Math 4400 Homework 5

Due: Friday, June 23rd, 2017

Feel free to work with your classmates, but everyone must turn in their own assignment. Please make a note of who you worked with on each problem. Let me know if you find a typo, or you're stuck on any of the problems.

1. (5 points) Prove that multiplication of 2×2 matrices with real entries is associative.
2. (a) (5 points) Let $GL_2(\mathbb{R})$ be the set of invertible 2×2 matrices with real entries. Prove that $GL_2(\mathbb{R})$ is a group under multiplication
   (b) (5 points) Let $SL_2(\mathbb{Z})$ be the set of 2×2 matrices with integer entries and determinant 1. Prove that $SL_2(\mathbb{Z})$ is a subgroup of $GL_2(\mathbb{R})$. This closely related to the so-called "modular group", which is one of the most interesting and important groups in number theory.
   (c) (5 points) Let $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R}, ac \neq 0 \right\}$. Prove that S is a subgroup of $GL_2(\mathbb{R})$.
   (d) (5 points) Let $T = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{R} \right\}$. Prove that T is a subgroup of S.
3. (a) (2 points) Let G be a group. Prove that if $a, x, y \in G$ and $ax = ay$, then $x = y$.
   (b) (2 points) Prove that the identity element of a group is unique. In other words, if G is a group, and $e, e' \in G$ are elements such that $eg = ge = g$ and $e'g = ge' = g$ for all $g \in G$, then $e = e'$.
   (c) (2 points) Prove that the inverse of a group element is unique. In other words, if G is a group, and $g, h, h' \in G$ are elements such that
   $$gh = hg = e$$
   $$gh' = h'g = e$$
   then $h = h'$.
4. Are the following groups? If yes, prove it. If not, say why not
   (a) (2 points) $\mathbb{Z}$ with the binary operation $\star$ defined by $a \star b = 2a + b$
   (b) (2 points) $\mathbb{N}$ under multiplication
   (c) (2 points) The set $\{1, -1\}$ under multiplication.
   (d) (2 points) $\mathbb{Z}/15\mathbb{Z}$ under addition
   (e) (2 points) $\mathbb{Z}/15\mathbb{Z}$ under multiplication
   (f) (2 points) $M_{2\times 2}(\mathbb{R})$, with the binary operation $\star$, defined by $A \star B = AB - BA$.
5. (5 points) Let $(G, \cdot)$ and $(H, *)$ be two groups. Show that $G \times H$ is a group, under the binary operation $\star$ defined by
   $$(g, h) \star (g', h') = (g \cdot g', h * h')$$
6. (5 points) Prove that cyclic groups are abelian
7. (a) (10 points) Let $a, b \in \mathbb{N}$. Show that $\operatorname{lcm}(a, b) = \dfrac{ab}{\gcd(a, b)}$.
   (b) (5 points) Let $a, n \in \mathbb{N}$ with $n \neq 0$. Prove that $o([a]) = \dfrac{n}{\gcd(a, n)}$ in $\mathbb{Z}/n\mathbb{Z}$
8. (5 points) Can a non-abelian group have an abelian subgroup? If yes, give an example. If not, prove why not.
9. (5 points) Let p be a prime number and let G be a group of order p. Prove that G is abelian.
10. (10 points) Let G be a group of order 4. Show that G is abelian. (Hint: we can write $G = \{e, a, b, c\}$ where e, a, b, c are all distinct. What can $o(a)$ be? Break the problem up into cases) It turns out there are only two different groups of order 4: $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.