Math 4400 Homework 2

Due: Wednesday, May 31st, 2017 (Quiz on Friday)

Feel free to work with your classmates, but everyone must turn in their own assignment. Please make a note of who you worked with on each problem. Let me know if you find a typo, or you're stuck on any of the problems.

1. Let $a_1, \ldots, a_n$ be nonzero integers, with $n \geq 2$. We define the greatest common denominator of this n-tuple recursively:
$$\gcd(a_1, \ldots, a_n) = \gcd\left(\gcd(a_1, \ldots, a_{n-1}), a_n\right)$$
and $\gcd(a_1, a_2)$ is the usual gcd. Prove the following generalization of Bezout's lemma: the equation
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$
has a solution with $x_1, \ldots, x_n \in \mathbb{Z}$ if and only if b is divisible by $\gcd(x_1, \ldots, x_n)$.

2. Prove the theorem we mentioned in class about how to get continued fractions expansions from the Euclidean algorithm. Namely, suppose $a, b \in \mathbb{Z}$ are integers with $a, b \geq 1$. Suppose the Euclidean algorithm applied to a and b goes as
$$\begin{aligned} b &= q_1a + r_1 \\ a &= q_2r_1 + r_2 \\ &\vdots \\ r_{n-1} &= q_{n+1}r_n \end{aligned}$$
for some $n \geq 0$. Show that $\dfrac{b}{a} = [q_1; q_2, \ldots, q_{n+1}]$

3. (a) Find all integer solutions of $13853x + 6951y = \gcd(13853, 6951)$.
   (b) Show that $427x + 259y = 13$ has no integer solutions

4. Suppose $a, b, c \in \mathbb{Z}$, $a \neq 0$. Suppose also that $c \mid a$ and $c \mid b$. Show that $c \mid \gcd(a, b)$.

5. Suppose $\gcd(a, b) = 1$, $a \mid c$, and $b \mid c$. Show $ab \mid c$.

6. Suppose $\gcd(a, b) = 1$ and $a \mid bc$. Show that $a \mid c$.

7. Let a and b be two positive integers. Let $S = \left\{c \in \mathbb{N} \middle| a \mid c, b \mid c\right\}$. Then S is nonempty, since it contains ab, so it has a minimal element. This minimal element is called the *lowest common multiple* of a and b and denoted $\operatorname{lcm}(a, b)$. Show that $\operatorname{lcm}(a, b)$ divides every other element of S. Hint: use the division algorithm.

8. Find a formula for all the points on the hyperbola
$$x^2 - y^2 = 1$$
whose coordinates are rational numbers