\|0\hat{x}\|_0 = 0$ by definition of N.
- Lastly, for any $\hat{x}, \hat{y} \in X/N$,

  $$\begin{aligned}
  \|\hat{x} + \hat{y}\|_0 = p(x+y) &\leq p(x) + p(y) \\
  &= \|\hat{x}\|_0 + \|\hat{y}\|_0.
  \end{aligned}$$

  Thus, (N4) is satisfied.

14. **(Quotient space)** Let Y be a closed subspace of a normed space $(X, \|\cdot\|)$. Show that a norm $\|\cdot\|_0$ on X/Y is defined by

$$\|\hat{x}\|_0 = \inf_{x\in\hat{x}} \|x\|.$$

where $\hat{x} \in X/Y$, that is, $\hat{x}$ is any coset of Y.

Solution: Define $\|\hat{x}\|_0$ as above. Also, recall that $X/Y = \{\hat{x} = x + Y : x \in X\}$ and its algebraic operations are defined by

$$\hat{u} + \hat{v} = (u + Y) + (v + Y) = (u + v) + Y = \widehat{u+v}.$$
$$\alpha\hat{u} = \alpha(u + Y) = \alpha u + Y = \widehat{\alpha u}.$$

- (N1) is obvious.
- If $\hat{x} = \hat{\mathbf{0}} = Y$, then $\|\hat{x}\|_0$=0 since $\mathbf{0} \in Y$. Conversely, suppose $\|\hat{x}\|_0 = \inf_{x\in\hat{x}} \|x\| = 0$. Properties of infimum gives that there exists a minimising sequence $(x_n) \in \hat{x}$ such that $\|x_n\|_0 \longrightarrow 0$, with limit $x = \mathbf{0}$. Since Y is closed, any $\hat{x} \in X/Y$ is closed, this implies that $\mathbf{0} \in \hat{x}$, and $\hat{x} = \hat{\mathbf{0}}$. Thus, (N2) is satisfied.
- For any nonzero scalars α,

$$\begin{aligned}\|\alpha\hat{x}\|_0 &= \inf_{y\in Y} \|\alpha x + y\| \\ &= |\alpha| \inf_{y\in Y} \left\|x + \frac{y}{\alpha}\right\| \\ &= |\alpha| \inf_{y\in Y} \|x + y\| \\ &= |\alpha|\|\hat{x}\|_0.\end{aligned}$$

  If $\alpha = 0$, then $\|0\hat{x}\|_0 = \|\widehat{0x}\|_0 = \|\hat{\mathbf{0}}\|_0 = 0 = 0\|\hat{x}\|_0$. Thus, (N3) is satisfie.d

- For any $\hat{u}, \hat{v} \in X/Y$,

$$\begin{aligned}\|\hat{u} + \hat{v}\|_0 &= \inf_{y_1,y_2\in Y} \|u + y_1 + v + y_2\| \\ &\le \inf_{y_1,y_2\in Y} \|u + y_1\| + \|v + y_2\| \\ &= \inf_{y_1\in Y} \|u + y_1\| + \inf_{y_2\in Y} \|v + y_2\| \\ &= \|\hat{u}\|_0 + \|\hat{v}\|_0.\end{aligned}$$

15. **(Product of normed spaces)** If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed spaces, show that the product vector space $X = X_1 \times X_2$ becomes a normed space if we define

$$\|x\| = \max\left\{\|x_1\|_1, \|x_2\|_2\right\}, \qquad \text{where } x = (x_1, x_2).$$

Solution: (N1) to (N3) are obvious. To verify (N4), for $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$,

$$\begin{aligned}
\|x + y\| &= \max\left\{\|x_1 + y_1\|_1, \|x_2 + y_2\|_2\right\} \\
&\leq \max\left\{\|x_1\|_1 + \|y_1\|_1, \|x_2\|_2 + \|y_2\|_2\right\} \\
&\leq \max\left\{\|x_1\|_1, \|x_2\|_2\right\} + \max\left\{\|y_1\|_1, \|y_2\|_2\right\} \\
&= \|x\| + \|y\|.
\end{aligned}$$

where we use the inequality $|a| \leq \max\{|a|, |b|\}$ for any $a, b \in \mathbb{R}$.

1.4 Finite Dimensional Normed Spaces and Subspaces.

Lemma 1.9. *Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). There exists a number $c > 0$ such that for every choice of scalars $\alpha_1, \ldots, \alpha_n$ we have*

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq c\Big(|\alpha_1| + \ldots + |\alpha_n|\Big).$$

- Roughly speaking, it states that in the case of linear independence of vectors, we cannot find a linear combination that involves large scalars but represents a small vector.

Theorem 1.10. *Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.*

Theorem 1.11. *Every finite dimensional subspace Y of a normed space X is closed in X.*

Definition 1.12. *A norm $\|\cdot\|$ on a vector space X is said to be* ***equivalent*** *to a norm $\|\cdot\|_0$ on X if there are positive constants a and b such that for all $x \in X$ we have*

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0.$$

- Equivalent norms on X define the same topology for X.

Theorem 1.13. *On a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.*

1. Given examples of subspaces of l^∞ and l^2 which are not closed.

   Solution:

2. What is the largest possible c in (1) if
   (a) $X = \mathbb{R}^2$ and $x_1 = (1, 0), x_2 = (0, 1)$,

   Solution:

   (b) $X = \mathbb{R}^3$ and $x_1 = (1, 0, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1)$.

   Solution:

3. Show that in the definition of equivalance of norms, the axioms of an equivalence relation hold.

Solution: We say that $\|\cdot\|$ on X is equivalent to $\|\cdot\|_0$ on X, denoted by $\|\cdot\| \sim \|\cdot\|_0$ if there exists positive constants $a, b > 0$ such that for all $x \in X$ we have

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0.$$

- Reflexivity is immediate.
- Suppose $\|\cdot\| \sim \|\cdot\|_0$. There exists $a, b > 0$ such that for all $x \in X$ we have

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0 \implies \frac{1}{b}\|x\| \leq \|x\|_0 \leq \frac{1}{a}\|x\|.$$

  This shows that $\|\cdot\|_0 \sim \|\cdot\|$, and symmetry is shown.
- Suppose $\|\cdot\| \sim \|\cdot\|_0$ and $\|\cdot\|_0 \sim \|\cdot\|_1$. There exists $a, b, c, d > 0$ such that for all $x \in X$ we have

$$\begin{aligned} a\|x\|_0 \leq \|x\| &\leq b\|x\|_0. \\ c\|x\|_1 \leq \|x\|_0 &\leq d\|x\|_1. \end{aligned}$$

  On one hand,

$$\|x\| \leq b\|x\|_0 \leq bd\|x\|_1.$$

  On the other hand,

$$\|x\| \geq a\|x\|_0 \geq ac\|x\|_1.$$

  Combining them yields for all $x \in X$

$$ac\|x\|_1 \leq \|x\|_0 \leq bd\|x\|_1.$$

  This shows tht $\|\cdot\| \sim \|\cdot\|_1$, and transitivity is shown.

4. Show that equivalent norms on a vector space X induce the same topology for X.

Solution: Suppose $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms on a vector space X. There exists positive constants $a, b > 0$ such that for all $x \in X$ we have

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0.$$

To show that they induce the same topology for X, we want to show that the open sets in $(X, \|\cdot\|)$ and $(X, \|\cdot\|_0)$ are the same. Consider the identity map

$$I : (X, \|\cdot\|) \longrightarrow (X, \|\cdot\|_0).$$

Let x_0 be any point in X. Given any $\varepsilon > 0$, choose $\delta = a\varepsilon > 0$. For all x satisfying $\|x - x_0\| < \delta$, we have

$$\|x - x_0\|_0 \leq \frac{1}{a}\|x - x_0\| < \frac{\cancel{a}\varepsilon}{\cancel{a}} = \varepsilon.$$

Since $x_0 \in X$ is arbitrary, this shows that I is continuous. Hence, if $M \subset X$ is open in $(X, \|\cdot\|_0)$, its preimage M again is also open in $(X, \|\cdot\|)$. Similarly, consider the identity map

$$\tilde{I}\colon (X, \|\cdot\|_0) \longrightarrow (X, \|\cdot\|).$$

Let x_0 be any point in X. Given any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{b} > 0$. For all x satisfying $\|x - x_0\|_0 < \delta$, we have

$$\|x - x_0\| \leq b\|x - x_0\|_0 < \frac{\cancel{b}\varepsilon}{\cancel{b}} = \varepsilon.$$

Since $x_0 \in X$ is arbitrary, this shows that $\tilde{I}$ is continuous. Hence, if $M \subset X$ is open in $(X, \|\cdot\|)$, its preimage M is also open in $(X, \|\cdot\|_0)$.

<u>*Remark*</u>: *The converse is also true, i.e. if two norms $\|\cdot\|$ and $\|\cdot\|_0$ on X give the same topology, then they are equivalent norms on X.*

5. If $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms on X, show that the Cauchy sequences in $(X, \|\cdot\|)$ and $(X, \|\cdot\|_0)$ are the same.

Solution: Suppose $\|\cdot\|$ on X is equivalent to $\|\cdot\|_0$ on X. There exists positive constants $a, b > 0$ such that for all $x \in X$ we have

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0.$$

Let (x_n) be any Cauchy sequence in $(X, \|\cdot\|)$. Given any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\|x_m - x_n\| < a\varepsilon \qquad \text{for all } m, n > N_1.$$

which implies

$$\|x_m - x_n\|_0 \leq \frac{1}{a}\|x_m - x_n\| < \frac{\cancel{a}\varepsilon}{\cancel{a}} = \varepsilon. \qquad \text{for all } m, n > N_1.$$

This shows that (x_n) is also a Cauchy sequence in $(X, \|\cdot\|_0)$. Conversely, let (x_n) be any Cauchy sequence in $(X, \|\cdot\|_0)$. Given any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\|x_m - x_n\|_0 < \frac{\varepsilon}{b} \qquad \text{for all } m, n > N_1.$$

which implies

$$\|x_m - x_n\| \leq b\|x_m - x_n\|_0 < \frac{\cancel{b}\varepsilon}{\cancel{b}} = \varepsilon \qquad \text{for all } m, n > N_2.$$

This shows that (x_n) is also a Cauchy sequence in $(X, \|\cdot\|)$.

6. Theorem 2.4.5 implies that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Give a direct proof of this fact.

Solution: Let $X = \mathbb{R}^n$, and $x = (\xi_j)$ be any element of X. On one hand,

$$\|x\|_\infty^2 = \Big(\max_{j=1,\ldots,n} |\xi_j|\Big)^2 \leq |\xi_1|^2 + \ldots + |\xi_n|^2 = \|x\|_2^2.$$

Taking square roots of both sides yields $\|x\|_\infty \leq \|x\|_2$. On the other hand,

$$\|x\|_2^2 = |\xi_1|^2 + \ldots + |\xi_n|^2 \leq n\Big(\max_{j=1,\ldots,n} |\xi_j|^2\Big) = n\|x\|_\infty^2.$$

Taking square roots of both sides yields $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$. Hence, combining these inequalities gives for all $x \in X$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

7. Let $\|\cdot\|_2$ be as in Problem 8, section 2.2, and let $\|\cdot\|$ be any norm on that vector space, call it X. Show directly that there is a $b > 0$ such that $\|x\| \leq b\|x\|_2$ for all x.

Solution: Let $X = \mathbb{R}^n$, and $\{e_1, \ldots, e_n\}$ be the standard basis of X defined by $\xi_j^n = \delta_{jn}$. Any $x = (\xi_j)$ in X has a unique representation $x = \xi_1 e_1 + \ldots + \xi_2 e_2$. Thus,

$$\begin{aligned}
\|x\| = \|\xi_1 e_1 + \ldots + \xi_n e_n\| &\leq |\xi_1|\|e_1\| + \ldots + |\xi_n|\|e_n\| \\
&= \sum_{j=1}^n |\xi_j|\|e_j\| \\
&\leq \left(\sum_{j=1}^n |\xi_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|e_j\|^2\right)^{\frac{1}{2}} \\
&= b\|x\|_2.
\end{aligned}$$

where we use **Cauchy-Schwarz inequality for sums** in the last inequality. Since $x \in X$ was arbitrary, the result follows.

8. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ satisfy

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

Solution: Let $X = \mathbb{R}^n$, and $x = (\xi_j)$ be any element of X. Note that if $x = \mathbf{0}$, the inequality is trivial since $\|x\|_1 = \|x\|_2 = 0$ by definition of a norm. So, pick any nonzero $x \in \mathbb{R}^n$. Using Cauchy-Schwarz inequality for sums,

$$\|x\|_1 = \sum_{j=1}^n |\xi_j| \leq \left(\sum_{j=1}^n 1^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n |\xi_j|^2\right)^{\frac{1}{2}} = \sqrt{n}\|x\|_2.$$

On the other hand, since $\|x\|_2 \neq 0$, define $y = (\eta_j)$, where $\eta_j = \dfrac{\xi_j}{\|x\|_2}$. Then

$$\begin{aligned}
\|y\|_2 = \left(\sum_{j=1}^n |\eta_j|^2\right)^{\frac{1}{2}} &= \left(\frac{1}{\|x\|_2^2}\sum_{j=1}^n |\xi_j|^2\right)^{\frac{1}{2}} \\
&= \frac{1}{\|x\|_2}\|x\|_2 = 1. \\
\|y\|_1 = \left(\sum_{j=1}^n |\eta_j|\right) &= \frac{1}{\|x\|_2}\sum_{j=1}^n |\xi_j| \\
&= \frac{\|x\|_1}{\|x\|_2}.
\end{aligned}$$

and

$$\begin{aligned}
\|y\|_2^2 = \sum_{j=1}^n |\eta_j|^2 \leq \left[\max_{i=1,\dots,n} |\eta_i|\right] \sum_{j=1}^n |\eta_j| &\leq \|y\|_1 \\
\implies 1 \leq \|y\|_1 &= \frac{\|x\|_1}{\|x\|_2} \\
\implies \|x\|_2 \leq \|x\|_1&.
\end{aligned}$$

To justfiy the second inequality on the first line, note that it suffices to prove that $|\eta_i| \leq 1$ for all $i = 1, \dots, n$, or equivalently, $|\xi_i| \leq \|x\|_2$ for all $i = 1, \dots, n$. From the definition of $\|\cdot\|_2$,

$$\|x\|_2^2 = \sum_{j=1}^n |\xi_j|^2 \geq |\xi_i|^2 \qquad \text{for all } i = 1, \dots, n.$$

Taking square roots of both sides yields $\|x\|_2 \geq |\xi_i|$ for all $i = 1, \dots, n$.

<u>Remark</u>: *Alternatively,*

$$\begin{aligned}
\|x\|_1^2 = \left(\sum_{j=1}^n |\xi_j|\right)^2 &= \left(\sum_{j=1}^n |\xi_j|^2\right) + \left(\sum_{i\neq j}^n |\xi_i||\xi_j|\right) \\
&\geq \left(\sum_{j=1}^n |\xi_j|^2\right) = \|x\|_2^2.
\end{aligned}$$

9. If two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a vector space X are equivalent, show that

$$\|x_n - x\| \longrightarrow 0 \Longleftrightarrow \|x_n - x\|_0 \longrightarrow 0.$$

Solution: Suppose two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a vector space X are equivalent, there exists positive constant $a, b > 0$ such that for all $x \in X$ we have

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0.$$

If $\|x_n - x\| \longrightarrow 0$, then

$$\|x_n - x\|_0 \leq \frac{1}{a}\|x_n - x\| \longrightarrow 0 \qquad \text{as } n \longrightarrow 0.$$

Conversely, if $\|x_n - x\|_0 \longrightarrow 0$, then

$$\|x_n - x\| \leq b\|x_n - x\|_0 \longrightarrow 0 \qquad \text{as } n \longrightarrow 0.$$

10. Show that all complex $m \times n$ matrices $A = (\alpha_{jk})$ with fixed m and n constitute an mn-dimensional vector space Z. Show that all norms on Z are equivalent. What would be the analogues of $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for the present space Z?

Solution:

1.5 Linear Operators.

Definition 1.14. *A **linear operator** T is an operator such that*

(a) the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field,

(b) for all $x, y \in \mathcal{D}(T)$ and scalars α, β,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

- Note that the above formula expresses the fact that a linear operator T is a **homomorphism** of a vector space (its domain) into another vector space, that is, T preserves the two operations of a vector space.
- On the LHS, we first apply a vector space operation (addition or scalar multiplication) and then map the resulting vector into Y, whereas on the RHS we first map x and y into Y and then perform the vector space operations in Y, the outcome being the same.

Theorem 1.15. *Let T be a linear operator. Then:*

(a) The range $\mathcal{R}(T)$ is a vector space.

(b) If $\dim \mathcal{D}(T) = n < \infty$, then $\dim \mathcal{R}(T) \leq n$.

(c) The null space $\mathcal{N}(T)$ is a vector space.

- An immediate consequence of part (b) is worth noting: *Linear operators preserve linear dependence.*

Theorem 1.16. *Let X, Y be a vector spaces, both real or both complex. Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. Then:*

(a) The inverse $T^{-1}\colon \mathcal{R}(T) \longrightarrow \mathcal{D}(T)$ exists if and only if $Tx = \mathbf{0} \implies x = \mathbf{0}$.

(b) If T^{-1} exists, it is a linear operator.

(c) If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathcal{R}(T) = \dim \mathcal{D}(T)$.

Theorem 1.17. *Let $T\colon X \longrightarrow Y$ and $S\colon Y \longrightarrow Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then the inverse $(ST)^{-1}\colon Z \longrightarrow X$ of the composition ST exists, and*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

1. Show that the identity operator, the zero operator and the differentiation operator (on polynomials) are linear.

Solution: For any scalars α, β and $x, y \in X$,

$$\begin{aligned} I_X(\alpha x + \beta y) &= \alpha x + \beta y = \alpha I_X x + \beta I_X y. \\ \mathbf{0}(\alpha x + \beta y) &= \mathbf{0} = \alpha \mathbf{0} x + \beta \mathbf{0} y. \\ T(\alpha x(t) + \beta y(t)) &= (\alpha x(t) + \beta y(t))' \\ &= \alpha x'(t) + \beta y'(t) = \alpha T x(t) + \beta T y(t). \end{aligned}$$

2. Show that the operators $T_1, \dots, T_4$ from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$\begin{aligned} T_1 &\colon (\xi_1, \xi_2) \mapsto (\xi_1, 0) \\ T_2 &\colon (\xi_1, \xi_2) \mapsto (0, \xi_2) \\ T_3 &\colon (\xi_1, \xi_2) \mapsto (\xi_2, \xi_1) \\ T_4 &\colon (\xi_1, \xi_2) \mapsto (\gamma\xi_1, \gamma\xi_2) \end{aligned}$$

respectively, are linear, and interpret these operators geometrically.

Solution: Denote $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$. For any scalars α, β,

$$\begin{aligned} T_1(\alpha x + \beta y) &= (\alpha\xi_1 + \beta\eta_1, 0) \\ &= \alpha(\xi_1, 0) + \beta(\eta_1, 0) = \alpha T_1(x) + \beta T_1(y). \\ T_2(\alpha x + \beta y) &= (0, \alpha\xi_2 + \beta\eta_2) \\ &= \alpha(0, \xi_2) + \beta(0, \eta_2) = \alpha T_2(x) + \beta T_2(y). \\ T_3(\alpha x + \beta y) &= (\alpha\xi_2 + \beta\eta_2, \alpha\xi_1 + \beta\eta_1) \\ &= (\alpha\xi_2, \alpha\xi_1) + (\beta\eta_2, \beta\eta_1) \\ &= \alpha(\xi_2, \xi_1) + \beta(\eta_2, \eta_1) = \alpha T_3(x) + \beta T_3(y). \\ T_4(\alpha x + \beta y) &= \Big(\gamma(\alpha\xi_1 + \beta\eta_1), \gamma(\alpha\xi_2 + \beta\eta_2)\Big) \\ &= (\alpha\gamma\xi_1, \alpha\gamma\xi_2) + (\beta\gamma\eta_1, \beta\gamma\eta_2) \\ &= \alpha(\gamma\xi_1, \gamma\xi_2) + \beta(\gamma\eta_1, \gamma\eta_2) = \alpha T_4(x) + \beta T_4(y). \end{aligned}$$

T_1 and T_2 are both projection to x-axis and y-axis respectively, while T_4 is a scaling transformation. T_3 first rotates the vector 90° anti-clockwise about the origin, then reflects across the y-axis.

3. What are the domain, range and null space of T_1, T_2, T_3 in Problem 2?

Solution: The domain of T_1, T_2, T_3 is $\mathbb{R}^2$, and the range is the x-axis for T_1, the y-axis for T_2 and $\mathbb{R}^2$ for T_3. The null space is the line $\xi_1 = 0$ for T_1, the line $\xi_2 = 0$ for T_2 and the origin $(0, 0)$ for T_3.

4. What is the null space of T_4 in Problem 2? Of T_1 and T_2 in 2.6-7? Of T in 2.6-4?

Solution: The null space of T_4 is $\mathbb{R}^2$ if $\gamma = 0$ and the origin $(0,0)$ if $\gamma \neq 0$. Fix a vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$. Consider the linear operators T_1 and T_2 defined by

$$T_1 x = x \times a = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \xi_2 a_3 - \xi_3 a_2 \\ \xi_3 a_1 - \xi_1 a_3 \\ \xi_1 a_2 - \xi_2 a_1 \end{bmatrix}.$$

$$T_2 x = x \cdot a = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3.$$

i.e. T_1 and T_2 are the cross product and the dot product with the fixed vector a respectively. The null space of T_1 is any scalar multiple of the vector a, while the null space of T_2 is the plane $\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 = 0$ in $\mathbb{R}^3$. For the differentiation operator, the null space is any constant functions $x(t)$ for $t \in [a,b]$.

5. Let $T\colon X \longrightarrow Y$ be a linear operator.

   (a) Show that the image of a subspace V of X is a vector space.

   Solution: Denote the image of a subspace V of X under T by $\text{Im}(V)$, it suffices to show that $\text{Im}(V)$ is a subspace of X. Choose any $y_1, y_2 \in \text{Im}(V)$, there exists $x_1, x_2 \in V$ such that $Tx_1 = y_1$ and $Tx_2 = y_2$. For any scalars α, β,

   $$\alpha y_1 + \beta y_2 = \alpha T x_1 + \beta T x_2 = T(\alpha x_1 + \beta x_2).$$

   This shows that $\alpha y_1 + \beta y_2 \in \text{Im}(V)$ since $\alpha x_1 + \beta x_2 \in V$ due to V being a subspace of X.

   (b) Show that the inverse image of a subspace W of Y is a vector space.

   Solution: Denote the inverse image of a subspace W of Y under T by $\text{PIm}(W)$, it suffices to show that $\text{PIm}(W)$ is a subspace of Y. Choose any $x_1, x_2 \in \text{PIm}(W)$, there exists $y_1, y_2 \in W$ such that $Tx_1 = y_1$ and $Tx_2 = y_2$. For any scalars α, β,

   $$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha y_1 + \beta y_2.$$

   This shows that $\alpha x_1 + \beta x_2 \in \text{PIm}(W)$ since $\alpha y_1 + \beta y_2 \in W$ due to W being a subspace of Y.

6. If the composition of two linear operators exists, show that it is linear.

Solution: Consider any two linear operators $T\colon X \longrightarrow Y$, $S\colon Y \longrightarrow Z$. For any $x, y \in X$ and scalars α, β,

$$ST(\alpha x + \beta y) = S(\alpha T x + \beta T y) \qquad \Big[\text{ by linearity of } T.\Big]$$

$$= \alpha(ST)x + \beta(ST)y \quad \Big[\text{by linearity of } S.\Big]$$

7. **(Commutativity)** Let X be any vector space and $S\colon X \longrightarrow X$ and $T\colon X \longrightarrow X$ any operators. S and T are said to **commute** if $ST = TS$, that is, $(ST)x = (TS)x$ for all $x \in X$. Do T_1 and T_3 in Problem 2 commute?

Solution: No. Choose $x = (1, 2)$, then

$$(T_1T_3)(1, 2) = T_1(2, 1) = (2, 0).$$
$$(T_3T_1)(1, 2) = T_3(1, 0) = (0, 1).$$

8. Write the operators in Problem 2 using 2×2 matrices.

Solution:

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad T_4 = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}.$$

9. In 2.6-8, write $y = Ax$ in terms of components, show that T is linear and give examples.

Solution: For any $j = 1, \ldots, r$, we have that $\eta_j = \displaystyle\sum_{k=1}^{n} a_{jk}\xi_k = a_{j1}\xi_1 + \ldots + a_{jn}\xi_n$. To see that T is linear, for any $j = 1, \ldots, r$,

$$\Big(A(\alpha x + \beta y)\Big)_j = \sum_{k=1}^{n} a_{jk}(\alpha\xi_k + \beta\eta_k)$$
$$= \alpha \sum_{k=1}^{n} a_{jk}\xi_k + \beta \sum_{k=1}^{n} a_{jk}\eta_k = \alpha(Ax)_j + \beta(Ay)_j.$$

10. Formulate the condition in 2.6-10(a) in terms of the null space of T.

Solution: Let X, Y be vector spaces, both real or both complex. Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator with domain $\mathcal{D}(T) \subset X$ and $\mathcal{R}(T) \subset Y$. The inverse $T^{-1}\colon \mathcal{R}(T) \longrightarrow \mathcal{D}(T)$ exists if and only if the null space of T, $\mathcal{N}(T) = \{\mathbf{0}\}$.

11. Let X be the vector space of all complex 2×2 matrices and define $T\colon X \longrightarrow X$ by $Tx = bx$, where $b \in X$ is fixed and bx denotes the usual product of matrices. Show that T is linear. Under what condition does T^{-1} exists?

Solution: For any $x, y \in X$ and scalars α, β,

$$\begin{aligned} T(\alpha x + \beta y) &= \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \alpha\xi_1 + \beta\eta_1 & \alpha\xi_2 + \beta\eta_2 \\ \alpha\xi_3 + \beta\eta_3 & \alpha\xi_4 + \beta\eta_4 \end{bmatrix} \\ &= \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \left\{ \alpha \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix} + \beta \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix} \right\} \\ &= \alpha bx + \beta by = \alpha Tx + \beta Ty. \end{aligned}$$

This shows that T is linear. T^{-1} exists if and only if b is a non-singular 2×2 complex matrix.

12. Does the inverse of T in 2.6-4 exist?

Solution: The inverse of the differentiation operator T does not exist because $\mathcal{N}(T) \neq \{\mathbf{0}\}$, the zero function.

13. Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator whose inverse exists. If $\{x_1, \ldots, x_n\}$ is a linearly independent set in $\mathcal{D}(T)$, show that the set $\{Tx_1, \ldots, Tx_n\}$ is linearly independent.

Solution: Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator whose inverse exists. Suppose

$$\alpha_1 Tx_1 + \ldots + \alpha_n Tx_n = \mathbf{0}_Y.$$

By linearity of T, the equation above is equivalent to

$$T(\alpha_1 x_1 + \ldots + \alpha_n x_n) = \mathbf{0}_Y.$$

Since T^{-1} exists, we must have " $Tx = \mathbf{0}_Y \implies x = \mathbf{0}_X$". Thus,

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = \mathbf{0}_X.$$

but $\{x_1, \ldots, x_n\}$ is a linearly independent set in $\mathcal{D}(T)$, so this gives $\alpha_1 = \ldots = \alpha_n = 0$.

14. Let $T\colon X \longrightarrow Y$ be a linear operator and dim X = dim $Y = n < \infty$. Show that $\mathcal{R}(T) = Y$ if and only if T^{-1} exists.

Solution: Suppose $\mathcal{R}(T) = Y$, by definition, for all $y \in Y$, there exists $x \in X$ such that $Tx = y$, i.e. T is surjective. We now show that T is injective. Since $\dim\,(X) = \dim\,(Y) = n < \infty$, the **Rank-Nullity Theorem** gives us

$$\dim(\mathcal{R}(T)) + \dim(\mathcal{N}(T)) = \dim(X).$$

but by assumption, $\mathcal{R}(T) = Y$, so

$$\begin{aligned} \dim(Y) + \dim(\mathcal{N}(T)) &= \dim(X). \\ \implies \dim(\mathcal{N}(T)) &= \dim(X) - \dim(Y) = 0. \\ \implies \mathcal{N}(T) &= \{\mathbf{0}_X\}. \end{aligned}$$

This shows that T is injective. Indeed, suppose for any x_1, x_2, we have $Tx_1 = Tx_2$. By linearity of T, $Tx_1 - Tx_2 = T(x_1 - x_2) = \mathbf{0}_Y \implies x_1 - x_2 = \mathbf{0}_X \implies x_1 = x_2$. Since T is both injective and surjective, we conclude that the inverse of T, T^{-1}, exists.

Conversely, suppose T^{-1} exists. From Problem 10, this means that $\mathcal{N}(T) = \{\mathbf{0}_X\} \implies \dim(\mathcal{N}(T)) = 0$. Invoking the **Rank-Nullity Theorem** gives

$$\begin{aligned} \dim(\mathcal{R}(T)) + \dim(\mathcal{N}(T)) &= \dim(X). \\ \implies \dim(\mathcal{R}(T)) &= \dim(X) = n. \end{aligned}$$

This implies that $\mathcal{R}(T) = Y$ since any proper subspace W of Y has dimension less than n.

15. Consider the vector space X of all real-valued functions which are defined on $\mathbb{R}$ and have derivatives of all orders everywhere on $\mathbb{R}$. Define $T\colon X \longrightarrow X$ by $y(t) = Tx(t) = x'(t)$. Show that $\mathcal{R}(T)$ is all of X but T^{-1} does not exist. Compare with Problem 14 and comment.

Solution: For any $y(t) \in \mathcal{R}(T)$, define $x(t) = \int_{-\infty}^{t} y(s)\,ds \in X$; **Fundamental Theorem of Calculus** gives that $x'(t) = Tx(t) = y(t)$. On the other hand, T^{-1} does not exist since the null space of T consists of every constant functions on $\mathbb{R}$. However, it doesn't contradict Problem 14 since X is an infinite-dimensional vector space.

1.6 Bounded and Continuous Linear Operators.

Definition 1.18.

1. *Let X and Y be normed spaces and $T\colon \mathcal{D}(T) \longrightarrow Y$ a linear operator, where $\mathcal{D}(T) \subset X$. The operator T is said to be **bounded** if there is a nonnegative number C such that for all $x \in \mathcal{D}(T)$, $\|Tx\| \leq C\|x\|$.*
   - *This also shows that a bounded linear operator maps bounded sets in $\mathcal{D}(T)$ onto bounded sets in Y.*
2. *The **norm** of a bounded linear operator T is defined as*

$$\|T\| = \sup_{\substack{x\in\mathcal{D}(T)\\ x\neq\mathbf{0}}} \frac{\|Tx\|}{\|x\|}.$$

   - *This is the smallest possible C for all nonzero $x \in \mathcal{D}(T)$.*
   - *With $C = \|T\|$, we have the inequality $\|Tx\| \leq \|T\|\|x\|$.*
   - *If $\mathcal{D}(T) = \{\mathbf{0}\}$, we define $\|T\| = 0$.*

Lemma 1.19. *Let T be a bounded linear operator. Then:*
(a) An alternative formula for the norm of T is

$$\|T\| = \sup_{\substack{x\in\mathcal{D}(T)\\ \|x\|=1}} \|Tx\|.$$

(b) $\|T\|$ is a norm.

Theorem 1.20. *If a normed space X is finite dimensional, then every linear operator on X is bounded.*

Theorem 1.21. *Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces.*
(a) T is continuous if and only if T is bounded.

(b) If T is continuous at a single point, it is continuous.

Corollary 1.22. *Let T be a bounded linear operator. Then:*
(a) $x_n \longrightarrow x$ [where $x_n, x \in \mathcal{D}(T)$] implies $Tx_n \longrightarrow Tx$.

(b) The null space $\mathcal{N}(T)$ is closed.

- It is worth noting that the range of a bounded linear operator may not be closed.

Definition 1.23.

1. *Two operators* T_1 *and* T_2 *are defined to be* **equal**, *written* $T_1 = T_2$ *if they have the same domain* $\mathcal{D}(T_1) = \mathcal{D}(T_2)$ *and if* $T_1x = T_2x$ *for all* $x \in \mathcal{D}(T_1) = \mathcal{D}(T_2)$.

2. *The* ***restriction*** *of an operator* $T\colon \mathcal{D}(T) \longrightarrow Y$ *to a subset* $B \subset \mathcal{D}(T)$ *is denoted by* $T|_B$ *and is the operator defined by* $T|_B\colon B \longrightarrow Y$, *satisfying*

$$T|_B x = Tx \qquad \text{for all } x \in B.$$

3. *The* ***extension*** *of an operator* $T\colon \mathcal{D}(T) \longrightarrow Y$ *to a superset* $M \supset \mathcal{D}(T)$ *is an operator* $\tilde{T}\colon M \longrightarrow Y$ *such that* $\tilde{T}|_{\mathcal{D}(T)} = T$, *that is,* $\tilde{T}x = Tx$ *for all* $x \in \mathcal{D}(T)$. *[Hence* T *is the restriction of* $\tilde{T}$ *to* $\mathcal{D}(T)$.*]*

    - *If* $\mathcal{D}(T)$ *is a proper subset of* M, *then a given* T *has many extensions; of practical interest are those extensions which preserve linearity or boundedness.*

Theorem 1.24 (Bounded linear extension)**.**
Let $T\colon \mathcal{D}(T) \longrightarrow Y$ *be a bounded linear operator, where* $\mathcal{D}(T)$ *lies in a normed space* X *and* Y *is a Banach space. Then* T *has an extension* $\tilde{T}\colon \overline{\mathcal{D}(T)} \longrightarrow Y$, *where* $\tilde{T}$ *is a bounded linear operator with norm* $\|\tilde{T}\| = \|T\|$.

- The theorem concerns an extension of a bounded linear operator T to the closure $\overline{\mathcal{D}(T)}$ of the domain such that the extended operator is again bounded and linear, and even has the same norm.
- This includes the case of an extension from a dense set in a normed space X to all of X.
- It also includes the case of an extension from a normed space X to its completion.

1. Prove $\|T_1T_2\| \leq \|T_1\|\|T_2\|$ and $\|T^n\| \leq \|T\|^n$ ($n \in \mathbb{N}$) for bounded linear operators $T_2\colon X \longrightarrow Y$, $T_1\colon Y \longrightarrow Z$ and $T\colon X \longrightarrow X$, where X, Y, Z are normed spaces.

Solution: Using boundedness of T_1 and T_2,

$$\|(T_1T_2)x\| = \|T_1(T_2x)\| \leq \|T_1\|\|T_2x\| \leq \|T_1\|\|T_2\|\|x\|.$$

The first inequality follows by taking supremum over all x of norm 1. A similar argument also shows the second inequality.

2. Let X and Y be normed spaces. Show that a linear operator $T\colon X \longrightarrow Y$ is bounded if and only if T maps bounded sets in X into bounded sets in Y.

Solution: Suppose $T\colon X \longrightarrow Y$ is bounded, there exists an $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. Take any bounded subset A of X, there exists $M_A > 0$ such that $\|x\| \leq M_A$ for all $x \in A$. For any $x \in A$,

$$\|Tx\| \leq C\|x\| \leq CM_A.$$

This shows that T maps bounded sets in X into bounded sets in Y.

Conversely, suppose a linear operator $T\colon X \longrightarrow Y$ maps bounded sets in X into bounded sets in Y. This means that for any fixed $R > 0$, there exists a constant $M_R > 0$ such that $\|x\| \leq R \implies \|Tx\| \leq M_R$. We now take any nonzero $y \in X$ and set

$$x = R\frac{y}{\|y\|} \implies \|x\| = R.$$

Thus,

$$\begin{aligned} \frac{R}{\|y\|}\|Ty\| = \left\|T\left(\frac{R}{\|y\|}y\right)\right\| &= \|Tz\| \leq M_R. \\ &\implies \|Ty\| \leq \frac{M_R}{R}\|y\|. \end{aligned}$$

where we crucially used the linearity of T. Rearranging and taking supremum over all y of norm 1 shows that T is bounded.

3. If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

Solution: We have $\|Tx\| \leq \|T\|\|x\| < \|T\|$.

4. Let $T\colon \mathcal{D}(T) \longrightarrow Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces. Show that if T is continuous at a single point, it is continuous on $\mathcal{D}(T)$.

Solution: Suppose T is continuous at an arbitrary $x_0 \in \mathcal{D}(T)$. This means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|Tx - Tx_0\| \leq \varepsilon$ for all $x \in \mathcal{D}(T)$ satisfying $\|x - x_0\| \leq \delta$. Fix an $y_0 \in \mathcal{D}(T)$, and set

$$x - x_0 = \delta\frac{y - y_0}{\|y - y_0\|} \implies \|x - x_0\| = \delta.$$

Since T is linear, for any $y \in \mathcal{D}(T)$ satisfying $\|y - y_0\| \leq \delta$,

$$\begin{aligned} \frac{\delta}{\|y - y_0\|}\|T(y - y_0)\| &= \left\|T\left(\frac{\delta}{\|y - y_0\|}(y - y_0)\right)\right\| = \|T(x - x_0)\| \leq \varepsilon \\ \implies \|T(y - y_0)\| &\leq \varepsilon\frac{\|y - y_0\|}{\delta} \leq \frac{\varepsilon\cancel{\delta}}{\cancel{\delta}} = \varepsilon. \end{aligned}$$

This shows that T is continuous at y_0. Since $y_0 \in \mathcal{D}(T)$ is arbitrary, the statement follows.

5. Show that the operator $T\colon l^\infty \longrightarrow l^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j/j$, $x = (\xi_j)$, is linear and bounded.

Solution: For any $x, z \in l^\infty$ and scalars α, β,

$$T(\alpha x + \beta z) = \left(\alpha \frac{\xi_j}{j} + \beta \frac{\kappa_j}{j}\right) = \alpha \left(\frac{\xi_j}{j}\right) + \beta \left(\frac{\kappa_j}{j}\right) = \alpha Tx + \beta Tz.$$

For any $x = (\xi_j) \in l^\infty$,

$$\left|\frac{\xi_j}{j}\right| \leq |\xi_j| \leq \sup_{j\in\mathbb{N}} |\xi_j| = \|x\|.$$

Taking supremum over $j \in \mathbb{N}$ on both sides yields $\|Tx\| \leq \|x\|$. We conclude that T is a bounded linear operator.

6. **(Range)** Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T\colon X \longrightarrow Y$ need not be closed in Y.

Solution: Define (x_n) to be a sequence in the space l^∞, where $x_n = (\xi_j^n)$ and

$$\xi_j^n = \begin{cases} \sqrt{j} & \text{if } j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

Consider the operator $T\colon l^\infty \longrightarrow l^\infty$ in Problem 5. Then $Tx_n = y_n = (\eta_j^n)$, where

$$\eta_j^n = \frac{\xi_j^n}{j} = \begin{cases} \dfrac{1}{\sqrt{j}} & \text{if } j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

We now have our sequence $(y_n) \in \mathcal{R}(T) \subset l^\infty$. We claim that it converges to y in l^∞, where y is a sequence in l^∞ defined as $y = (\eta_j)$, $\eta_j = \dfrac{1}{\sqrt{j}}$. Indeed,

$$\|y_n - y\|_{l^\infty} = \sup_{j\in\mathbb{N}} |\eta_j^n - \eta_j| = \frac{1}{\sqrt{n+1}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

However, $y \notin \mathcal{R}(T)$. Indeed, if there exists an $x \in l^\infty$ such that $Tx = y$, then x must be the sequence (ξ_j), with $\xi_j = \sqrt{j}$, which is clearly not in the space l^∞. Hence, $\mathcal{R}(T)$ is not closed in l^∞.

7. **(Inverse operator)** Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$\|Tx\| \geq b\|x\| \qquad \text{for all } x \in X,$$

show that then $T^{-1}\colon Y \longrightarrow X$ exists and is bounded.

Solution: We first show that T is injective, and therefore the inverse T^{-1} exists since T is bijective (T is surjective by assumption). Indeed, choose any $x_1, x_2 \in X$ and suppose $x_1 \neq x_2$, then $\|x_1 - x_2\| > 0$. Since T is linear,

$$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq b\|x_1 - x_2\| > 0 \implies Tx_1 \neq Tx_2.$$

We are left to show there exists an $C > 0$ such that $\|T^{-1}y\| \leq C\|y\|$ for all $y \in Y$. Since T is surjective, for any $y \in Y$, there exists $x \in X$ such that $y = Tx$; existence of T^{-1} then implies $x = T^{-1}y$. Thus,

$$\|T(T^{-1}(y))\| \geq b\|T^{-1}y\| \implies \|T^{-1}y\| \leq \frac{1}{b}\|y\|.$$

where $C = \dfrac{1}{b} > 0$.

8. Show that the inverse $T^{-1}\colon \mathcal{R}(T) \longrightarrow X$ of a bounded linear operator $T\colon X \longrightarrow Y$ need not be bounded.

Solution: Consider the operator $T\colon l^\infty \longrightarrow l^\infty$ defined by $y = Tx = (\eta_j)$, $\eta_j = \xi_j/j$, $x = (\xi_j)$. We shown in Problem 6 that T is a bounded linear operator. We first show that T is injective. For any $x_1, x_2 \in l^\infty$, suppose $Tx_1 = Tx_2$. For any $j \in \mathbb{N}$,

$$(Tx_1)_j = (Tx_2)_j \implies \frac{\xi_j^1}{j} = \frac{\xi_j^2}{j} \implies \xi_j^1 = \xi_j^2 \quad \text{since } \frac{1}{j} \neq 0.$$

This shows that $x_1 = x_2$ and T is injective. Thus, there exists an inverse $T^{-1}\colon \mathcal{R}(T) \longrightarrow l^\infty$ defined by $x = T^{-1}y = (\xi_j)$, $\xi_j = j\eta_j$, $y = (\eta_j)$. Let's verify that this is indeed the inverse operator.

$$T^{-1}(Tx) = T^{-1}y = T^{-1}\left(\frac{\xi_j}{j}\right) = \left(j\frac{\xi_j}{j}\right) = (\xi_j) = x.$$
$$T(T^{-1}y) = Tx = T((j\eta_j)) = \left(\frac{j\eta_j}{j}\right) = (\eta_j) = y.$$

We claim that T^{-1} is not bounded. Indeed, let $y_n = (\delta_{jn})_{j=1}^\infty$, where δ_{jn} is the Kronecker delta function. Then $\|y_n\| = 1$ and

$$\|T^{-1}y_n\| = \|(j\delta_{jn})\| = n \implies \frac{\|T^{-1}y_n\|}{\|y_n\|} = n.$$

Since $n \in \mathbb{N}$ is arbitrary, this shows that there is no fixed number $C > 0$ such that $\dfrac{\|T^{-1}y_n\|}{\|y_n\|} \leq C$, i.e. T^{-1} is not bounded.

9. Let $T\colon C[0,1] \longrightarrow C[0,1]$ be defined by

$$y(t) = \int_0^t x(s)\,ds.$$

Find $\mathcal{R}(T)$ and $T^{-1}\colon \mathcal{R}(T) \longrightarrow C[0,1]$. Is T^{-1} linear and bounded?

Solution: First, **Fundamental Theorem of Calculus** yields

$$\mathcal{R}(T) = \{y(t) \in C[0,1] \colon y(t) \in C^1[0,1], y(0) = 0\} \subset C[0,1].$$

Next, we show that T is injective and thus the inverse $T^{-1}\colon \mathcal{R}(T) \longrightarrow C[0,1]$ exists. Indeed, suppose for any $x_1, x_2 \in C[0,1]$, $Tx_1 = Tx_2$. Then

$$\begin{aligned} Tx_1 = Tx_2 &\implies \int_0^t x_1(s)\,ds = \int_0^t x_2(s)\,ds \\ &\implies \int_0^t \Big[x_1(s) - x_2(s)\Big]\,ds = 0 \\ &\implies x(s) = y(s) \text{ for all } s \in [0,t]. \end{aligned}$$

where the last implication follows from $x - y$ being a continuous function in $[0,t] \subset [0,1]$. The inverse operator T^{-1} is defined by $T^{-1}y(t) = y'(t)$, i.e. T^{-1} is the differentiation operator. Since differentiation is a linear operation, so is T^{-1}. However, T^{-1} is not bounded. Indeed, let $y_n(t) = t^n$, where $n \in \mathbb{N}$. Then $\|y_n\| = 1$ and

$$\|T^{-1}y_n\| = \|nt^{n-1}\| = n \implies \frac{\|T^{-1}y_n\|}{\|y_n\|} = n.$$

Since $n \in \mathbb{N}$ is arbitrary, this shows that there is no fixed number $C > 0$ such that $\dfrac{\|T^{-1}y_n\|}{\|y_n\|} \le C$, i.e. T^{-1} is not bounded.

10. On $C[0,1]$ define S and T

$$Sx(t) = y(t) = t\int_0^1 x(s)\,ds \qquad Tx(t) = y(t) = tx(t).$$

respectively. Do S and T commute? Find $\|S\|$, $\|T\|$, $\|ST\|$ and $\|TS\|$.

Solution: They do not commute. Take $x(t) = t \in C[0,1]$. Then

$$\begin{aligned} (ST)x(t) &= S(t^2) = t\int_0^1 s^2\,ds = \frac{t}{3}. \\ (TS)x(t) &= T\left(\frac{t}{2}\right) = \frac{t^2}{2}. \end{aligned}$$

11. Let X be the normed space of all bounded real-valued functions on $\mathbb{R}$ with norm defined by

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)|,$$

and let $T\colon X \longrightarrow X$ defined by

$$y(t) = Tx(t) = x(t - \Delta)$$

where $\Delta > 0$ is a constant. (This ia model of a *delay line*, which is an electric device whose output y is a delayed version of the input x, the time delay being Δ.) Is T linear? Bounded?

Solution: For any $x, z \in X$ and scalars α, β,

$$T(\alpha x + \beta z) = \alpha x(t - \Delta) + \beta z(t - \Delta) = \alpha Tx + \beta Tz.$$

This shows that T is linear. T is bounded since

$$\|Tx\| = \sup_{t \in \mathbb{R}} |x(t - \Delta)| = \|x\|.$$

12. **(Matrices)** We know that an $r \times n$ matrix $A = (\alpha_{jk})$ defines a linear operator from the vector space X of all ordered n-tuples of numbers into the vector space Y of all ordered r-tuples of numbers. Suppose that any norm $\|\cdot\|_1$ is given on X and any norm $\|\cdot\|_2$ is given on Y. Remember from Problem 10, Section 2.4, that there are various norms on the space Z of all those matrices (r and n fixed). A norm $\|\cdot\|$ on Z is said to be *compatible* with $\|\cdot\|_1$ and $\|\cdot\|_2$ if

$$\|Ax\|_2 \leq \|A\|\|x\|_1.$$

Show that the norm defined by

$$\|A\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_1}$$

is compatible with $\|\cdot\|_1$ and $\|\cdot\|_2$. This norm is often called the *natural norm* defined by $\|\cdot\|_1$ and $\|cdot\|_2$. IF we choose $\|x\|_1 = \max_j |\xi_j|$ and $\|y\|_2 = \max_j |\eta_j|$, show that the natural norm is

$$\|A\| = \max_j \sum_{k=1}^{n} |\alpha_{jk}|.$$

Solution:

13. Show that in 2.7-7 with $r = n$, a compatible norm is defined by

$$\|A\| = \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_{jk}^2 \right)^{\frac{1}{2}},$$

but for $n > 1$ this is *not* the natural norm defined by the Euclidean norm on $\mathbb{R}^n$.

Solution:

14. If in Problem 12, we choose

$$\|x\|_1 = \sum_{k=1}^{n} |\xi_k| \qquad \|y\|_2 = \sum_{j=1}^{r} |\eta_j|,$$

show that a compatible norm is defined by

$$\|A\| = \max_k \sum_{j=1}^{r} |\alpha_{jk}|.$$

Solution:

15. Show that for $r = n$, the norm in Problem 14 is the natural norm corresponding to $\|\cdot\|_1$ and $\|\cdot\|_2$ as defined in that problem.

Solution:

1.7 Linear Functionals.

Definition 1.25.

1. *A **linear functional** f is a linear operator with domain in a vector space X and range in the scalar field K of X; thus, $f\colon \mathcal{D}(f) \longrightarrow K$, where $K = \mathbb{R}$ if X is real and $K = \mathbb{C}$ if X is complex.*

2. *A **bounded linear functional** f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $\mathcal{D}(f)$ lies. Thus there exists a nonnegative number C such that for all $x \in \mathcal{D}(f)$, $|f(x)| \le C\|x\|$. Furthermore, the **norm** of f is*

$$\|f\| = \sup_{\substack{x\in\mathcal{D}(f)\\ x\neq \mathbf{0}}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in\mathcal{D}(f)\\ \|x\|=1}} |f(x)|.$$

   - *As before, we have that $|f(x)| \le \|f\|\|x\|$.*

Theorem 1.26. *A linear functional f with domain $\mathcal{D}(f)$ in a normed space X is continuous if and only if f is bounded.*

Definition 1.27.

1. *The set of all linear functionals defined on a vector space X can itself be made into a vector space. This space is denoted by X^* and is called the **algebraic dual space** of X. Its algebraic operations of vector space are defined in a natural way as follows.*

   (a) The sum $f_1 + f_2$ of two functionals f_1 and f_2 is the functional s whose value at every $x \in X$ is

$$s(x) = (f_1 + f_2)(x) = f_1(x) + f_2(x).$$

   (b) The product αf of a scalar α and a functional f is the functional p whose value at every $x \in X$ is

$$p(x) = (\alpha f)(x) = \alpha f(x).$$

2. *We may also consider the algebraic dual $(X^*)^*$ of X^*, whose elements are the linear functionals defined on X^*. We denote $(X^*)^*$ by X^{**} and call it the **second algebraic dual space** of X.*

   - *We can obtain an interesting and important relation between X and X^{**}, as follows. We can obtain a $g \in X^{**}$, which is a linear functional defined on X^*, by choosing a **fixed** $x \in X$ and setting*

$$g(f) = g_x(f) = f(x) \qquad \text{where } x \in X \text{ fixed, } f \in X^* \text{ variable.}$$

   - *g_x is linear. Indeed,*

$$g_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = \alpha g_x(f_1) + \beta g_x(f_2).$$

   Hence, g_x is an element of X^**, by the definition of X^{**}.

- *To each $x \in X$ there corresponds a $g_x \in X^{**}$. This defines a mapping $C\colon X \longrightarrow X^{**}$, $C\colon x \mapsto g_x$; C is called the* ***canonical mapping*** *of X into X^{**}. C is linear since its domain is a vector space and we have*

$$\begin{aligned}(C(\alpha x+\beta y))(f) &= g_{\alpha x+\beta y}(f)\\ &= f(\alpha x+\beta y)\\ &= \alpha f(x)+\beta f(y)\\ &= \alpha g_x(f)+\beta g_y(f)\\ &= \alpha(Cx)(f)+\beta(Cy)(f).\end{aligned}$$

  C is also called the ***canonical embedding*** *of X into X^{**}.*

3. *An* ***metric space isomorphism*** *T of a metric space $X=(X,d)$ onto a metric space $\tilde{X}=(\tilde{X},\tilde{d})$ is a bijective mapping which preserves distance, that is, for all $x,y\in X$, $\tilde{d}(Tx,Ty)=d(x,y)$. $\tilde{X}$ is then called* ***isomorphic*** *with X.*

4. *An* ***vector space isomorphism*** *T of a vector space X onto a vector space $\tilde{X}$ over the same field is a bijective mapping which preserves the two algebraic operations of vector space; thus, for all $x,y\in X$ and scalars α,*

$$T(x+y)=Tx+Ty \qquad \text{and } T(\alpha x)=\alpha Tx,$$

   that is, $T\colon X\longrightarrow\tilde{X}$ is a bijective linear operator. $\tilde{X}$ is then called ***isomorphic*** *with X, and X and $\tilde{X}$ are called* ***isomorphic vector spaces****.*

5. *If X is isomorphic with a subspace of a vector space Y, we say that X is* ***embeddable*** *in Y.*

   - *It can be shown that the canonical mapping C is injective. Since C is linear, it is a vector space isomorphism of X onto the range $\mathcal{R}(C)\subset X^{**}$.*
   - *Since $\mathcal{R}(C)$ is a subspace of X^{**}, X is embeddable in X^{**}, and C is also called the* ***canonical embedding*** *of X into X^{**}.*
   - *If C is surjective (hence bijective), so that $\mathcal{R}(C)=X^{**}$, then X is said to be* ***algebraically reflexive****.*

1. Show that the functionals in 2.8-7 and 2.8-8 are linear.

Solution: Choose a fixed $t_0\in J=[a,b]$ and set $f_1(x)=x(t_0)$, where $x\in C[a,b]$. For any $x,y\in C[a,b]$ and scalars α,β,

$$f_1(\alpha x+\beta y)=(\alpha x+\beta y)(t_0)=\alpha x(t_0)+\beta y(t_0)=\alpha f_1(x)+\beta f_1(y).$$

Choose a fixed $a=(a_j)\in l^2$ and set $f(x)=\displaystyle\sum_{j=1}^{\infty}\xi_j\alpha_j$, where $x=(\xi_j)\in l^2$. For any $x=(\xi_j), y=(\eta_j)\in l^2$ and scalars α,β,

$$f(\alpha x+\beta y)=\sum_{j=1}^{\infty}(\alpha\xi_j+\beta\eta_j)\alpha_j=\alpha\sum_{j=1}^{\infty}.\xi_j\alpha_j+\beta\sum_{j=1}^{\infty}\eta_j\alpha_j=\alpha f(x)+\beta f(y).$$

Note we can split the infinite sum because the two infinite sums are convergent by Cauchy-Schwarz inequality.

2. Show that the functionals defined on $C[a,b]$ by

$$\begin{aligned} f_1(x) &= \int_a^b x(t)y_0(t)\,dt \qquad (y_0 \in C[a,b] \text{ fixed.}) \\ f_2(x) &= \alpha x(a) + \beta x(b) \qquad (\alpha, \beta \text{ fixed.}) \end{aligned}$$

are linear and bounded.

Solution: For any $x, y \in C[a,b]$ and scalars γ, δ,

$$\begin{aligned} f_1(\gamma x + \delta y) &= \int_a^b \Big[\gamma x(t) + \delta y(t)\Big] y_0(t)\,dt \\ &= \gamma \int_a^b x(t)y_0(t)\,dt + \delta \int_a^b y(t)y_0(t)\,dt. \\ &= \gamma f_1(x) + \delta f_1(y). \\ f_2(\gamma x + \delta y) &= \alpha(\gamma x + \delta y)(a) + \beta(\gamma x + \delta y)(b) \\ &= \alpha(\gamma x(a) + \delta y(a)) + \beta(\gamma x(b) + \delta y(b)) \\ &= \gamma(\alpha x(a) + \beta x(b)) + \delta(\alpha y(a) + \beta y(b)) \\ &= \gamma f_2(x) + \delta f_2(y). \end{aligned}$$

To show that f_1 and f_2 are bounded, for any $x \in C[a,b]$,

$$\begin{aligned} |f_1(x)| = \left|\int_a^b x(t)y_0(t)\,dt\right| &\le \max_{t\in[a,b]} |x(t)| \int_a^b y_0(t)\,dt \\ &= \left(\int_a^b y_0(t)\,dt\right)\|x\|. \\ |f_2(x)| = |\alpha x(a) + \beta x(b)| &\le \alpha \max_{t\in[a,b]} |x(t)| + \beta \max_{t\in[a,b]} |x(t)| \\ &= (\alpha + \beta)\|x\|. \end{aligned}$$

3. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^0 x(t)\,dt - \int_0^1 x(t)\,dt.$$

Solution:

$$\begin{aligned}|f(x)| &= \left|\int_{-1}^{0} x(t)\,dt - \int_{0}^{1} x(t)\,dt\right| \\ &\leq \left|\int_{-1}^{0} x(t)\,dt\right| + \left|\int_{0}^{1} x(t)\,dt\right| \\ &\leq \|x\|\left|\int_{-1}^{0} dt\right| + \|x\|\left|\int_{0}^{1} dt\right| \\ &= 2\|x\|\end{aligned}$$

Taking the supremum over all x of norm 1, we obtain $\|f\| \leq 2$. To get $\|f\| \geq 2$, we choose the particular $x(t) \in C[-1,1]$ defined by

$$x(t) = \begin{cases} -2t-2 & \text{if } -1 \leq t \leq -\dfrac{1}{2}, \\ 2t & \text{if } -\dfrac{1}{2} \leq t \leq \dfrac{1}{2}, \\ -2t+2 & \text{if } \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

Note that $\|x\| = 1$ and

$$\|f\| \leq \frac{|f(x)|}{\|x\|} = |f(x)| = 2.$$

4. Show that for $J = [a,b]$,

$$f_1(x) = \max_{t\in J} x(t) \qquad f_2(x) = \min_{t\in J} x(t)$$

define functionals on $C[a,b]$. Are they linear? Bounded?

Solution: Both f_1 and f_2 define functionals on $C[a,b]$ since continuous functions attains its maximum and minimum on closed interval. They are not linear. Choose $x(t) = \dfrac{t-a}{b-a}$ and $y(t) = \dfrac{-(t-a)}{b-a}$. Then

$$\begin{aligned} f_1(x+y) &= 0 \quad \text{but} \quad f_1(x) + f_1(y) = 1 + 0 = 1. \\ f_2(x+y) &= 0 \quad \text{but} \quad f_2(x) + f_2(y) = 0 - 1 = -1. \end{aligned}$$

They are, however, bounded, since for any $x \in C[a,b]$,

$$\begin{aligned} |f_1(x)| &= \max_{t\in J} x(t) \leq \max_{t\in J} |x(t)| = \|x\|. \\ |f_2(x)| &= \min_{t\in J} x(t) \leq \max_{t\in J} |x(t)| = \|x\|. \end{aligned}$$

5. Show that on any sequence space X we can define a linear functional f by setting $f(x) = \xi_n$ (n fixed), where $x = (\xi_j)$. Is f bounded if $X = l^\infty$?

> **Solution:** For any $x = (\xi_j), y = (\eta_j) \in X$ and scalars α, β,
>
> $$f(\alpha x + \beta y) = \alpha \xi_n + \beta \eta_n = \alpha f(x) + \beta f(y).$$
>
> f is bounded if $X = l^\infty$. Indeed, for any $x \in l^\infty$,
>
> $$|f(x)| = |\xi_n| \leq \sup_{j \in \mathbb{N}} |\xi_j| = \|x\|.$$
>
> Remark: *In fact, we can show that $\|f\| = 1$. Taking supremum over all x of norm 1 on previous equation yields $\|f\| \leq 1$. To get $\|f\| \geq 1$, we choose the particular $x = (\xi_j) = (\delta_{jn})$, note that $\|x\| = 1$ and*
>
> $$\|f\| \geq \frac{|f(x)|}{\|x\|} = |f(x)| = 1.$$

6. **(Space $C^1[a, b]$)** The space $C^1[a, b]$ is the normed space of all continuously differentiable functions on $J = [a, b]$ with norm defined by

$$\|x\| = \max_{t \in J} |x(t)| + \max_{t \in J} |x'(t)|.$$

(a) Show that the axioms of a norm are satisfied.

> **Solution:** (N1) and (N3) are obvious. For (N2), if $x(t) \equiv 0$, then $\|x\| = 0$. On the other hand, if $\|x\| = 0$, since both $\max_{t \in J} |x(t)|$ and $\max_{t \in J} |x'(t)|$ are nonnegative, we must have $|x(t)| = 0$ and $|x'(t)| = 0$ for all $t \in [a, b]$ which implies $x(t) \equiv 0$. Finally, (N4) follows from
>
> $$\begin{aligned} \|x + y\| &= \max_{t \in J} |x(t) + y(t)| + \max_{t \in J} |x'(t) + y'(t)| \\ &\leq \max_{t \in J} |x(t)| + \max_{t \in J} |y(t)| + \max_{t \in J} |x'(t)| + \max_{t \in J} |y'(t)| \\ &= \|x\| + \|y\|. \end{aligned}$$

(b) Show that $f(x) = x'(c)$, $c = \dfrac{a + b}{2}$, defines a bounded linear functional on $C^1[a, b]$.

> **Solution:** For any $x, y \in C^1[a, b]$ and scalars α, β,
>
> $$f(\alpha x + \beta y) = \alpha x'(c) + \beta y'(c) = \alpha f(x) + \beta f(y).$$
>
> To see that f is bounded,
>
> $$|f(x)| = |x'(c)| \leq \max_{t \in J} |x'(t)| \leq \|x\|.$$

(c) Show that f is not bounded, considered as functional on the subspace of $C[a, b]$ which consists of all continuously differentiable functions.

Solution:

7. If f is a bounded linear functional on a complex normed space, is $\bar{f}$ bounded? Linear? (The bar denotes the complex conjugate.)

Solution: $\bar{f}$ is bounded since $|f(x)| = |\overline{f(x)}|$, but it is not linear since for any $x \in X$ and complex numbers α, $\overline{f(\alpha x)} = \overline{\alpha f(x)} = \bar{\alpha}\overline{f(x)} \neq \alpha \overline{f(x)}$.

8. **(Null space)** The *null space* $\mathcal{N}(M^*)$ of a set $M^* \subset X^*$ is defined to be the set of all $x \in X$ such that $f(x) = 0$ for all $f \in M^*$. Show that $\mathcal{N}(M^*)$ is a vector space.

Solution: Since X is a vector space, it suffices to show that $\mathcal{N}(M^*)$ is a subspace of X. Note that all element of M^* are linear functionals. For any $x, y \in \mathcal{N}(M^*)$, we have $f(x) = f(y) = 0$ for all $f \in M^*$. Then for any $f \in M^*$ and scalars α, β,

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = 0. \qquad \left[\text{ by linearity of } f \in M^*\right]$$

This shows that $\alpha x + \beta y \in \mathcal{N}(M^*)$ and the statement follows.

9. Let $f \neq 0$ be any linear functional on a vector space X and x_0 any fixed element of $X \setminus \mathcal{N}(f)$, where $\mathcal{N}(f)$ is the null space of f. Show that any $x \in X$ has a unique representation $x = \alpha x_0 + y$, where $y \in \mathcal{N}(f)$.

Solution: Let $f \neq 0$ be any linear functional on X and x_0 any fixed element of $X \setminus \mathcal{N}(f)$. We claim that for any $x \in X$, there exists a scalar α such that $x = \alpha x_0 + y$, where $y \in \mathcal{N}(f)$. First, applying f on both sides yields

$$f(x) = f(\alpha x_0 + y) = \alpha f(x_0) + f(y) \implies f(y) = f(x) - \alpha f(x_0).$$

By choosing $\alpha = \dfrac{f(x)}{f(x_0)}$ (which is well-defined since $f(x_0) \neq 0$), we see that

$$f(y) = f(x) - \frac{f(x)}{\cancel{f(x_0)}}\cancel{f(x_0)} = 0 \implies y \in \mathcal{N}(f).$$

To show uniqueness, suppose x has two representations $x = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2$, where α_1, α_2 are scalars and $y_1, y_2 \in \mathcal{N}(f)$. Subtracting both representations yields

$$(\alpha_1 - \alpha_2)x_0 = y_2 - y_1.$$

Applying f on both sides gives

$$f((\alpha_1 - \alpha_2)x_0) = f(y_2 - y_1)$$

$$(\alpha_1 - \alpha_2)f(x_0) = f(y_2) - f(y_1) = 0$$

by linearity of f and $y_1, y_2 \in \mathcal{N}(f)$. Since $f(x_0) \neq 0$, we must have $\alpha_1 - \alpha_2 = 0$. This also implies $y_1 - y_2 = 0$.

10. Show that in Problem 9, two elements $x_1, x_2 \in X$ belong to the same element of the quotient space $X/\mathcal{N}(f)$ if and only if $f(x_1) = f(x_2)$. Show that codim $\mathcal{N}(f) = 1$.

Solution: Suppose two elements $x_1, x_2 \in X$ belong to the same element of the quotient space $X/\mathcal{N}(f)$. This means that there exists an $x \in X$ and $y_1, y_2 \in \mathcal{N}(f)$ such that

$$x_1 = x + y_1 \qquad \text{and} \qquad x_2 = x + y_2.$$

Substracting these equations and applying f yields

$$\begin{aligned} x_1 - x_2 = y_1 - y_2 &\implies f(x_1 - x_2) = f(y_1 - y_2) \\ &\implies f(x_1) - f(x_2) = f(y_1) - f(y_2) = 0. \end{aligned}$$

where we use the linearity of f and $y_1, y_2 \in \mathcal{N}(f)$. Conversely, suppose $f(x_1) - f(x_2) = 0$; linearity of f gives

$$f(x_1 - x_2) = 0 \implies x_1 - x_2 \in \mathcal{N}(f).$$

This means that there exists $y \in \mathcal{N}(f)$ such that $x_1 - x_2 = \mathbf{0} + y$, which implies that $x_1, x_2 \in X$ must belong to the same coset of $X/\mathcal{N}(f)$.

Codimension of $\mathcal{N}(f)$ is defined to be the dimension of the quotient space $X/\mathcal{N}(f)$. Choose any $\hat{x} \in X/\mathcal{N}(f)$, there exists an $x \in X$ such that $\hat{x} = x + \mathcal{N}(f)$. Since $f \neq 0$, there exists an $x_0 \in X \setminus \mathcal{N}(f)$ such that $f(x_0) \neq 0$. Looking at Problem 9, we deduce that $\hat{x}$ has a unique representation $\hat{x} = \alpha x_0 + \mathcal{N}(f) = \alpha(x_0 + \mathcal{N}(f))$. This shows that $x_0 + \mathcal{N}(f)$ is a basis for $X/\mathcal{N}(f)$ and codim $\mathcal{N}(f) = 1$.

11. Show that two linear functionals $f_1 \neq 0$ and $f_2 \neq 0$ which are defined on the same vector space and have the same null space are proportional.

Solution: Let $x, x' \in X$ and consider $z = xf_1(x') - x'f_1(x)$. Clearly, $f_1(z) = 0 \implies z \in \mathcal{N}(f_1) = \mathcal{N}(f_2)$. Thus,

$$0 = f_2(z) = f_2(x)f_1(x') - f_2(x')f_1(x).$$

Since $f_1 \neq 0$, there exists some $x' \in X \setminus \mathcal{N}(f_1)$ such that $f_1(x') \neq 0$; we also have $f_2(x') \neq 0$ since $\mathcal{N}(f_1) = \mathcal{N}(f_2)$. Hence, for such an x', we have

$$f_2(x) = \frac{f_2(x')}{f_1(x')} f_1(x).$$

Since $x \in X$ is arbitrary, the result follows.

12. **(Hyperplane)** If Y is a subspace of a vector space X and $\text{codim} Y = 1$, then every element of X/Y is called a *hyperplane parallel to* Y. Show that for any linear functional $f \neq 0$ on X, the set $H_1 = \{x \in X : f(x) = 1\}$ is a hyperplane parallel to the null space $\mathcal{N}(f)$ of f.

Solution: Since $f \neq 0$ on X, H_1 is not empty. Fix an $x_0 \in H_1$, and consider the coset $x_0 + \mathcal{N}(f)$. Note that this is well-defined irrespective of elements in H_1. Indeed, for any $y \in H_1, y \neq x_0$, $y - x_0 \in \mathcal{N}(f)$ since $f(y - x_0) = f(y) - f(x_0) = 1 - 1 = 0$; this shows that $x + \mathcal{N}(f) = y + \mathcal{N}(f)$ for any $x, y \in H_1$.

- For any $x \in x_0 + \mathcal{N}(f)$, there exists an $y \in \mathcal{N}(f)$ such that $x = x_0 + y$. Since f is linear, $f(x) = f(x_0 + y) = f(x_0) + f(y) = 1 \implies x \in H_1$. This shows that $x_0 + \mathcal{N}(f) \subset H_1$.
- For any $x \in H_1$, $x = x + 0 = x + x_0 - x_0 = x_0 + (x - x_0) \in x_0 + \mathcal{N}(f)$ since $f(x - x_0) = f(x) - f(x_0) = 1 - 1 = 0$. This shows that $H_1 \subset x_0 + \mathcal{N}(f)$.

Finally, combining the two set inequality gives $H_1 = x_0 + \mathcal{N}(f)$ and the statement follows.

13. If Y is a subspace of a vector space X and f is a linear functional on X such that $f(Y)$ is not the whole scalar field of X, show that $f(y) = 0$ for all $y \in Y$.

Solution: The statement is trivial if f is the zero functional, so suppose $f \neq 0$. Suppose, by contradiction, that $f(y) \neq 0$ for all $y \in Y$, then there exists an $y_0 \in Y$ such that $f(y_0) = \alpha$ for some nonzero $\alpha \in K$. Since Y is a subspace of a vector space X, $\beta y_0 \in Y$ for all $\beta \in K$. By linearity of f,

$$f(\beta y_0) = \beta f(y_0) = \beta\alpha \in f(Y).$$

Since $\beta \in K$ is arbitrary, this implies that $f(Y) = K$; this is a contradiction to the assumption that $f(Y) \neq K$. Hence, by proof of contradiction, $f(y) = 0$ for all $y \in Y$.

14. Show that the norm $\|f\|$ of a bounded linear functional $f \neq 0$ on a normed space X can be interpreted geometrically as the reciprocal of the distance $\tilde{d} = \inf\{\|x\| : f(x) = 1\|$ of the hyperplane $H_1 = \{x \in X : f(x) = 1\}$ from the origin.

Solution:

15. **(Half space)** Let $f \neq 0$ be a bounded linear functional on a real normed space X. Then for any scalar c we have a hyperplane $H_c = \{x \in X : f(x) = c\}$, and H_c determines the two *half spaces*

$$X_{c1} = \{x \in X : f(x) \leq c\} \qquad \text{and} \qquad X_{c2} = \{x \in X : f(x) \geq c\}.$$

Show that the closed unit ball lies in X_{c1}, where $c = \|f\|$, but for no $\varepsilon > 0$, the half space X_{c1} with $c = \|f\| - \varepsilon$ contains that ball.

Solution:

1.8 Linear Operators and Functionals on Finite Dimensional Spaces.

- A linear operator $T\colon X \longrightarrow Y$ determines a unique matrix representing T with respect to a given basis for X and a given basis for Y, where the vectors of each of the bases are assumed to be arranged in a fixed order. Conversely, any matrix with r rows and n columns determines a linear operator which it represents with respect to given bases for X and Y.

- Let us now turn to **linear functionals** on X, where dim $X = n$ and $\{e_1, \ldots, e_n\}$ is a basis for X. For every $f \in X^*$ and every $x = \sum \xi_j e_j \in X$, we have

$$f(x) = f\left(\sum_{j=1}^{n} \xi_j e_j\right) = \sum_{j=1}^{n} \xi_j f(e_j) = \sum_{j=1}^{n} \xi_j \alpha_j.$$

  where $\alpha_j = f(e_j)$ for $j = 1, \ldots, n$. We see that f is uniquely determined by its values α_j ath the n basis vectors of X. Conversely, every n-tuple of scalars $\alpha_1, \ldots, \alpha_n$ determines a linear functional on X.

Theorem 1.28. *Let X be an n-dimensional vector space and $E = \{e_1, \ldots, e_n\}$ a basis for X. Then $F = \{f_1, \ldots, f_n\}$ given by $f_k(e_j) = \delta_{jk}$ is a basis for the algebraic dual X^* of X, and dim $X^* =$ dim $X = n$.*

- $\{f_1, \ldots, f_n\}$ is called the **dual basis** of the basis $\{e_1, \ldots, e_n\}$ for X.

Lemma 1.29. *Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$.*

Theorem 1.30. *A finite dimensional vector space is algebraically reflexive.*

1. Determine the null space of the operator $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ represented by

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$

Solution: Performing Gaussian elimination on the matrix yields

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ -2 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 7 & 4 & 0 \end{array}\right].$$

This yields a solution of the form $(\xi_1, \xi_2, \xi_3) = t(-2, -4, 7)$ where $t \in \mathbb{R}$ is a free variable. Hence, the null space of T is the span of $(-2, -4, 7)$.

2. Let $T \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by $(\xi_1, \xi_2, \xi_3) \mapsto (\xi_1, \xi_2, -\xi_1 - \xi_2)$. Find $\mathcal{R}(T), \mathcal{N}(T)$ and a matrix which represents T.

Solution: Consider the standard basis for X, given by $e_1 = (1,0,0), e_2 = (0,1,0)$, $e_3 = (0,0,1)$. The matrix representing T with respect to $\{e_1, e_2, e_3\}$ is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

The range of T, $\mathcal{R}(T)$ is the plane $\xi_1 + \xi_2 + \xi_3 = 0$. The null space of T, $\mathcal{N}(T)$ is span of $(0,0,1)$.

3. Find the dual basis of the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ for $\mathbb{R}^3$.

Solution: Consider a basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ defined by $e_j = (\xi_n^j) = \delta_{jn}$ for $j = 1,2,3$. Given any $x = (\eta_j)$ in $\mathbb{R}^3$, let $f_k(x) = \displaystyle\sum_{j=1}^{3} \alpha_j^k \eta_j$ be the dual basis of $\{e_1, e_2, e_3\}$. From the definition of a dual basis, we require that $f_k(e_j) = \delta_{jk}$. More precisely, for f_1, we require that

$$\begin{aligned} f_1(e_1) &= \alpha_1^1 = 1. \\ f_1(e_2) &= \alpha_2^1 = 0. \\ f_1(e_3) &= \alpha_3^1 = 0. \end{aligned}$$

which implies that $f_1(x) = \eta_1$. Repeating the same computation for f_2 and f_3, we find that $f_2(x) = \eta_2$ and $f_3(x) = \eta_3$. Hence,

$$f_1 = (1,0,0) \qquad f_2 = (0,1,0) \qquad f_3 = (0,0,1).$$

4. Let $\{f_1, f_2, f_3\}$ be the dual basis of $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, where $e_1 = (1,1,1), e_2 = (1,1,-1), e_3 = (1,-1,-1)$. Find $f_1(x), f_2(x), f_3(x)$, where $x = (1,0,0)$.

Solution: Note that we can write x as

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{2}e_1 + \frac{1}{2}e_3.$$

Thus, using the defintiion of a dual basis $f_k(e_j) = \delta_{jk}$, we have

$$f_1(x) = \frac{1}{2}f_1(e_1) + \frac{1}{2}f_1(e_3) = \frac{1}{2}.$$

$$f_2(x) = \frac{1}{2}f_2(e_1) + \frac{1}{2}f_2(e_3) = 0.$$

$$f_3(x) = \frac{1}{2}f_3(e_1) + \frac{1}{2}f_3(e_3) = \frac{1}{2}.$$

where we use linearity of f_1, f_2, f_3.

5. If f is a linear functional on an n-dimensional vector space X, what dimension can the null space $\mathcal{N}(f)$ have?

Solution: The **Rank-Nullity theorem** states that

$$\dim\left(\mathcal{N}(f)\right) = \dim\left(X\right) - \dim\left(\mathcal{R}(f)\right) = n - \dim\left(\mathcal{R}(f)\right).$$

If f is the zero functional, then $\mathcal{N}(f) = X$ and $\mathcal{N}(f)$ has dimension n; if f is not the zero functional, then $\mathcal{R}(f) = K$ which has dimension 1, so $\mathcal{N}(f)$ has dimension $n-1$. Hence, $\mathcal{N}(f)$ has dimension n or $n-1$.

6. Find a basis for the null space of the functional f defined on $\mathbb{R}^3$ by $f(x) = \xi_1 + \xi_2 - \xi_3$, where $x = (\xi_1, \xi_2, \xi_3)$.

Solution: Let $x = (\xi_1, \xi_2, \xi_3)$ be any point in the null space of f, they must satisfy the relation $\xi_1 + \xi_2 - \xi_3 = 0$. Thus,

$$x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_1 + \xi_2 \end{bmatrix} = \xi_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \xi_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, a basis for $\mathcal{N}(f)$ is given by $\{(1,0,1),(0,1,1)\}$.

7. Same task as in Problem 6, if $f(x) = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$, where $\alpha_1 \neq 0$.

Solution: Let $x = (\xi_1, \xi_2, \xi_3)$ be any point in $\mathcal{N}(f)$, they must satisfy the relation

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 = 0 \iff \xi_1 = -\frac{\alpha_2}{\alpha_1}\xi_2 - \frac{\alpha_3}{\alpha_1}\xi_3.$$

Rewriting x using this relation yields

$$x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -\frac{\alpha_2}{\alpha_1}\xi_2 - \frac{\alpha_3}{\alpha_1}\xi_3 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{\xi_2}{\alpha_1}\begin{bmatrix} -\alpha_2 \\ \alpha_1 \\ 0 \end{bmatrix} + \frac{\xi_3}{\alpha_1}\begin{bmatrix} -\alpha_3 \\ 0 \\ \alpha_1 \end{bmatrix}.$$

Hence, a basis for $\mathcal{N}(f)$ is given by $\{(-\alpha_2, \alpha_1, 0), (-\alpha_3, 0, \alpha_1)\}$.

8. If Z is an $(n-1)$-dimensional subspace of an n-dimensional vector space X, show that Z is the null space of a suitable linear functional f on X, which is uniquely determined to within a scalar multiple.

Solution: Let X be an n-dimensional vector space, and Z an $(n-1)$-dimensional subspace of X. Choose a basis $A = \{z_1, \ldots, z_{n-1}\}$ of Z, here we can obtain a basis $B = \{z_1, \ldots, z_{n-1}, z_n\}$ of X, where B is obtained by extending A using sifting method. Any $z \in Z$ can be written uniquely as $z = \alpha_1 z_1 + \ldots + \alpha_{n-1} z_{n-1}$. If we want to find $f \in X^*$ such that $\mathcal{N}(f) = Z$, using linearity of f this translates to

$$f(z) = f(\alpha_1 z_1 + \ldots + \alpha_{n-1} z_{n-1}) = \alpha_1 f(z_1) + \ldots + \alpha_{n-1} f(z_{n-1}) = 0.$$

Since this must be true for all $z \in Z$, it enforces the condition $f(z_j) = 0$ for all $j = 1, \ldots, n-1$. A similar argument shows that $f(z_n) \neq 0$, otherwise $\mathcal{N}(f) = X \neq Z$. Thus, the functional we are looking for must satisfy the following two conditions:

(a) $f(z_j) = 0$ for all $j = 1, \ldots, n-1$.

(b) $f(z_n) \neq 0$.

Consider a linear functional $f \colon X \longrightarrow K$ defined by $f(z_j) = \delta_{jn}$ for all $j = 1, \ldots, n$. We claim that $\mathcal{N}(f) = Z$. Indeed,

- Choose any $z \in Z$, there exists a unique sequence of scalars (β_j) such that $z = \beta_1 z_1 + \ldots + \beta_{n-1} z_{n-1}$. Using linearity of f, we have
$$f(z) = f\left(\sum_{j=1}^{n-1} \beta_j z_j\right) = \sum_{j=1}^{n-1} \beta_j f(z_j) = 0.$$
This shows that $Z \subset \mathcal{N}(f)$.

- Choose any $x \in X \setminus Z$, there exists a unique sequence of scalars (γ_j) such that $x = \gamma_1 z_1 + \ldots + \gamma_n z_n$ and $\gamma_n \neq 0$. (otherwise $x \in Z$.) Using linearity of f, we have
$$f(x) = f\left(\sum_{j=1}^{n} \gamma_j z_j\right) = \sum_{j=1}^{n} \gamma_j f(z_j) = \gamma_n f(z_n) \neq 0.$$
This shows that $(X \setminus Z) \not\subset \mathcal{N}(f)$ or equivalently $\mathcal{N}(f) \subset Z$.

Hence, $Z \subset \mathcal{N}(f)$ and $\mathcal{N}(f) \subset Z$ implies $Z = \mathcal{N}(f)$.

We are left to show that f is uniquely determined up to scalar multiple. Let λ be any nonzero scalars and consider the linear functional λf. Any $x \in X$ can be written uniquely as $x = \alpha_1 z_1 + \ldots + \alpha_n z_n$. Using linearity of λf, we have

$$(\lambda f)(x) = \lambda f(x) = \lambda f\left(\sum_{j=1}^{n} \alpha_j z_j\right) = \lambda\left(\sum_{j=1}^{n} \alpha_j f(z_j)\right) = \lambda \alpha_n f(z_n).$$

If $\alpha_n = 0$, then $x \in Z$ and $(\lambda f)(x) = 0$; if $\alpha_n \neq 0$, then $x \in X \setminus Z$ and $(\lambda f)(x) = \lambda \alpha_n \neq 0$.

9. Let X be the vector space of all real polynomials of a real variable and of degree less than a given n, together with the polynomial $x = 0$ (whose degree is left undefined in the usual discussion of degree). Let $f(x) = x^{(k)}(a)$, the value of the kth derivative (k fixed) of $x \in X$ at a fixed $a \in \mathbb{R}$. Show that f is a linear functional on X.

Solution: This follows from the algebra rules of differentiation. More precisely, for any $x, y \in X$ and scalars α, β,

$$f(\alpha x + \beta y) = (\alpha x + \beta y)^{(k)}(a) = \alpha x^{(k)}(a) + \beta y^{(k)}(a) = \alpha f(x) + \beta f(y).$$

10. Let Z be a proper subspace of an n-dimensional vector space X, and let $x_0 \in X \setminus Z$. Show that there is a linear functional f on X such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in Z$.

Solution: Fix an nonzero $x_0 \in X \setminus Z$, note that $x_0 = \mathbf{0}$ then such a functional doesn't exist since $f(x_0) = 0$. Let X be an n-dimensional vector space, and Z be an m-dimensional subspace of X, with $m < n$. Choose a basis $A = \{z_1, \ldots, z_m\}$ of Z, here we can obtain a basis $B = \{z_1, \ldots, z_m, z_{m+1}, \ldots, z_n\}$ of X with $z_{m+1} = x_0$, where B is obtained by extending A using sifting method. Any $x \in X$ can be written uniquely as

$$x = \alpha_1 z_1 + \ldots + \alpha_m z_m + \alpha_{m+1} x_0 + \alpha_{m+2} z_{m+2} + \ldots + \alpha_n z_n.$$

Consider the linear functional $f \colon X \longrightarrow K$ defined by $f(x) = \alpha_{m+1} + \ldots + \alpha_n$, where α_j is the j-th scalar of x with respect to the basis B for all $j = m+1, \ldots, n$. We claim that $f(x_0) = 1$ and $f(Z) = 0$. Indeed,

- If $x = x_0$, then $\alpha_{m+1} = 1$ and $\alpha_j = 0$ for all $j \neq m + 1$.
- If $x \in Z$, then $\alpha_j = 0$ for all $j = m + 1, \ldots, n$.
- If $x \in X \setminus Z$, at least one of $\{\alpha_{m+1}, \ldots, \alpha_n\}$ is non-zero.

<u>Remark</u>: *The functional f we constructed here is more tight, in the sense that $\mathcal{N}(f) = Z$. If we only require that $Z \subset \mathcal{N}(f)$, then $f(x) = \alpha_{m+1}$ will do the job.*

11. If x and y are different vectors in a finite dimensional vector space X, show that there is a linear functional f on X such that $f(x) \neq f(y)$.

Solution: Let X be an n-dimensional vector space and $\{e_1, \ldots, e_n\}$ a basis of X. Any $x, y \in X$ can be written uniquely as

$$x = \sum_{j=1}^{n} \alpha_j e_j \qquad \text{and} \qquad y = \sum_{j=1}^{n} \beta_j e_j.$$

Since $x \neq y$, there exists at least one $j_0 \in \{1, \dots, n\}$ such that $\alpha_{j_0} \neq \beta_{j_0}$. Consider the linear functional $f\colon X \longrightarrow K$ defined by $f(e_j) = \delta_{jj_0}$. Using linearity of f,

$$f(x) = f\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \sum_{j=1}^{n} \alpha_j f(e_j) = \alpha_{j_0}.$$
$$f(y) = f\left(\sum_{j=1}^{n} \beta_j e_j\right) = \sum_{j=1}^{n} \beta_j f(e_j) = \beta_{j_0}.$$

Clearly, $f(x) \neq f(y)$.

12. If $f_1, \dots, f_p$ are linear functionals on an n-dimensional vector space X, where $p < n$, show that there is a vector $x \neq 0$ in X such that $f_1(x) = 0, \dots, f_p(x) = 0$. What consequences does this result have with respect to linear equations?

Solution:

13. **(Linear extension)** Let Z be a proper subspace of an n-dimensional vector space X, and let f be a linear functional on Z. Show that f can be *extended linearly* to X, that is, there is a a linear functional $\tilde{f}$ on X such that $\tilde{f}|_Z = f$.

Solution:

14. Let the functional f on $\mathbb{R}^2$ be defined by $f(x) = 4\xi_1 - 3\xi_2$, where $x = (\xi_1, \xi_2)$. Regard $\mathbb{R}^2$ as the subspace of $\mathbb{R}^3$ given by $\xi_3 = 0$. Determine all linear extensions $\tilde{f}$ of f from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Solution:

15. Let $Z \subset \mathbb{R}^3$ be the subspace represented by $\xi_2 = 0$ and let f on Z be defined by $f(x) = \dfrac{\xi_1 - \xi_3}{2}$. Find a linear extension $\tilde{f}$ of f to $\mathbb{R}^3$ such that $\tilde{f}(x_0) = k$ (a given constant), where $x_0 = (1, 1, 1)$. Is $\tilde{f}$ unique?

Solution: