WORKSHEET #2

DUE FRIDAY, SEPTEMBER 8TH

You may turn in this assignment in a(n Abelian?) group of up to 4 people.

1. Determine if the following subsets H are *subgroups* of the given group G.

(a) G is the group of all bijective functions $f : \mathbb{Z} \to \mathbb{Z}$ which is a group under composition. H is the set of all elements $f \in G$ such that $f(7) = 7$.

(b) $G = \mathbb{R}_{>0}$ is the group of positive real numbers under multiplication. H is the set of integer powers of 2, that is $H = \{2^i \mid i \in \mathbb{Z}\}$.

(c) G is the group of of bijective functions $f : \{1,2,3\} \to \{1,2,3\}$ where the operation is composition and H is the subset of G made up of bijective functions such that $f(i) = i$ for some $i = 1,2,3$. That is $H = \big\{f \in G \mid \exists i \in \{1,2,3\} \text{ such that } f(i) = i\big\}$.

The following 8 transformations make up the group D4. (You may think of them as a certain set of functions from the vertices of the square to the vertices of the square).

We consider this as a set with law of composition – composition of functions. Closure is not obvious, though you will check it below.

You may assume that D4 is indeed a group (for the moment anyways, we'll check closure and inverses later).

2. Prove that D4 is *not* Abelian, that is, show that the function composition is not commutative.

3. Find the cyclic subgroups generated by:

(a) e
(b) $r90$,
(c) $f3$.

4. Identify the smallest subgroup of D4 which contains both $r180$ and $f1$.

Hint: It must contain $r180 \circ f1$ and $f1 \circ r180$ and $f1 \circ f1$ and ... and inverses to all these elements.

5. Write down a complete group multiplication table for D4. For example, in row $r270$ and column $f1$, you should put $f1 \circ r270$ (it is one of the other things in D_4).

What do you notice about every column and row?

Hint: You are working in groups. Do one row and column together, and then divide the labor.

	e	$r90$	$r180$	$r270$	$f1$	$f2$	$f3$	$f4$	
e									
$r90$									
$r180$									
$r270$									
$f1$									
$f2$									
$f3$									
$f4$									

6. Use the multiplication table you made on the previous page to conclude that $D4$ is indeed a group.

7. Find all the subgroups of $D4$.

8. Suppose $n > 0$ is an integer and $U = \{1 \leq x < n \mid \gcd(x, n) = 1\}$. That is, U is the integers between 1 and n that are relatively prime to n. Show that for any two elements $x, y \in U$, that $x \cdot_n y \in U$ also. Again, $\cdot_n$ is multiplication mod n.

Hint: There is more than one one to do this. One approach is to use the fact that p is a prime factor of xy if and only if p is a factor of x or p is a factor of y.

9. With notation as above, suppose $x \in U$. Show that there exists $y \in U$ such that $x \cdot_n y = 1$. As a consequence, assuming $\cdot_n$ is associative (which you are free to assume). Deduce that U is an Abelian group under $\cdot_n$.

Hint: Again, $\gcd(z, n) = 1$ if and only if there exist $s, t \in \mathbb{Z}$ such that $sz + tn = 1$.

10. [Open ended] Work with your group to compute example of U for various n. Collect data via computation by hand, or if someone in your group has some computer background, perhaps with a computer. Try determine for which n is the group U cyclic. Don't prove your answer, but have some data to justify your guess.

Hint: Some interesting n to consider: $n = 4, 8, 3, 9, 27, 6, 18, 12, 20$. (Please do not try to google the answer, but if somehow you already know it, that's ok).