WORKSHEET #5

DUE MONDAY, OCTOBER 30TH, IN GRADESCOPE

You may turn in this assignment in a group of up to 4 people.

Definition. Suppose G is a group and S is a set. A (left) *group action/operation (of G on S)* is a map

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, s) &\longmapsto g * s \end{aligned}$$

that satisfies the following conditions.

(1) $1 * s = s$ for all $s \in S$ (here $1 \in G$ is the identity element).
(2) $(gg') * s = g * (g' * s)$ for all $g, g' \in G$ and $s \in S$. (Note gg' is just multiplication in g).

Eventually, we'll stop writing the asterisk, but for now, it's just a fancy way to write a smiley face.

1. Consider the following groups G and sets S. Determine if the given operation is a group action or not.

(a) $G = O_n$ ($n \times n$ orthogonal matrices) and $S = \mathbb{R}^n$. The operation is $(A, v) \mapsto Av$ (ie, matrix times a vector).

(b) $G = (\mathbb{Z}, +)$, $S = \mathbb{R}_{>0}$ (positive real numbers). The operation is $(n, x) \mapsto x^n$.

(c) G is any group, H is a subgroup, and $S = \{aH \mid a \in G\}$ (the set of left cosets of H). The actions is $(g, aH) \mapsto gaH$.

Definition. Suppose G is a group acting on a set S. For any $s \in S$ we define the *orbit of* s, denoted O_s, to be

$$\{s' \in S \mid s' = gs \text{ for some } g \in G\}.$$

2. Suppose that G is a group acting on a set S. Prove that the distinct orbits partition S.

3. For the following group actions, and $s \in S$, compute the orbit O_s.

(a) $G = O_2$, $S = \mathbb{R}^2$, the action is multiplication of matrices times vectors $v \mapsto Av$, $s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) G is the cyclic subgroup of O_2 generated by the matrix $\begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix}$. $S = \mathbb{R}^2$ and the action is multiplication of matrices times vectors $v \mapsto Av$. $s = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

(c) G is a group, H is a subgroup and $S = \{aH \mid a \in G\}$ is the set of left cosets of H. The action is as in 1(c). $s = eH$.

Definition. Suppose G is acting on a set S. Fix $s \in S$. The *stabilizer of* s, denoted G_s, is the set

$$\{g \in G \mid g * s = s\}.$$

4. With notation as above, prove that the stabilizer G_s is a subgroup of G.

5. Compute the stabilizers of $s \in S$ with respect to the given group action.

(a) $G = O_2$, $S = \mathbb{R}^2$, the action is as in 3(a), and $s = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

(b) $G = \left\langle \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix} \right\rangle$, $S = \mathbb{R}^2$ and $s = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ are as in 3(b)

(c) G is a group, N is a normal subgroup, and $S = \{aN \mid a \in G\}$ is the set of left cosets of N. The action is as in 1(c). Finally, let $s = aN$ for some $a \in G$.

6. Suppose A is an orthogonal matrix with determinant -1. Prove that A has an eigenvalue of -1.

Hint: Note that to show A has eigenvalue -1, it suffices to show that $\det(A+I)=0$. Compute $\det(A^t(A+I))$ in several ways. Recall A^t is the transpose of A.

7. Describe *geometrically* the action of a 3×3 orthogonal matrix A with determinant -1.

Hint: We know there is an eigenvalue of -1. Let v be the associated eigenvector and $W=v^\perp$ the linear space orthogonal to v. Let B be the matrix with determinant -1 that reflects across W. Consider BA, an orthogonal matrix. We should already know $\det(BA)$ and a geometric interpretation of BA, use that.