Jing-Song Huang and Pavle Pandžić

Dirac Operators in Representation Theory

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Preface

We begin with the origin of the Dirac equation and give a brief introduction of the Dirac operators due to Parthasarathy, Vogan and Kostant. Then we explain a conjecture of Vogan on Dirac cohomology, which we proved in [HP1], its applications and the organization of the book.

0.1 The Dirac equation. The Dirac equation has an interesting connection to $E = mc^2$, the Einstein's equation from his special theory of relativity. This equation relates the energy E of a particle at rest to its mass m through a conversion factor, the square of the speed c of light. However, this way of writing the equation obscures the underlying four-dimensional geometry. The relation for a particle in motion is a hyperbolic equation:

$$E^2 - c^2\mathbf{p}\cdot\mathbf{p} = (mc^2)^2. \tag{0.1}$$

Here the energy E is the first component of a vector $(E, c\mathbf{p}) = (E, cp_1, cp_2, cp_3)$ and $\mathbf{p}$ is the vector that describes the momentum of the particle. This more general equation exhibits the mechanism of the conversion of mass into relative motion.

For describing relativistic spin $\frac{1}{2}$ particles Dirac was to rewrite the quadratic Einstein relation (0.1) as a linear relation. This would seem impossible. But Dirac came up with a new idea by writing the relation as follows:

$$\gamma_0 E + c\sum_{j=1}^{3}\gamma_j p_j = mc^2 I, \tag{0.2}$$

where γ_k $(k = 0, 1, 2, 3)$ are 4×4 matrices and I is the 4×4 identity matrix. The four matrices $\gamma_0, \gamma_1, \gamma_2$ and γ_3 are anticommutative: $\gamma_j\gamma_k = -\gamma_k\gamma_j$ for $j \neq k$. Furthermore, they satisfy $\gamma_0^2 = I$ and $\gamma_j^2 = -I$ for $j \neq 0$. In quantum mechanics energy and momentum are expressed by differential operators:

$$E = i\hbar\frac{\partial}{\partial t}, \quad \mathbf{p} = -i\hbar\nabla,$$

where $\hbar$ is the Planck constant. Substituting the above differentials for E and $\mathbf{p}$ into (0.2), one obtains the Dirac equation:

$$i\hbar\gamma_0\frac{\partial}{\partial t} - i\hbar c\sum_{j=1}^{3}\gamma_j\frac{\partial}{\partial x_j} = mc^2 I, \tag{0.3}$$

This matrix valued first order differential equation has had a remarkable success in describing many elementary particles that make up matter. The various analogs of the corresponding differential operator are called Dirac operators. The impact of the Dirac operators on the development of mathematics is also significant. The extension of the definition of Dirac operator to a differentiable manifold and a proof of the corresponding index theorem by Atiyah and Singer is one of the most influential theories of mathematics in the twentieth century.

0.2 Group representations and discrete series. Representations of finite groups were studied by Dedekind, Frobenius, Hurwitz and Schur at the beginning of the 20th century. In the 1920s, the focus of investigations was representation theory of compact Lie groups and its relations to invariant theory. Cartan and Weyl obtained the well-known classification of equivalence classes of irreducible unitary representations of connected compact Lie groups in terms of highest weights. In the 1930s, Dirac and Wigner initiated the investigation of infinite-dimensional representations of noncompact Lie groups.

Harish-Chandra was a Ph. D. student of Dirac at the University of Cambridge from 1945 to 1947. After receiving Ph. D. Harish-Chandra began a systematical investigation of infinite-dimensional representations of semisimple Lie groups and laid down the foundation for further development. Let G denote a semisimple Lie group with finite center, the discrete series representations are the irreducible representations contained in the decomposition of the regular representations on $L^2(G)$. In 1965 and 1966, Harish-Chandra published two papers which gave a complete classification of discrete series representations. Later he used this classification to prove the Plancherel formula. This classification of discrete series is also crucial to Langlands classification of admissible representations. However, Harish-Chandra did not give explicit construction of discrete series. His work was parallel to that of Cartan-Weyl for irreducible unitary representations of compact Lie groups.

0.3 Dirac cohomology and Vogan's conjecture. The Dirac operator was used for construction of the discrete series representations by Parthasarathy and Atiyah-Schmid. Denote by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ the Lie algebras of G and K, where K is a maximal compact subgroup of G. We drop the subscript for their complexifications. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. The Killing form on $\mathfrak{g}$, which is non-degenerate on $\mathfrak{p}$, defines the Clifford algebra $C(\mathfrak{p})$ as an associative algebra with unit. Given an orthonormal basis Z_i of $\mathfrak{p}$, Vogan defined an algebraic version of the Dirac operator to be

$$D = \sum_i Z_i \otimes Z_i \quad \in \quad U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

It is easy to see that D is independent of the choice of basis Z_i and K-invariant (for the adjoint action of K on both factors). Then it can be shown that

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C,$$

where C is a constant and $\mathfrak{k}_\Delta$ denotes a diagonal embedding of $\mathfrak{k}$ into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

If S is a space of spinors (a simple $C(\mathfrak{p})$-module), then D acts on $X \otimes S$. The Dirac cohomology is defined to be

$$H_D(X) = \text{ Ker } D / \text{ Ker } D \cap \text{ Im } D.$$

The following statement was conjectured by Vogan and proved in [HP1]. For any $z \in Z(\mathfrak{g})$ there is a unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$, and there are K-invariant elements $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, such that

$$z \otimes 1 = \zeta(z) + Da + bD.$$

Moreover, $\zeta \colon Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta)$ is an algebra homomorphism having a simple explicit description in terms of Harish-Chandra isomorphisms.

This allows us to identify the infinitesimal character of an irreducible $(\mathfrak{g}, K)$-module that has nonzero Dirac cohomology. Kostant extends this result to his cubic Dirac operator defined in a more general setting of a pair of quadratic Lie algebras.

0.4 Applications and organization of the book. Determination of infinitesimal character by Dirac cohomology enables us to simplify proofs of a few classical theorems and even sharpen some. After some preliminaries in Chapters 1 and 2, we explain our proof of the Vogan's conjecture in Chapter 3. In Chapter 4 we obtain a simpler proof of a generalized Weyl Character formula due to Gross, Kostant, Ramond and Steinberg as well as a generalized Bott-Borel-Weil theorem. Chapters 5 and 6 provide the necessary background of cohomological parabolic induction. In Chapter 7 we give a simpler proof of the construction and classification of the discrete series representations. In Chapter 8 we sharpen the Langlands formula on automorphic forms and obtain the relation of Dirac cohomology to $(\mathfrak{g}, K)$-cohomology. Chapter 9 is on the relation of Dirac cohomology to Lie algebra cohomology. In chapter 10 we prove an analogue of the Vogan's conjecture for basic classical Lie superalgebras.

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Jing-Song Huang
Pavle Pandžić

Contents

1

Lie groups, Lie algebras and representations

In this preliminary chapter we will outline an introduction to basic structure and representation theory of Lie groups and algebras. For those who are not already acquainted with this material, the hope is that the little we will say, perhaps with a little supplementing from the quoted literature, could be enough to proceed without plunging into a long and serious study of the many things involved in this theory. For those who are already familiar with the matter, this chapter can either be skipped, or can serve as a quick reminder of some of the main points. To keep it as simple as possible, we will mostly explain things in the case of matrix groups, which in any case contains the main examples.

1.1 Lie groups and algebras

Definition 1.1.1. A *Lie group* G is a group which is also a smooth manifold, in such a way that the group operations are smooth. In more words, the multiplication map from $G \times G$ into G and the inverse map from G into G are required to be smooth.

Morphisms between two Lie groups G and H are smooth maps which are also group homomorphisms. A *Lie subgroup* of G is a Lie group H together with a one-to-one immersion $\phi : H \to G$ (an immersion is a smooth map whose differential is one-to-one at every point). An especially nice case is when the image of ϕ is closed - then H is called a *closed subgroup* . Let us point out that it will not create confusion to say just "closed subgroup" instead of "closed Lie subgroup" in view of the Theorem 1.1.3 below.

Examples 1.1.2. Here are the main examples of Lie groups. Consider the group $GL(n, \mathbb{R})$ of invertible $n \times n$ real matrices. The space $M_n(\mathbb{R})$ of all $n \times n$ real matrices can be identified with $\mathbb{R}^{n^2}$ in the obvious way, by putting the matrix elements into one vector, row by row. So we can consider the standard Euclidean topology on $M_n(\mathbb{R})$. Then $GL(n, \mathbb{R})$ is an open set in

$M_n(\mathbb{R})$, because it is defined by an open condition $\det g \neq 0$. In particular, it is a differentiable manifold. Moreover, it is clear that both multplication and inverting are smooth operations - in fact, their components are rational functions in the matrix entries. Thus $GL(n,\mathbb{R})$ is a Lie group. In a very similar way one sees that $GL(n,\mathbb{C})$ is a Lie group, as an open subset of $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$.

Further examples are most easily obtained via the following fundamental theorem:

Theorem 1.1.3. (Cartan). *Every closed subgroup of a Lie group is a Lie subgroup in a unique way.*

For a proof of this theorem see [War] or [**?**]. All of the groups we define below are obviously closed subgroups of $GL(n,\mathbb{C})$; therefore they are Lie subgroups, and in particular Lie groups. In the following $\mathbb{F}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$.

Examples 1.1.4. The group $SL(n,\mathbb{F})$ of $n\times n$ matrices over $\mathbb{F}$ of determinant 1 is a closed subgroup of $GL(n,\mathbb{F})$ and therefore a Lie group. Further examples are obtained using bilinear forms.

Let $(x,y) = \sum x_i y_i$ denote the standard bilinear form on $\mathbb{F}^n$. Then all bilinear forms B on $\mathbb{F}^n$ are in one to one correspondence with matrices $H \in M_n(\mathbb{F})$: the relationship is

$$B(x,y) = (Hx,y), \qquad x,y \in \mathbb{F}^n.$$

The group $G = GL(n,\mathbb{F})$ acts on the space of all bilinear forms on $\mathbb{F}^n$: $g \in G$ sends B to the form

$$B^g(x,y) = B(gx,gy), x,y \in \mathbb{F}^n).$$

(Note that this is a right action, not a left one, but it is the customary one.)

Since $B(gx,gy) = (Hgx,gy) = (g^\tau Hgx,y)$, where g^τ denotes the transpose of the matrix g, the matrix corresponding to B^g is $g^\tau Hg$. Thus g preserves the form B if and only if $g^\tau Hg = H$.

On $\mathbb{C}^n$, we can also consider Hermitian forms B; using the standard Hermitian form $\langle\, , \rangle$, these forms correspond to matrices H in exactly the same way as above. The only difference is that the action of $g \in GL(n,\mathbb{C})$ is now given by $H \mapsto g^*Hg$, where g^* denotes the conjugate transpose of g.

In particular, if B is a nondegenerate symmetric bilinear form on $\mathbb{C}^n$, then in a suitable basis its matrix is the identity matrix I. Thus $g \in GL(n,\mathbb{C})$ preserves B if and only if $g^\tau g = I$ and the Lie group of such matrices is denoted by $O(n,\mathbb{C})$. Over $\mathbb{R}$, if B has signature (p,q), then the corresponding diagonal matrix $H_{p,q}$ has p one's and q minus one's on the diagonal. The resulting Lie group consists of matrices satisfying $g^\tau H_{p,q} g = H_{p,q}$ and is denoted by $O(p,q)$. In particular, $O(n) = O(n,0)$ is the group of (real) orthogonal matrices.

Similarly, if we consider a nondegenerate hermitian form of signature (p, q) on $\mathbb{C}^n$, we arrive at the Lie group $U(p,q)$. In particular, $U(n) = U(n,0)$ denotes the group of unitary matrices.

If besides preserving the form we also impose the condition $\det g = 1$, we get the Lie groups $SO(n,\mathbb{C})$, $SO(p,q)$, $SO(n)$, $SU(p,q)$ and $SU(n)$.

By considering symplectic (i.e., bilinear, nondegenerate, skew symmetric) forms instead of symmetric ones, we arrive at symplectic groups $Sp(2n,\mathbb{C})$ and $Sp(2n,\mathbb{R})$. These can only be defined on even dimensional spaces, and their elements are automatically of determinant 1.

Other important examples are the groups $B(n,\mathbb{F})$ of upper triangular matrices and their various subgroups, in particular the groups $N(n,\mathbb{F})$ of unipotent matrices, i.e., upper triangular matrices with 1's on the diagonal. There are also abelian groups like $\mathbb{C}^n$, $\mathbb{R}^n$ or the torus $\mathbb{T}^n$. In the context of matrix groups, these show up as diagonal matrices.

1.1.5. The Lie algebra of a Lie group. Let G be a Lie group and let us denote the tangent space to G at the identity by $\mathfrak{g}$. Then $\mathfrak{g}$ is not merely a vector space, for we can define an operation $[\,,]$ on $\mathfrak{g}$ as follows. First, any $g \in G$ defines an inner automorphism of G called $\mathrm{Int}\,(g)$:

$$\mathrm{Int}\,(g)h = ghg^{-1}, \qquad h \in G.$$

Taking the differential of $\mathrm{Int}\,(g)$ at $h = e$, the identity of G, we obtain a vector space automorphism of $\mathfrak{g}$ which we denote by $\mathrm{Ad}\,(g)$. In this way we get a Lie group morphism $\mathrm{Ad}\, : G \to GL(\mathfrak{g})$. Differentiating Ad at $g = e$, we obtain a linear map from $\mathfrak{g}$ into $\mathrm{End}\,(\mathfrak{g})$ which we denote by ad . Namely, as $GL(\mathfrak{g})$ is open in $\mathrm{End}\,(\mathfrak{g})$, the tangent space to $GL(\mathfrak{g})$ at the identity is all of $\mathrm{End}\,(\mathfrak{g})$. Now $[\,,]$ is defined by

$$[X,Y] = \mathrm{ad}\,(X)Y, \qquad X, Y \in \mathfrak{g}.$$

The operation $[\,,]$ is called the commutator or bracket. With this operation, $\mathfrak{g}$ becomes a Lie algebra in the sense of the following definition.

Definition 1.1.6. A Lie algebra over $\mathbb{F}$ is a vector space $\mathfrak{g}$ with a bilinear anticommutative operation $[\,,]$, such that for every $X \in \mathfrak{g}$, the operator $\mathrm{ad}\, X$ on $\mathfrak{g}$ sending Y to $[X,Y]$ is a derivation, i.e.,

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]], \qquad X,Y,Z \in \mathfrak{g}.$$

Another common way to formulate the last equality is the so called Jacobi identity, $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = 0$. We will only work with finite-dimensional Lie algebras in this book, so whenever we say "a Lie algebra" we mean a finite-dimensional one.

One defines notions as morphisms, Lie subalgebras, ideals, etc. in the usual way.

Example 1.1.7. The vector space $M_n(\mathbb{F})$ with the commutator

$$[A,B] = AB - BA, \qquad A,B \in M_n(\mathbb{F})$$

is a Lie algebra over $\mathbb{F}$. We denote this algebra by $\mathfrak{gl}(n,\mathbb{F})$.

Proposition 1.1.8. *If G is a Lie group, then $\mathfrak{g}$ with the operation $[\,,]$ as defined in 1.1.5 is a Lie algebra.*

For a general proof of this proposition, see [**?**], pp. 46-47. We will explain it for matrix groups, where $\mathfrak{g}$ turns out to be a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{F})$.

Examples 1.1.9. Let G be a Lie subgroup of $GL(n,\mathbb{F})$ and denote by $\mathfrak{g}$ the Lie algebra of G. Then $\mathfrak{g}$ is a subspace of $M_n(\mathbb{F})$, since $GL(n,\mathbb{F})$ is an open subset of $M_n(\mathbb{F})$.

Let $X \in \mathfrak{g}$. We say that a curve α in G corresponds to X if $\alpha(0) = I$ and $\alpha'(0) = X$. In that case, for any $g \in G$,

$$\mathrm{Ad}\,(g)X = \frac{\mathrm{d}}{\mathrm{d}t} g\alpha(t)g^{-1}\big|_{t=0} = g\alpha'(0)g^{-1} = gXg^{-1}.$$

Namely, we interpreted the calculation in $M_n(\mathbb{F})$, where we used the Leibniz rule for the matrix product and observed that g is constant with respect to t.

Let now $X, Y \in \mathfrak{g}$ and let α correspond to X as above. Using the Leibniz rule again, we see that

$$\begin{aligned}[X,Y] = \ \mathrm{ad}\,(X)Y = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}\,(\alpha(t))Y\big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)Y\alpha(t)^{-1}\big|_{t=0} = \\ &\alpha'(0)Y + Y\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)^{-1}\big|_{t=0} = XY - YX;\end{aligned}$$

namely, differentiating $\alpha(t)\alpha(t)^{-1} = I$ using the Leibniz rule, we see that $\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)^{-1}\big|_{t=0} = -X$. So we see that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{F})$; in particular, it is a Lie algebra.

In general, whenever H is a Lie subgroup of G, the Lie algebra $\mathfrak{h}$ of H embeds into the Lie algebra $\mathfrak{g}$ of G as a Lie subalgebra. This is a special case of the following proposition, which asserts that the correspondence $G \mapsto \mathfrak{g}$ is functorial. The proof is straightforward.

Proposition 1.1.10. *Let $\varphi : G \to H$ be a morphism of Lie groups and let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Then the differential $d\varphi : \mathfrak{g} \to \mathfrak{h}$ of φ at the identity is a morphism of Lie algebras.*

Remark 1.1.11. The Lie algebra $\mathfrak{g}$ of G can also be defined to consist of left invariant (smooth) vector fields on G. The left invariance condition means the following: let $l_g : G \to G$ be the left translation by $g \in G$, i.e., $l_g(h) = gh$. A vector field X on G is left invariant if $(dl_g)_h X_h = X_{gh}$ for all $g,h \in G$. The Lie algebra operation in this setting is the bracket of vector fields: $[X,Y]f =$

$X(Yf)-Y(Xf)$, for $X, Y \in \mathfrak{g}$ and f a smooth function on G. Here we identify vector fields with derivations of the algebra $C^\infty(G)$, i.e., think of them as first order differential operators.

To relate the two constructions, notice that to any left invariant vector field one can attach its value at e and conversely, a tangent vector at e can be translated to all other points of G to obtain a left invariant vector field. One shows that this identification also respects the commutators; see e.g. [**?**], pp. 47-48.

We now want to identify the Lie algebras of the matrix groups described in Examples 1.1.4.

Examples 1.1.12. Suppose G is one of the groups $O(p,q)$, $O(n,\mathbb{C})$, $Sp(n,\mathbb{F})$. So G is the set of matrices g such that $g^\tau Hg = H$, where H is the matrix of the bilinear form B defining G. Let $X \in \mathfrak{g}$ and let α be a curve in G corresponding to X. Then differentiating $\alpha(t)^\tau H\alpha(t) = H$ at $t = 0$ we obtain

$$X^\tau H + HX = 0. \tag{1.1}$$

So, if X is in $\mathfrak{g}$, then X is skew symmetric with respect to H. Conversely, suppose that $X \in \mathfrak{gl}(n,\mathbb{F})$ satisfies (1.1). To see that $X \in \mathfrak{g}$, it suffices to exhibit a curve α in G corresponding to X. We claim that such a curve is given by

$$\alpha(t) = \mathrm{e}^{tX}, \qquad t \in \mathbb{R}.$$

Indeed, it is clear that $\alpha(0) = I$ and $\alpha'(0) = X$, so we only need to check that $\mathrm{e}^{tX} \in G$ for all t. But (1.1) implies that

$$(tX^\tau)^n H = H(-tX)^n$$

for every $n \geq 0$ and every $t \in \mathbb{R}$, and hence

$$\mathrm{e}^{tX^\tau} H = \left(\sum_{n=0}^{\infty} \frac{1}{n!}(tX^\tau)^n\right) H = H \left(\sum_{n=0}^{\infty} \frac{1}{n!}(-tX)^n\right) = H\mathrm{e}^{-tX}.$$

This implies $\mathrm{e}^{tX^\tau} H\mathrm{e}^{tX} = H$, and since $\mathrm{e}^{tX^\tau} = (\mathrm{e}^{tX})^\tau$, this means $\mathrm{e}^{tX} \in G$ as claimed. So we get that the Lie algebras of $O(p,q)$, $O(n,\mathbb{C})$ and $Sp(n,\mathbb{F})$ are respectively $\mathfrak{o}(p,q)$, $\mathfrak{o}(n,\mathbb{C})$ and $\mathfrak{sp}(n,\mathbb{F})$, the Lie subalgebras of $\mathfrak{gl}(n,\mathbb{F})$ defined by (1.1) for the appropriate choice of H.

In a completely analogous way one sees that the Lie algebra of $U(p,q)$ is

$$\mathfrak{u}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \,|\, X^*H + HX = 0\},$$

where $H = H_{p,q}$ is the matrix of the hermitian form defining $U(p,q)$ and $*$ denotes the conjugate transpose.

Let us now examine the condition $\det g = 1$. Let α be any curve in $GL(n,\mathbb{F})$ with $\alpha(0) = I$. We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \alpha(t)\Big|_{t=0} = \operatorname{tr} \alpha'(0).$$

To see this, write

$$\det \alpha(t) = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} \alpha(t)_{i\sigma(i)},$$

with S_n the symmetric group on n letters. When we differentiate this expression with respect to t and set $t = 0$, the only nonzero terms in the above sum will be the ones with σ equal to the identity (because of $\alpha(0) = I$). Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \alpha(t)\Big|_{t=0} = \sum_{i=1}^{n} \alpha'(0)_{ii} = \operatorname{tr} \alpha'(0).$$

We conclude that the Lie algebra of $SL(n, \mathbb{F})$ is contained in

$$\mathfrak{sl}(n, \mathbb{F}) = \{X \in \mathfrak{gl}(n, \mathbb{F}) \bigm| \operatorname{tr} X = 0\}.$$

Conversely, if $X \in \mathfrak{sl}(n, \mathbb{F})$, then e^{tX} is a curve in $SL(n, \mathbb{F})$ with velocity X at I. Namely, e^{tX} is in $SL(n, \mathbb{F})$ for all t, because

$$\det \mathrm{e}^{A} = \mathrm{e}^{\operatorname{tr} A}, \qquad A \in \mathfrak{gl}(n, \mathbb{F}),$$

as is easily seen by replacing A by its Jordan form.

It now also follows that the Lie algebras of the groups $SU(p, q)$, $SO(p, q)$ and $SO(n, \mathbb{C})$ are respectively the Lie algebras $\mathfrak{su}(p, q)$, $\mathfrak{so}(p, q)$ and $\mathfrak{so}(n, \mathbb{C})$, obtained from $\mathfrak{u}(p, q)$, $\mathfrak{o}(p, q)$ and $\mathfrak{o}(n, \mathbb{C})$ by imposing an additional condition $\operatorname{tr} = 0$.

Finally, using similar methods it is not difficult to check that the Lie algebra of the group $B(n, \mathbb{F})$ of invertible upper triangular matrices is the Lie algebra $\mathfrak{b}(n, \mathbb{F})$ of all upper triangular matrices, while the Lie algebra of the group $N(n, \mathbb{F})$ of unipotent matrices is the Lie algebra $\mathfrak{n}(n, \mathbb{F})$ of all strictly upper triangular matrices.

Note how in understanding the examples an important role was played by the exponentials e^{tX}; a crucial property was that whenever $X \in \mathfrak{g}$, e^{tX} is in G. The exponential map can be defined and plays a crucial role also in the general situation. It is closely related to the notion of *one-parameter subgroups* in G; these are Lie group morphisms from $\mathbb{R}$ into G.

Theorem 1.1.13. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $X \in \mathfrak{g}$. Then there is a unique one-parameter subgroup in G with velocity X at $t = 0$.*

This theorem is proved by using the theory of ordinary differential equations; in fact, a one parameter subgroup corresponding to X is an integral curve for the left invariant vector field on G defined by X.[1]

[1]The theorem can also be derived from the more general facts about subgroups, subalgebras and mappings, like in [War]. These more general facts are however again obtained using differential equations.

For example, if $G = GL(n, \mathbb{F})$, we can differentiate the condition

$$\varphi(t+s) = \varphi(t)\varphi(s)$$

with respect to s and then set $s = 0$ to obtain $\varphi'(t) = X\varphi(t)$. This differential equation with the initial condition $\varphi(0) = I$ has e^{tX} as the only solution.

Getting back to the general situation, we denote the one-parameter subgroup corresponding to X by $\exp_X$. Putting all the one-parameter subgroups together, one gets the *exponential map* $\exp : \mathfrak{g} \to G$, defined by

$$\exp(X) = \exp_X(1), \qquad X \in \mathfrak{g}.$$

It now follows from the uniqueness of one-parameter subgroups that $\exp_X(t) = \exp(tX)$, for every $t \in \mathbb{R}$, so the one-parameter subgroups are all given as $t \mapsto \exp(tX)$ for various X. Moreover, one shows that exp is smooth and that it maps a neighborhood of 0 in $\mathfrak{g}$ diffeomorphically onto a neighborhood of the identity in G.

For all matrix groups, exp is just the ordinary exponential map, the one we used in our examples. This is a special case of the following functoriality principle: if $\varphi : G \to H$ is a Lie group morphism, then the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{d}\varphi} & \mathfrak{h} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\varphi} & H \end{array}$$

commutes. This again follows from the uniqueness of one-parameter subgroups.

If we apply this functoriality principle to $\varphi = \operatorname{Int}(g) : G \to G$, we get the useful formula

$$g \exp X\, g^{-1} = \exp \operatorname{Ad}(g)X, \qquad g \in G, X \in \mathfrak{g}.$$

For $\varphi = \operatorname{Ad} : G \to GL(\mathfrak{g})$, we get the formula

$$\operatorname{Ad}(\exp X) = \mathrm{e}^{\operatorname{ad} X}, \qquad X \in \mathfrak{g}.$$

1.1.14. Subgroups and subalgebras. After seeing examples, let us now mention some general results. Most of these are in fact easy to prove assuming everything we have mentioned above.

If H is a Lie subgroup of G, then its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ can be characterized as the set of all $X \in \mathfrak{g}$ with the property that $\exp(tX) \in H$ for every $t \in \mathbb{R}$.

If on the other hand $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$. H is generated by the image of $\mathfrak{h}$ under the exponential map.

If H is a normal subgroup of G, then its Lie algebra $\mathfrak{h}$ is an *ideal* in $\mathfrak{g}$, i.e., $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. Conversely, if $\mathfrak{h}$ is an ideal then the connected subgroup H corresponding to $\mathfrak{h}$ is a normal subgroup of G.

The center $Z(G)$ of G is a closed and therefore Lie subgroup. If G is connected, then the Lie algebra of $Z(G)$ is the center of $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g})$, consisting of all $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Also, if G is connected, $Z(G)$ is the kernel of $\mathrm{Ad} : G \to GL(\mathfrak{g})$. On the other hand, it is clear that $\mathfrak{z}(\mathfrak{g})$ is the kernel of $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

It follows that a connected Lie group is abelian if and only if its Lie algebra is abelian.

1.1.15. Some remarks about classification. Finally, let us say a few words about classifying Lie groups and algebras.

The first question to settle is: to what extent is a Lie group G determined by its Lie algebra $\mathfrak{g}$? First, if two groups G and G' have the same identity component, then their Lie algebras are obviously the same. For example, the group $O(n)$ has two connected components, one of which is the idenity component $SO(n)$. (It is an easy exercise to see that $SO(n)$ is connected, and then the above fact is clear from the fact that an orthogonal matrix has determinant ± 1.) Consequently, the Lie algebras of $O(n)$ and $SO(n)$ coincide, i.e., $\mathfrak{o}(n) = \mathfrak{so}(n)$. This is also obvious from the algebraic point of view: a skew symmetric matrix has zeros on the diagonal, hence its trace is automatically zero.

Second, even if G and G' are both connected, their Lie algebras can still coincide. An example of this situation is the well known double covering map $SU(2) \to SO(3)$, obtained by letting $SU(2)$ viewed as the unit quaternions act on $\mathbb{R}^3$ viewed as purely imaginary quaternions via (quaternionic) conjugation. (We will later on explain this example in more detail, when we encounter the *Spin* group.) In the presence of a covering map, the two Lie algebras must coincide; in the above example, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, as can also be easily checked algebraically.

This is where the ambiguity ends: a connected Lie group is determined by its Lie algebra up to coverings. That is, there is a unique connected simply connected Lie group $\tilde{G}$ with a given Lie algebra $\mathfrak{g}$, and all connected G with Lie algebra $\mathfrak{g}$ are covered by $\tilde{G}$. Furthermore, the kernel of a covering $\tilde{G} \to G$ is a discrete central subgroup of $\tilde{G}$, which can be identified with the fundamental group of G.

An important feature of simply connected groups is the following lifting property; for a proof, see [War], p. 101, or [?], p. 53. The idea is that one gets the graph of φ as the Lie subgroup of $G \times H$ corresponding to the graph of ψ, which is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$.

Theorem 1.1.16. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Let H be any other Lie group with Lie algebra $\mathfrak{h}$. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra morphism. Then there is a unique Lie group morphism $\varphi : G \to H$ with differential equal to ψ.*

The second topic we wish to mention is the question of the kind of Lie groups and algebras one wishes to study. We will formulate some relevant definitions for Lie algebras and omit the discussion of analogous group theoretic definitions; let us just say that more or less a Lie group is "of the same kind as its Lie algebra".

For a Lie algebra $\mathfrak{g}$, define $C^0\mathfrak{g} = D^0\mathfrak{g} = \mathfrak{g}$, and inductively $C^{i+1}\mathfrak{g} = [\mathfrak{g}, C^i\mathfrak{g}]$, $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$. Then $\mathfrak{g}$ is called *nilpotent* if $C^i\mathfrak{g} = 0$ for large i and $\mathfrak{g}$ is called *solvable* if $D^i\mathfrak{g} = 0$ for large i. A typical example of a nilpotent Lie algebra is the Lie algebra $\mathfrak{n}(n, \mathbb{F})$ of strictly upper triangular matrices. A typical example of a solvable Lie algebra is the Lie algebra $\mathfrak{b}(n, \mathbb{F})$ of upper triangular matrices. All subalgebras and quotients of nilpotent (respectively solvable) Lie algebras are themselves nilpotent (respectively solvable). Furthermore, if $\mathfrak{s} \subset \mathfrak{g}$ is a solvable ideal such that the quotient $\mathfrak{g}/\mathfrak{s}$ is also solvable, then $\mathfrak{g}$ is solvable.

A Lie algebra $\mathfrak{g}$ is called *simple* if it has only trivial ideals. For example, the algebras $\mathfrak{sl}(n, \mathbb{F})$, $\mathfrak{so}(n, \mathbb{F})$ and $\mathfrak{sp}(2n, \mathbb{F})$ are simple, except for $\mathfrak{sl}(1, \mathbb{F})$ and $\mathfrak{so}(n, \mathbb{F})$ when n equals 1,2 or 4. For $\mathbb{F} = \mathbb{C}$, these almost exhaust the examples of simple Lie algebras; there are just five more examples - the so called exceptional Lie algebras. For $\mathbb{F} = \mathbb{R}$, there are also the examples $\mathfrak{su}(p, q)$ and $\mathfrak{so}(p, q)$ we mentioned, and some more that we have not mentioned.

A Lie algebra $\mathfrak{g}$ is called *semisimple* if it is a direct sum of simple ideals, and *reductive* if it is the direct sum of its center and a semisimple ideal. For example, $\mathfrak{gl}(n, \mathbb{F})$ is reductive - it has a one-dimensional center consisting of scalar matrices, and a direct complement to the center is the simple Lie algebra $\mathfrak{sl}(n, \mathbb{F})$.

Any Lie algebra is a semidirect product of its largest solvable ideal (the radical) and a semisimple subalgebra. This is the so called *Levi decomposition.* It means that to some extent, understanding solvable and semisimple Lie algebras is enough to understand all Lie algebras.

A Lie group G is called *semisimple* if the Lie algebra of G is semisimple. Following Wallach [W] we define a real reductive group or a reductive Lie group as follows. Let $f_1, \cdots, f_m$ be complex polynomials on $M(n, \mathbb{C})$ such that each f_i is real valued on $M(n, \mathbb{R})$ and such that the set of simultaneous zeros of f_i in $GL(n, \mathbb{C})$ is a subgroup $G_{\mathbb{C}}$ of $GL(n, \mathbb{C})$. Then $G_{\mathbb{C}}$ is called an *affine algebraic group defined over* $\mathbb{R}$. The subgroup $G_{\mathbb{R}} = G_{\mathbb{C}} \cap GL(n, \mathbb{C})$ is called the group of real points. By a *real reductive group* or a *reductive Lie group* we mean a finite covering group G of an open subgroup G_0 of $G_{\mathbb{R}}$. For example, $GL(n, \mathbb{F})$ is reductive and every connected semisimple Lie group with finite center is reductive. Thus, we can define a *Cartan involution* on Lie algebra $\mathfrak{g}$ of a reductive Lie group G by $\theta(X) = -\bar{X}^t$.

We are primarily interested in studying semisimple or reductive Lie algebras (and groups), and we do not mind assuming the groups are connected whenever it is convenient to do so. One can not however completely ignore other situations; for example, as we will see, in the representation theory of semisimple Lie algebras, a crucial role is played by the maximal solvable subalgebras.

1.2 Finite-dimensional representations

Definition 1.2.1. Let V be a complex n-dimensional vector space. A *representation* of a Lie group G on V is a continuous group homomorphism

$$\pi : G \to GL(V).$$

A representation of a (real or complex) Lie algebra $\mathfrak{g}$ $\mathfrak{g}$ on V is a morphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

We have already met some representations: $\mathrm{Ad}\ : G \to GL(\mathfrak{g})$ is a representation of G on its Lie algebra, while $\mathrm{ad}\ : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation of $\mathfrak{g}$ on itself. Bot of these are called adjoint representations.

If $\pi : G \to GL(V)$ is a representation, it follows that π is smooth, i.e., it is a morphism of Lie groups. Namely, any continuous group homomorphism φ between Lie groups is automatically smooth, as follows from Cartan's Theorem 1.1.3 applied to the graph of φ. Hence we can differentiate π at e and obtain a homomorphism

$$\mathrm{d}\pi : \mathfrak{g} \to \mathfrak{gl}(V)$$

of Lie algebras, i.e., a representation of the Lie algebra $\mathfrak{g}$ of G. We will often denote $\mathrm{d}\pi$ just by π; this should not create confusion.

The main idea of passing from G to $\mathfrak{g}$ is turning a harder, analytic problem of studying representations of G into an easier, purely algebraic (or even combinatorial in some sense) problem of studying representations of $\mathfrak{g}$. Actually, since we are only considering complex representations, we can as well complexify $\mathfrak{g}$ and study representations of $\mathfrak{g}_{\mathbb{C}}$. Thus we will from now on speak mostly about complex Lie algebras and their representations. To simplify notation, and avoid writing $\mathfrak{g}_{\mathbb{C}}$ many times, we will from now on denote the (real) Lie algebra of G by $\mathfrak{g}_0$, and its complexification by $\mathfrak{g}$.

Definition 1.2.2. A representation π of a Lie group G (respectively a Lie algebra $\mathfrak{g}$) on V is *irreducible* if it has no nontrivial *subrepresentations*, i.e., V has only trivial subspaces $\{0\}$ and V which are invariant under G (respectively under $\mathfrak{g}$).

An important special class of representations of G consists of *unitary representations*. A representation π of G on V is unitary if V has an inner product such that all the operators $\pi(g)$, $g \in G$, are unitary. Then for any $X \in \mathfrak{g}_0$, the Lie algebra of G, $\pi(X)$ is skew hermitian.

1.2.3. Complete reducibility If π is unitary, then it is *completely reducible*, i.e., every invariant subspace has an invariant direct complement. Namely, if $W \subset V$ is invariant for G, then $W^\perp$ is also invariant for G: if $v \in W^\perp$ and $g \in G$, then

$$\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle = 0, \qquad w \in W,$$

so $\pi(g)v \in W^{\perp}$.

We will now describe the so called *unitarian trick* due to Weyl; it is a short way of showing that finite-dimensional representations of semisimple Lie algebras or groups are always completely reducible. The point is to make use of compact Lie groups, whose representation theory is relatively easy. First, any compact Lie group K has a finite measure $\mathrm{d}k$ invariant under left translations, i.e., satisfying

$$\int_K f(k'k)\mathrm{d}k = \int_K f(k)\mathrm{d}k,$$

for any $k' \in K$. To see this, take a basis of the cotangent space at $e \in G$, and using translations produce left invariant 1-forms, defining a basis of the cotangent space at any point. Taking the exterior product of all these forms will produce a left invariant volume form on K. (This actually works for any Lie group G, but the resulting measure is not finite unless G is compact.)

Having a finite left invariant measure, we can now proceed to conclude that any finite-dimensional representation (π, V) of K is unitary, in the sense that there is a K-invariant Hermitian inner product on V. Namely, taking any inner product $(\,,)$ on V, we can average it over K to produce an invariant inner product $\langle\,,\rangle$:

$$\langle v, w\rangle = \int_K (\pi(k)v, \pi(k)w)\mathrm{d}k, \qquad v, w \in V.$$

The new inner product is clearly K-invariant by the K-invariance of the measure. Now since V is in fact unitary, it is completely reducible as we saw above.

Let now $\mathfrak{g}$ be any complex semisimple Lie algebra. Then there exists a compact Lie group G_c whose complexified Lie algebra is $\mathfrak{g}$. The (real) Lie algebra $\mathfrak{g}_0$ of G_c is called the *compact form* of $\mathfrak{g}$. For a proof of the existence of compact forms, see e.g. [He]. Moreover, G_c can be taken to be simply connected, since the universal covering group of a compact semisimple Lie group is compact. This is a fundamental theorem of Weyl. Its proof can be found e.g. in [**?**], Chapter 3.

For example, if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we can take $G_c = SU(n)$, while for $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ we can take G_c to be the group $Spin(n)$, the universal (double) cover of $SO(n)$.

Now by 1.1.16 any representation V of $\mathfrak{g}_0$ on a complex vector space, which is the same as a representation of $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$, can be lifted to a representation of G_c. It follows that the representation V of $\mathfrak{g}$ is completely reducible. This result is also called Weyl's Theorem.

In view of complete reducibility, to understand finite-dimensional representations of semisimple Lie algebras, it is enough to understand the irreducible representations. This is what we will outline in the rest of this section.

Example 1.2.4. The most basic example and the first one to study is the description of irreducible finite-dimensional representations of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. It is not only an example, but also an extremely useful tool in the study of more complicated representations of larger Lie algebras.

There is an obvious basis for $\mathfrak{g}$: take

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The commutator relations in this basis are readily calculated to be

$$[h,e] = 2e; \qquad [h,f] = -2f; \qquad [e,f] = h.$$

We are going to show the following: for every positive integer k, there is up to isomorphism exactly one irreducible representation of $\mathfrak{g}$ of dimension k. Moreover, this representation has a distinguished basis v_{-n}, v_{-n+2},..., v_{n-2}, v_n, where $n = k-1$ and each v_i is an eigenvector of $\pi(h)$ with the eigenvalue i. Furthermore, $\pi(f)v_{n-2j} = v_{n-2j-2}$ for $j = 0, 1, \ldots, n-1$, while $\pi(f)v_{-n} = 0$. Finally,

$$\pi(e)v_{n-2j} = j(n-j+1)v_{n-2j+2} \tag{1.2}$$

for $j = 1, 2, \ldots, n$, while $\pi(e)v_n = 0$. Here is a picture describing our representation $V = V_n$:

$$\mathbb{C}v_{-n} \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C}v_{-n+2} \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C}v_{n-2} \underset{f}{\overset{e}{\rightleftarrows}} \mathbb{C}v_n$$

The numbers $-n, -n+2, \ldots, n-2, n$, i.e., the eigenvalues of $\pi(h)$, are called the weights of V_n. The weight n is called the highest weight, and the vector v_n, which is unique up to scalar, is called a highest weight vector. It is characterized by the condition $\pi(e)v_n = 0$. We will see later that while in an irreducible representation of a general semisimple Lie algebra there can be many vectors of a given weight, there is always only one highest weight vector up to scalar.

Let us now prove the above claims. Let V be an irreducible finite-dimensional representation of $\mathfrak{g}$. The operator $\pi(h)$ on V has at least one eigenvalue $\lambda \in \mathbb{C}$. Let us fix λ and a corresponding eigenvector v_λ in the eigenspace V_λ.

Let now $v \in V$ be an eigenvector of $\pi(h)$ with any eigenvalue $\mu \in \mathbb{C}$. Then since $\pi(h)\pi(e) - \pi(e)\pi(h) = \pi([h,e]) = 2\pi(e)$, it follows that

$$\pi(h)\pi(e)v = \pi(e)\pi(h)v + 2\pi(e)v = (\mu+2)\pi(e)v.$$

In other words, $\pi(e)v$ is an eigenvector of $\pi(h)$ with the eigenvalue $\mu + 2$. By an analogous calculation, $\pi(f)v$ is an eigenvector of $\pi(h)$ with the eigenvalue $\mu - 2$. This shows that if we add up all eigenspaces of $\pi(h)$ with eigenvalues $\mu \in \lambda + 2\mathbb{Z}$, we are getting a $\mathfrak{g}$-invariant subspace of V. This subspace is nonzero since we assumed $V_\lambda \neq 0$; thus it has to be all of V since V is irreducible. Furthermore, only finitely many V_μ can be nonzero, since V is finite-dimensional. Replacing λ with $\lambda + 2k$ for the largest possible k such that $V_{\lambda+2k} \neq 0$, we can thus assume that $V_\mu \neq 0$ implies $\mu = \lambda - 2j$ for some $j \in \mathbb{Z}_+$. It follows that $\pi(e)V_\lambda = 0$, and in particular $\pi(e)v_\lambda = 0$ for our fixed eigenvector with the eigenvalue λ.

Consider now the vectors $\pi(f)^j v_\lambda \in V_{\lambda-2j}$ for $j = 0, 1, 2, \ldots$. Since all these vectors are linearly independent and V is finite-dimensional, there must be some j such that $\pi(f)^j v_\lambda = 0$. We fix the smallest such j, $j_0 > 0$. On the other hand, we claim that for any $i \in \mathbb{Z}_+$,

$$\pi(e)\pi(f)^i v_\lambda = i(\lambda - i + 1)\pi(f)^{i-1}v_\lambda. \tag{1.3}$$

where the right hand side is defined to be zero if $i = 0$.

This claim is proved by induction on i. It is true for $i = 0$. Assuming it is true for some i, we use the relation $[e, f] = h$ to calculate

$$\begin{aligned}\pi(e)\pi(f)^{i+1}v_\lambda &= (\pi(f)\pi(e) + \pi(h))\,\pi(f)^i v_\lambda \\ = \pi(f)\left(i(\lambda - i + 1)\pi(f)^{i-1}v_\lambda\right) &+ (\lambda - 2i)\pi(f)^i v_\lambda \\ &= (i+1)(\lambda - i)\pi(f)^i v_\lambda,\end{aligned}$$

and this is the claim for $i + 1$.

In particular, for $i = j_0$, we conclude that $j_0(\lambda - j_0 + 1) = 0$, i.e., $\lambda = j_0 - 1$. Denoting $j_0 - 1$ by $n \in \mathbb{Z}_+$, we see that the vectors

$$v_n = v_\lambda,\ v_{n-2} = \pi(f)v_n, \ldots, v_{-n+2} = \pi(f)^{n-1}v_n,\ v_{-n} = \pi(f)^n v_n$$

span a nonzero $\mathfrak{g}$-invariant subspace of V which thus has to be all of V. In particular, $\dim V$ is $n + 1$, and we have exhibited a basis with the required properties; (1.2) is exactly the claim (1.3).

We have now shown the uniqueness of a representation of given dimension. Existence can be proclaimed obvious, in the sense that if we take a vector space with the action of h, e and f given on the basis elements as above, then one can check the commutation relations and thus the representation is constructed. On the other hand, these representations also arise in many concrete realizations in examples. Here is one such realization; we invite the reader to check that this indeed is the irreducible representation V_n.

Let V be the space of complex polynomials in two variables z_1 and z_2 of degree n. Denote by ∂_1 and ∂_2 the partial derivatives with respect to the variables. Then

$$h \mapsto z_2\partial_2 - z_1\partial_1, \qquad e \mapsto z_2\partial_1, \qquad f \mapsto z_1\partial_2$$

defines an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on V.

In order to describe irreducible finite-dimensional representations of a general semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we first need to describe the structure of

$\mathfrak{g}$; this can be viewed as studying the adjoint representation. Namely, we need analogues of the elements h, e, f of $\mathfrak{sl}(2,\mathbb{C})$ and their commutation relations. All of the things we are going to say (and omit) can be found in many books; we recommend [Kn1], Chapter IV, [**?**], Chapter 5, or [Hum]. What we will do is state the results in general, and illustrate them for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, where everything can be seen directly.

1.2.5. Cartan subalgebras and roots. The analogues of h are the elements of a *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$. By definition, $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ consisting of semisimple elements, i.e., those $H \in \mathfrak{g}$ for which $\operatorname{ad} H$ is a semisimple operator on $\mathfrak{g}$. This means that we can simultaneously diagonalize all $\operatorname{ad} H$, $H \in \mathfrak{h}$. The nonzero joint eigenvalues of these are called the *roots* of $\mathfrak{g}$ and the corresponding joint eigenspaces are called the *root spaces*. The set of all roots is denoted by Δ; it is a subset of $\mathfrak{h}^*$. The root spaces are denoted by $\mathfrak{g}_\alpha$, $\alpha \in \Delta$. Thus we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ denotes the joint eigenspace with eigenvalue zero. In fact, one shows that $\mathfrak{g}_0 = \mathfrak{h}$.

For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, we can choose $\mathfrak{h}$ to be the Lie subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Let us diagonalize $\operatorname{ad} H$ for $H = \operatorname{diag}(h_1, \dots, h_n) \in \mathfrak{h}$. Let E_{ij} be the matrix whose all entries are zero except for the ij entry which is equal to one. Then a simple calculation shows that

$$[H, E_{ij}] = (h_i - h_j)E_{ij}.$$

We see that the roots are the functionals on $\mathfrak{h}$ given by $h \mapsto h_i - h_j$, $i \neq j$. We will denote them by $\epsilon_i - \epsilon_j$, where ϵ_i denotes the functional $h \mapsto h_i$ on $\mathfrak{h}$. The root space corresponding to $\epsilon_i - \epsilon_j$ is spanned by E_{ij}. Observe that for every root $\alpha = \epsilon_i - \epsilon_j$, $-\alpha = \epsilon_j - \epsilon_i$ is also a root.

For general $\mathfrak{g}$, one shows that a Cartan subalgebra always exists (and is unique up to an inner automorphism). It is still true in general that all $\mathfrak{g}_\alpha$ are one-dimensional, and that $-\alpha$ is a root whenever α is. Moreover, for every α we can pick elements $h_\alpha \in \mathfrak{h}$, $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ spanning a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, with h_α, e_α and f_α corresponding respectively to h, e, f under the isomorphism.

For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, if $\alpha = \epsilon_i - \epsilon_j$, we can choose $h_\alpha = E_{ii} - E_{jj}$, $e_\alpha = E_{ij}$ and $f_\alpha = E_{ji}$.

The real span of all h_α, $\alpha \in \Delta$, is a real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$. For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{h}_\mathbb{R}$ consists of real diagonal matrices. The roots take real values on $\mathfrak{h}_\mathbb{R}$ and thus span a real form $\mathfrak{h}_\mathbb{R}^*$ of $\mathfrak{h}^*$.

1.2.6. Killing form. There is a symmetric bilinear form on a semisimple Lie algebra $\mathfrak{g}$ called the *Killing form*:

$$B(X, Y) = \operatorname{tr} (\operatorname{ad} X \operatorname{ad} Y), \qquad X, Y \in \mathfrak{g}.$$

The form B is nondegenerate and invariant under any automorphism φ of $\mathfrak{g}$, i.e.,

$$B(\varphi X, \varphi Y) = B(X, Y), \qquad X, Y \in \mathfrak{g}.$$

This follows immediately from the observation that $\operatorname{ad}(\varphi X) = \varphi \circ \operatorname{ad} X \circ \varphi^{-1}$ for any $X \in \mathfrak{g}$.

In particular, setting $\varphi = \mathrm{e}^{t \operatorname{ad} Z}$ for $Z \in \mathfrak{g}$ and differentiating with respect to $t \in \mathbb{R}$, we see that

$$B(\operatorname{ad}(Z)X, Y) = -B(X, \operatorname{ad}(Z)Y), \qquad X, Y, Z \in \mathfrak{g}.$$

Also, if G is any Lie group with complexified Lie algebra $\mathfrak{g}$, then B is invariant under $\operatorname{Ad}(g)$ for any $g \in G$.

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, one can replace B with a simpler form with the same properties: $B_1(X, Y) = \operatorname{tr} XY$. In fact, B_1 and B are proportional.

It immediately follows from the invariance that for $\alpha, \beta \in \Delta \cup \{0\}$,

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \qquad \text{unless} \quad \alpha + \beta = 0.$$

It follows that B is nondegenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ for any $\alpha \in \Delta \cup \{0\}$. In particular, B is nondegenerate on $\mathfrak{h}$. It is also nondegenerate on $\mathfrak{h}_\mathbb{R}$ and in fact induces a positive definite inner product there. Since we can use B to identify $\mathfrak{h}_\mathbb{R}$ with $\mathfrak{h}_\mathbb{R}^*$, we can also transfer this inner product to $\mathfrak{h}_\mathbb{R}^*$. We denote the transferred inner product by $(\, ,)$. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, if we identify the space of all real diagonal matrices with $\mathbb{R}^n$ in the obvious way, the form B_1 becomes the standard inner product. The algebra $\mathfrak{h}_\mathbb{R}$ then becomes the orthogonal of the vector $(1, 1, \ldots, 1)$. The same is true for the dual spaces.

1.2.7. Positive roots and simple roots. A system of *positive roots* is a subset Δ^+ of Δ such that for every $\alpha \in \Delta$, exactly one of the elements $\pm\alpha$ is in Δ^+, and if $\alpha, \beta \in \Delta^+$ then $\alpha + \beta$ is either in Δ^+, or is not a root at all.

One way to construct a system of positive roots is to pick some $H \in \mathfrak{h}_\mathbb{R}$ not annihilated by any of the roots, and then proclaim α to be in Δ^+ if $\alpha(H) > 0$. Such H are called regular.

For $\mathfrak{g} = (n, \mathbb{C})$ $(n \geq 2)$, if we take $H = \operatorname{diag}(n-1, n-3, \ldots, -n+1)$, then the positive roots are $\epsilon_i - \epsilon_j$ with $i < j$.

A positive root is called *simple* if it can not be written as a sum of two positive roots. For our choice of positive roots for $\mathfrak{sl}(n, \mathbb{C})$, the corresponding simple roots are $\epsilon_1 - \epsilon_2$, $\epsilon_2 - \epsilon_3$, $\ldots$, $\epsilon_{n-1} - \epsilon_n$.

Simple roots form a basis for $\mathfrak{h}^*$; moreover, every positive root can be written as a linear combination of simple roots with all coefficients in $\mathbb{Z}_+$. For example if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the for any $i < j$

$$\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i+1}) + \cdots + (\epsilon_{j-1} - \epsilon_j).$$

Every positive root system defines a positive chamber in $\mathfrak{h}_{\mathbb{R}}$, consisting of all X such that $\alpha(X) > 0$ for all $\alpha \in \Delta^+$. For example, the element H we used to define Δ^+ is in the positive chamber. More generally, a *Weyl chamber* is a connected component of the complement in $\mathfrak{h}_{\mathbb{R}}$ of the union of $\operatorname{Ker} \alpha$ for $\alpha \in \Delta$. Every Weyl chamber is the positive chamber for exactly one choice of a positive root system. It is clear that also in general the Weyl chambers are open cones with vertex at 0.

Identifying $\mathfrak{h}_{\mathbb{R}}^*$ with $\mathfrak{h}_{\mathbb{R}}$ by means of our inner product $(\,,)$, we can define Weyl chambers also in $\mathfrak{h}_{\mathbb{R}}^*$; they are connected components of what is left of $\mathfrak{h}_{\mathbb{R}}^*$ after removing the hyperplanes $\alpha^\perp$, $\alpha \in \Delta$.

1.2.8. Weyl group. To each root α we can associate the orthogonal reflection s_α of $\mathfrak{h}_{\mathbb{R}}^*$ with respect to $\alpha^\perp$. It is defined by

$$s_\alpha \lambda = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \alpha, \qquad \lambda \in \mathfrak{h}_{\mathbb{R}}^*.$$

Then Δ is preserved by all s_α, $\alpha \in \Delta$. Moreover, for any $\alpha, \beta \in \Delta$, the number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ appearing in the definition of $s_\alpha \beta$ is an integer; namely, this number is equal to $\beta(h_\alpha)$, so it is a weight in a representation of $\mathfrak{sl}(2, \mathbb{C})$. (These properties say that Δ is a root system in $\mathfrak{h}_{\mathbb{R}}^*$ in the abstract sense.)

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the reflection $s_{\epsilon_i - \epsilon_j}$ sends $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}_{\mathbb{R}}^* \cong (1, \ldots, 1)^\perp \subset \mathbb{R}^n$ to

$$\lambda - (\epsilon_i - \epsilon_j, \lambda)(\epsilon_i - \epsilon_j).$$

This has the same components as λ, except that the i-th and j-th components exchanged places. It is thus easily seen that Δ is preserved, and the numbers $(\epsilon_i - \epsilon_j, \epsilon_r - \epsilon_s)$ are obviously integers.

The finite group of reflections generated by s_α for $\alpha \in \Delta$ is called the *Weyl group* of Δ and it is denoted by W. For $\mathfrak{sl}(n, \mathbb{C})$, W is S_n, the symmetric group on n letters. Namely we saw that $s_{\epsilon_i - \epsilon_j}$ induces the transposition $i \leftrightarrow j$ on the coordinates.

The group W acts simply transitively on the set of all possible positive root systems, or equivalently, on the set of all Weyl chambers.

1.2.9. Triangular decomposition. Since for any two positive roots α and β one has $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and the sum of two positive roots is either a positive root or not a root at all, we see that the positive root spaces span a subalgebra of $\mathfrak{g}$ which we denote by $\mathfrak{n}$. The Lie algebra $\mathfrak{n}$ is nilpotent.

The same applies to the Lie algebra $\mathfrak{n}^-$ spanned by the negative root spaces. These two together with $\mathfrak{h}$ give a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with our choice of $\mathfrak{h}$ and of positive roots, $\mathfrak{n}$ consists of the strictly upper triangular matrices and $\mathfrak{n}^-$ consists of the strictly lower triangular matrices.

After describing the structure of $\mathfrak{g}$, we are now ready to describe the irreducible finite-dimensional representations (π, V) of $\mathfrak{g}$. We will mostly follow the approach of [GW].

The first basic fact is that $\pi(H)$, $H \in \mathfrak{h}$ can be simultaneously diagonalized on V. To see this, note first that there is a joint eigenspace of all $\pi(H)$, on which each $\pi(H)$ acts by a scalar $\lambda(H)$. The resulting functional $\lambda \in \mathfrak{h}^*$ is called a weight of π, and the joint eigenspace is denoted by V_λ and called a *weight space* corresponding to λ. It is immediate that for any root α and weight λ, $\pi(\mathfrak{g}_\alpha)V_\lambda \subset V_{\lambda+\alpha}$. So if we choose any weight λ, then the sum of all the weight spaces V_μ with $\mu - \lambda$ equal to a combination of roots with integer coefficients forms a subrepresentation of V. By irreducibility, this is all of V. Hence V decomposes as a direct sum of all weight spaces.

Note that in this terminology, roots are nothing else but the nonzero weights of the adjoint representation.

In view of finite dimensionality, we can now choose a maximal weight λ for the following partial order on weights: $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots. Then λ is called a *highest weight* for V. Let us fix a nonzero $v_\lambda \in V_\lambda$. By irreducibility, v_λ is a *cyclic vector* for V, i.e., the elements

$$\pi(X_1)\pi(X_2)\dots\pi(X_k)v_\lambda, \qquad k \in \mathbb{Z}_+, \quad X_1,\dots X_k \in \mathfrak{g} \tag{1.4}$$

span V. We can assume that all X_i in (1.4) are either in $\mathfrak{h}$, or are root vectors. In fact, the elements of $\mathfrak{h}$ can be skipped: if $H \in \mathfrak{h}$ and if X_{α_i} are root vectors, then

$$\pi(H)\pi(X_{\alpha_1})\dots\pi(X_{\alpha_r})v_\lambda = (\lambda + \alpha_1 + \dots + \alpha_r)(H)\pi(X_{\alpha_1})\dots\pi(X_{\alpha_r})v_\lambda.$$

Furthermore, it is a fact that whenever α, β and $\alpha+\beta$ are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. This means that we can assume all the X_i in the expressions (1.4) are actually e_α or f_α where α is a simple root.

We claim that in fact already

$$\pi(f_{\alpha_1})\dots\pi(f_{\alpha_k})v_\lambda, \qquad k \in \mathbb{Z}_+, \quad \alpha_1,\dots\alpha_k \text{ simple roots}$$

span V. To see this, it is enough to show that the span of these vectors – call it Z – is a subrepresentation (Z is nonzero since it contains v_λ). Z is obviously invariant under $\mathfrak{n}^-$ and $\mathfrak{h}$, so we only need to see it is invariant under $\mathfrak{n}$. For that, it is enough to see that Z is invariant under e_α for every simple root α.

But if α and β are simple roots, then $[e_\alpha, f_\beta]$ is either 0, if $\alpha \neq \beta$, or h_α if $\alpha = \beta$. Indeed, it is clear that the difference of two simple roots cannot be a root: if $\alpha - \beta = \gamma$, then if γ is positive we get $\alpha = \beta + \gamma$, so alpha is not simple, and if γ is negative, then $\beta = \alpha + (-\gamma)$, so β is not simple.

So we see that in calculating $\pi(e_\alpha)\pi(f_{\alpha_1})\dots\pi(f_{\alpha_k})v_\lambda$, we can commute e_α to the right, where it kills v_λ, and the commutators that are left contain f_β's and perhaps a few h_α's – but these we already saw can be eliminated.

Hence we conclude that all weights of V are of the form $\lambda - \sum_i c_i\alpha_i$, where α_i are simple roots and c_i are nonnegative integers. In particular, λ is the unique highest weight of V, i.e., $\lambda \geq \mu$ for any weight μ of V. We also proved that v_λ is up to scalar the only vector of weight λ, i.e., $\dim V_\lambda = 1$.

The next thing we want to show is that V is uniquely determined by its highest weight. Indeed, suppose W is another representation with the same highest weight λ and with a highest weight vector w_λ. Then $z_\lambda = (v_\lambda, w_\lambda)$ generates an irreducible subrepresentation Z of $V \oplus W$. Restricting to Z the projections of $V \oplus W$ to V respectively W, we obtain nonzero maps from Z to V respectively W. These maps have to be isomorphisms by the following simple and basic fact, and hence $V \cong W$.

1.2.10. Schur's Lemma. *Let M and N be irreducible modules for a Lie algebra $\mathfrak{g}$ and let $\varphi : M \to N$ be a nonzero map respecting the $\mathfrak{g}$-actions. Then φ is an isomorphism. Moreover, this isomorphism is unique up to a scalar multiple. The same is true for representations of a group.*

Proof. The kernel and the image of φ are $\mathfrak{g}$-invariant, so they have to be zero respectively N by irreducibility of M and N. For the second statement, if $\varphi_1, \varphi_2 : M \to N$ are isomorphisms, then $\varphi_2^{-1}\varphi_1$ is an automorphism of M. This map has an eigenspace (because our modules are always over $\mathbb{C}$). This eigenspace is $\mathfrak{g}$-invariant, hence it must be all of M and the map is a scalar.

1.2.11. Dominant weights. We now want to describe which $\lambda \in \mathfrak{h}^*$ can show up as highest weights of irreducible finite-dimensional representations. It is easy to obtain the necessary conditions using the $\mathfrak{sl}(2,\mathbb{C})$ theory. Namely, the highest weight vector v_λ will also be a highest weight vector for every copy of $\mathfrak{sl}(2,\mathbb{C})$ corresponding to a positive root α. Thus, the corresponding eigenvalue of h_α, that is, $\lambda(h_\alpha) = \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}$, is a nonnegative integer.

It follows that for any weight μ of any finite-dimensional representation, $\frac{2(\alpha,\mu)}{(\alpha,\alpha)}$ is an integer for every root α. All μ satisfying this condition are called weights for $\mathfrak{g}$. The lattice of all integer combinations of weights is called the weight lattice of $\mathfrak{g}$. The weights that in addition satisfy $\frac{2(\alpha,\mu)}{(\alpha,\alpha)} \geq 0$ for all positive roots α are called dominant (with respect to our fixed system of positive roots Δ^+). Another way to express dominance is as belonging to the closure of the positive Weyl chamber.

1.2.12. Construction of irreducible finite-dimensional representations The remaining question is if every dominant weight is the highest weight of an irreducible finite-dimensional representation. The answer is yes, and the representations in question can be constructed in several ways. One way is to first construct certain universal representations with highest weight λ, the so called Verma modules, and then obtain the irreducible finite-dimensional modules as their quotients. This is for example done in [Hum] and [Kn1]; we will define Verma modules in 1.4.2. In examples, one can instead take the approach of [GW] and first construct the so called *fundamental representations*. These correspond to the *fundamental weights*, which are the smallest possible weights, in the sense that if the simple roots are $\alpha_1, \dots, \alpha_l$, then the fundamental weights $\omega_1, \dots, \omega_l$ satisfy

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{ij}.$$

The fundamental weights generate the weight lattice over $\mathbb{Z}$, hence the name.

Once the fundamental representations are constructed, one can construct other (larger) representations as follows. Let $(\pi_\lambda, V(\lambda))$ and $(\pi_\mu, V(\mu))$ be the irreducible finite-dimensional representations with highest weights λ respectively μ and highest weight vectors v_λ respectively v_μ. We can consider the tensor product representation $V(\lambda) \otimes V(\mu)$; the action of $X \in \mathfrak{g}$ is given by

$$\pi(X)(v \otimes w) = (\pi_\lambda(X)v) \otimes w + v \otimes \pi_\mu(X)w, \qquad v \in V_\lambda,\ w \in V_\mu.$$

It is quite hard to decompose tensor product representations in general; we will say a little more about this in 1.5. There is however one obvious component, the highest one, with highest weight $\lambda + \mu$, generated by the highest weight vector $v_\lambda \otimes w_\mu$. So we get to construct $V(\lambda+\mu)$ starting from $V(\lambda)$ and $V(\mu)$. Iterating this process, we can start from $V(\omega_1), \dots, V(\omega_l)$ and obtain $V(\lambda)$, for an arbitrary dominant weight $\lambda = n_1\omega_1 + \dots + n_l\omega_l$.

Let us show how to construct the fundamental representations for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. First, it is easy to see that in this case the fundamental weights are

$$\begin{aligned} \omega_i &= \frac{n-i}{n}(\epsilon_1 + \dots + \epsilon_i) - \frac{i}{n}(\epsilon_{i+1} + \dots + \epsilon_n) \\ &= \epsilon_1 + \dots + \epsilon_i - \frac{i}{n}(\epsilon_1 + \dots + \epsilon_n), \end{aligned}$$

for $i = 1, \dots, n-1$. So ω_i acts on $\mathfrak{h}$ as the restriction of $\epsilon_1 + \dots + \epsilon_i$ to the subspace $\mathfrak{h}$ of the diagonal matrices.

To construct a representation with this highest weight, we start from the standard representation π on $\mathbb{C}^n$; this is the representation of $\mathfrak{sl}(n, \mathbb{C})$ given by its very definition. Then we consider $\bigwedge^i \mathbb{C}^n$ for $i = 1, \dots, n-1$. The action on this space is induced by π, using the Leibniz rule:

$$\begin{aligned} \pi(X)(v_1 \wedge \dots \wedge v_i) = (\pi(X)v_1) \wedge v_2 \wedge \dots \wedge v_i \\ +v_1 \wedge (\pi(X)v_2) \wedge \dots \wedge v_i \\ + \dots + v_1 \wedge \dots \wedge (\pi(X)v_i). \end{aligned}$$

for $X \in \mathfrak{g}$ and $v_1, \dots, v_i \in \mathbb{C}^n$.

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{C}^n$. Then

$$e_{k_1} \wedge \dots \wedge e_{k_i}, \qquad k_1 < \dots < k_i$$

form a basis of $\bigwedge^i \mathbb{C}^n$. These are all clearly weight vectors, with weights respectively equal to the restrictions of $\epsilon_{k_1} + \dots + \epsilon_{k_i}$ to $\mathfrak{h}$.

A highest weight vector is characterized by the fact that it is annihilated by $\mathfrak{n}$, or equivalently, by all simple root vectors $E_{i\,i+1}$, $i = 1, \dots, n-1$. It is now clear that among the above weight vectors the only highest weight vector is $e_1 \wedge \cdots \wedge e_i$. The corresponding highest weight is the restriction of $\epsilon_1 + \cdots + \epsilon_i$ to $\mathfrak{h}$, and this is exactly the fundamental weight ω_i.

1.2.13. The case of reductive $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$ be a reductive Lie algebra with center $\mathfrak{z}$ and semisimple part $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Let (π, V) be an irreducible finite-dimensional representation of $\mathfrak{g}$. By Schur's Lemma 1.2.10, $\mathfrak{z}$ acts on V by scalars. These scalars are given by some $\lambda \in \mathfrak{z}^*$. The action of $\mathfrak{g}_1$ and λ determine the action of $\mathfrak{g}$. So to understand the representations of $\mathfrak{g}$ it is enough to understand representations of $\mathfrak{g}_1$. A similar argument works also for infinite-dimensional representations.

1.2.14. Integrating representations. Let now G be a Lie group with Lie algebra $\mathfrak{g}_0$ and let as usual $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$. The representations of $\mathfrak{g}$ are the same as representations of $\mathfrak{g}_0$. We want to determine when they come from representations of the group G. In other words, the question is which representations of $\mathfrak{g}_0$ can be "integrated" or "exponentiated" to G.

We already know from Theorem 1.1.16 that any representation of $\mathfrak{g}_0$ can be integrated to the universal cover $\tilde{G}$ of G. The condition for a representation $\pi : \tilde{G} \to GL(V)$ to factor through G is that π is trivial on the kernel of the covering map.

To illustrate this situation (and also Theorem 1.1.16) better, let us consider the simplest possible case of one-dimensional (abelian) Lie groups. There are two connected one-dimensional groups: the real line $\mathbb{R}$ and the circle group $\mathbb{T}^1$. Both have the same Lie algebra $\mathbb{R}$.

Consider the one-dimensional representations of the Lie algebra $\mathbb{R}$. Each of them is given by $t \mapsto t\lambda$ for some $\lambda \in \mathbb{C}$ (we identify 1×1 complex matrices with complex numbers). All of these representations exponentiate to the group $\mathbb{R}$, and give all possible characters of $\mathbb{R}$:

$$t \longmapsto e^{t\lambda}, \quad \lambda \in \mathbb{C}.$$

However, among these characters only the periodic ones will be well defined on $\mathbb{T}^1$, and $e^{t\lambda}$ is periodic if and only if $\lambda \in 2\pi i\mathbb{Z}$.

In the semisimple case, the weights which parametrize the representations of G, the so called analytically integral weights, form a finite index subgroup (sublattice) of the weight lattice. The quotient can be identified with the kernel of the covering, i.e., with the fundamental group of G.

1.2.15. Some further properties of finite-dimensional representations. Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and let W be the Weyl group of $\mathfrak{g}$ (see 1.2.8). The weights of a finite dimensional representation (π, V) of $\mathfrak{g}$ form a finite set in $\mathfrak{h}^*$, invariant under W. A distinguished role is played by extremal weights: these are the highest weights with respect to various choices of positive roots for $(\mathfrak{g}, \mathfrak{h})$, and they form one W-orbit. All of them are of multiplicity one. The multiplicities of other weights can be expressed in terms of a partition function; this is Kostant's multiplicity formula, see for example [Hum], Section 24.2.

If V is a representation of a connected compact Lie group G (with complexified Lie algebra $\mathfrak{g}$), then one can consider the *character* of V, i.e., the function $\chi : G \to \mathbb{C}$ defined as

$$\chi(g) = \operatorname{tr} \pi(g), \qquad g \in G.$$

These functions are important in harmonic analysis on G. They are determined by their restrictions to a maximal torus T in G. These restrictions are given explicitly in terms of extremal weights by the well known Weyl character formula. See for example [Hum], Section 24.3, [W], Section 2.5, or [Kn1], Section IV.10.

1.2.16. Cartan decomposition. There is a *Cartan decomposition* of $\mathfrak{g}_0$:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Here $\mathfrak{k}_0$ and $\mathfrak{p}_0$ can be defined as the eigenspaces of the so called *Cartan involution* θ of $\mathfrak{g}_0$ with the eigenvalues 1 respectively -1.

There is also a Cartan involution Θ and a Cartan decomposition $G = KP$ on the group level, with K the subgroup of fixed points of Θ. The Cartan involution θ of $\mathfrak{g}_0$ is the differential of Θ at the identity, $\mathfrak{k}_0$ is the Lie algebra of K and $P = \exp(\mathfrak{p}_0)$.

In most cases we are interested in, K is a *maximal compact subgroup* of G; for example, this is true if G is semisimple connected with finite center.

Rather than defining the Cartan involution in general, let us note that for all the matrix examples we have encountered so far, $\Theta(g) = (g^*)^{-1}$, the inverse of the conjugate transpose. Hence for $X \in \mathfrak{g}_0$, $\theta(X)$ is minus the conjugate transpose of X. So e.g. for $G = SL(n, \mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R})$, K is $SO(n)$, $\mathfrak{k}_0$ is $\mathfrak{so}(n)$, $\mathfrak{p}_0$ is the space of symmetric matrices in $\mathfrak{g}_0$ and $P = \exp(\mathfrak{p}_0)$ is the set of positive definite symmetric matrices. Note that in this case $G = KP$ describes the polar decomposition of matrices.

It is clear that $\mathfrak{k}_0$ is a subalgebra, $[\mathfrak{k}_0, \mathfrak{p}_0] \subset \mathfrak{p}_0$ and $[\mathfrak{p}_0, \mathfrak{p}_0] \subset \mathfrak{k}_0$. Furthermore, the Killing form B of 1.2.6 is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$.

Finally, we will also use the complexified version of the Cartan involution on $\mathfrak{g}$, denoted again by θ, and the corresponding complexified Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

As part of proving the existence of Cartan decompositions, one shows that $\exp : \mathfrak{p}_0 \to P$ is a diffeomorphism. In the example $G = SL(n, \mathbb{R})$, this is clear from the fact that symmetric matrices can be diagonalized. So P is contractible and it follows that G is homotopically equivalent to K. It also follows that a finite-dimensional representation of $\mathfrak{g}$ will exponentiate to G if and only if it exponentiates to K.

1.3 Infinite-dimensional representations

In this section G is a connected reductive group with Cartan decomposition $G = KP$ (see 1.2.16), and we assume that K is compact. A lot of the following will however work also for more general groups.

A representation of G is a continuous action by linear operators on a complex topological vector space $\mathcal{V}$. More precisely:

Definition 1.3.1. A *representation* of a Lie group G on a topological vector space $\mathcal{V}$ is a group homomorphism π from G into the group of linear automorphisms of $\mathcal{V}$, such that the map

$$G \times \mathcal{V} \to \mathcal{V}, \qquad (g, v) \mapsto \pi(g)v$$

is continuous. In particular, each $\pi(g)$ is a continuous linear operator on $\mathcal{V}$.

Typically, one puts additional conditions on $\mathcal{V}$ and requires it to be at least a Fréchet space, or even a Banach or a Hilbert space. In these cases, one can show that the continuity requirement is equivalent to a seemingly weaker one, that for each $v \in \mathcal{V}$ the map $g \mapsto \pi(g)v$ is a continuous map for G into V.

One can define a morphism between two representations $(\pi, \mathcal{V})$ and $(\rho, \mathcal{W})$ of G to be a continuous linear map $T : \mathcal{V} \to \mathcal{W}$ which intertwines the G-actions, i.e., $T\pi(g) = \rho(g)T$ for every $g \in G$. The category of representations obtained in this way is however too big to be a reasonable setting to work with.

Example 1.3.2. Let X be a smooth manifold with a smooth action of G; a typical example is a quotient G/H where H is a closed subgroup of G and where G acts by left translations. Then G also acts on functions on X, by

$$(\pi(g)f)(x) = f(g^{-1} \cdot x), \qquad g \in G, x \in X.$$

This defines many representations of G, depending on which function space we take. For example, we can consider the continuous or smooth functions, we can in addition require the functions to be compactly supported, or we can consider various L^p-spaces. We can also consider "generalized functions" like measures or distributions, again with or without the compact support condition. Each of these spaces has a natural topology, and one can check that the above defined action of G satisfies the required continuity condition. All these spaces are quite different as topological vector spaces and there is no chance they could be isomorphic. Still, one can not help feeling that there should be a strong relationship among these representations.

1.3.3. Unitary representations. An important special class of representations are representations by unitary operators on (separable) Hilbert spaces. Two such representations $(\pi, \mathcal{H})$ and $(\rho, \mathcal{H}')$ are considered to be equivalent if there is a Hilbert space isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ intertwining π and ρ. These representations are interesting for applications to harmonic analysis. It is however a still unsolved problem to classify irreducible unitary representations of a given group G up to equivalence. (Irreducibility in the infinite-dimensional setting means that there are no closed invariant subspaces.)

1.3.4. Smooth vectors. When studying finite-dimensional representations, we made substantial use of the fact that the Lie algebra of G acts naturally on representations of G. We would thus like to differentiate also infinite-dimensional representations of G to get representations of the (complexified) Lie algebra $\mathfrak{g}$. This is however not quite possible in general. Actually, $\mathfrak{g}$ does act, but only on the subspace $\mathcal{V}_\infty$ of *smooth vectors*, i.e., vectors $v \in \mathcal{V}$ such that the map $g \mapsto \pi(g)v$ from G into $\mathcal{V}$ is smooth. One can show that $\mathcal{V}_\infty$ is dense in $\mathcal{V}$.

1.3.5. K-finite vectors. For the groups we are interested in, there is a better choice of a dense subspace on which $\mathfrak{g}$ acts: the space of vectors finite under the maximal compact subgroup K of G. A vector $v \in \mathcal{V}$ is K-finite if the orbit Kv spans a finite-dimensional subspace of $\mathcal{V}$. The reason why it is a good idea to consider the action of K is the fact that the representations of a compact group are relatively easy to study. We already saw in 1.2.3 that finite-dimensional representations of K are all unitary and therefore completely reducible. This is also true for any infinite-dimensional representation $(\pi, \mathcal{H})$ on a Hilbert space; we can again integrate the given inner product over K and thus make it K-invariant. Furthemore, any unitary representation of K is a Hilbert direct sum of irreducibles, and all irreducibles are finite-dimensional. These facts can be proved e.g. using the basic facts about compact operators. See for example [W], Section 1.4.

So if we have a representation of G on a Hilbert space $\mathcal{H}$ (not necessarily unitary), it can be assumed to be unitary for K by changing the inner product if necessary, and then decomposed as

$$\mathcal{H} = \widehat{\bigoplus}_{\delta \in \hat{K}} \mathcal{H}(\delta).$$

Here $\hat{K}$ denotes the set of unitary equivalence classes of irreducible unitary representations of K, and for each $\delta \in \hat{K}$, $\mathcal{H}(\delta)$ denotes the sum of all irreducible subrepresentations of $\mathcal{H}$ isomorphic to δ. The symbol $\hat{\oplus}$ denotes the Hilbert direct sum: its elements are the series $\sum_\delta v_\delta$ (with $v_\delta \in \mathcal{H}(\delta)$), which are convergent in $\mathcal{H}$. Those δ for which $\mathcal{H}(\delta) \neq 0$ are called the K-types of $(\pi, \mathcal{H})$, and the space $\mathcal{H}(\delta)$ is called the isotypic component of $\mathcal{H}$ corresponding to δ.

The space of K-finite vectors in $\mathcal{H}$ is then the algebraic direct sum

$$\mathcal{H}_K = \bigoplus_{\delta \in \hat{K}} \mathcal{H}(\delta)$$

of finite sums $\sum_\delta v_\delta$. Clearly, $\mathcal{H}_K$ is dense in $\mathcal{H}$.

In general, if $(\pi, \mathcal{V})$ is a representation of G on a not necessarily Hilbert space, it is still true that the space of K-finite vectors $\mathcal{V}_K = \oplus_{\delta \in \hat{K}} \mathcal{V}(\delta)$ is dense in $\mathcal{V}$. The representation is called *admissible* if $\dim \mathcal{V}(\delta) < \infty$ for every $\delta \in \mathcal{V}$. A basic fact (cf. [Kn1]) is

Theorem 1.3.6. (Harish-Chandra) *Any irreducible unitary representation of G is admissible.*

If $\mathcal{V}$ is irreducible, then the K-finite vectors are all smooth (in fact even analytic). This follows from the regularity theorem for certain elliptic differential equations (these equations come from the Casimir operators described in 1.4.6). It is a fact that all finitely generated admissible representations are of finite length, i.e., have finite descending filtrations with irreducible quotients. It follows that K-finite vectors in any finitely generated admissible representation are smooth. This means that $\mathfrak{g}$ acts on $\mathcal{V}_K$. The group G does not act on $\mathcal{V}_K$, but of course the subgroup K does. In this way we are led to the following concept.

1.3.7. $(\mathfrak{g}, K)$-modules. A vector space V is a $(\mathfrak{g}, K)$*-module* if it is simultaneously a representation of $\mathfrak{g}$ and a finite representation of K, in such a way that the two actions of $\mathfrak{k}_0$, one obtained by differentiating the K-action and the other by restricting the $\mathfrak{g}$-action, agree.

In case we want to allow disconnected groups, we also need the $\mathfrak{g}$-action to be K-equivariant, i.e., $\pi(k)\pi(X)v = \pi(Ad(k)X)\pi(k)v$, for all $k \in K$, $X \in \mathfrak{g}$ and $v \in V$. One usually also requires some finiteness conditions, like finite generation, or admissibility defined in the same way as above.

Morphisms of $(\mathfrak{g}, K)$-modules are linear maps which intertwine the actions of $\mathfrak{g}$ and K. Submodules, direct products and sums etc. are defined in the obvious way.

The $(\mathfrak{g}, K)$-module corresponding to a finitely generated admissible representation $(\pi, \mathcal{V})$ of G is called the *Harish-Chandra module* of π.

1.3.8. Infinitesimal equivalence. Two representations of G are said to be *infinitesimally equivalent* if their $(\mathfrak{g}, K)$-modules are isomorphic. This is a much weaker condition than the existence of a continuous G-intertwining isomorphism. On the other hand, Harish-Chandra proved that if two unitary representations are infinitesimally equivalent, then they are in fact unitarily equivalent (cf. [Kn1]).

1.3.9. Globalizations. Going back from $(\mathfrak{g}, K)$-modules to representations of G is hard. We call a representation $(\pi, \mathcal{V})$ of G a globalization of a Harish-Chandra module V, if V is isomorphic to $\mathcal{V}_K$. Every irreducible V has globalizations. In fact there are many of them; one can choose e.g. a Hilbert space globalization (not necessarily unitary), or a smooth globalization. There are also notions of minimal and maximal globalizations (cf. [S4]). A few names to mention here are Harish-Chandra, Lepowsky, Rader, Casselman, Wallach and Schmid. We will not use globalizations in this book; in fact, for the most part we will work only with $(\mathfrak{g}, K)$-modules.

1.3.10. Irreducible $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-modules. To finish this section, let us describe an example where it is easy to explicitly write down all irreducible $(\mathfrak{g}, K)$-modules. This is the case $G = SL(2,\mathbb{R})$, whose representations correspond to $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-modules. The approach is taken from [V1].

We can not start like we did for finite-dimensional representations, by diagonalizing the action of the basic element h. Namely, there is no reason for h to act semisimply - in fact, one can see from the description of representations given below that h actually never acts finitely on infinite-dimensional irreducible representations. We are however assuming that K is acting finitely, and thus we will be able to diagonalize the action of $\mathfrak{k}$ which is another Cartan subalgebra of $\mathfrak{g}$.

We choose a basic element W of $\mathfrak{k}$ and root vectors X and Y for $(\mathfrak{g}, \mathfrak{k})$ as follows:

$$W = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \qquad X = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}; \qquad Y = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}.$$

In this way we got another basis of $\mathfrak{g}$ satisfying the same commutation relations as h, e and f:

$$[W, X] = 2X, \qquad [W, Y] = -2Y, \qquad [X, Y] = W.$$

Let us now define a large family of $(\mathfrak{g}, K)$-modules. For a fixed choice of $\lambda \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$, we define a $(\mathfrak{g}, K)$-module $V_{\lambda,\epsilon}$ as follows: a basis of $V_{\lambda,\epsilon}$ is given by v_n, $n \in \mathbb{Z}$ of parity ϵ. The action of $\mathfrak{g}$ is given by

$$\begin{aligned} \pi(W)v_n = nv_n; \qquad \pi(X)v_n &= \frac{1}{2}(\lambda + (n+1))v_{n+2}; \\ \pi(Y)v_n &= \frac{1}{2}(\lambda - (n-1))v_{n-2}. \end{aligned}$$

The action of K is then determined by the action of W:

$$\pi\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) v_n = e^{in\theta} v_n.$$

We leave it as an exercise for the reader to check that in this way we indeed obtained a $(\mathfrak{g}, K)$-module.

The picture is similar to the one that we had for finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{C})$, but now it is infinite:

$$\dots \;\underset{Y}{\overset{X}{\rightleftarrows}}\; \mathbb{C}v_{n-2} \;\underset{Y}{\overset{X}{\rightleftarrows}}\; \mathbb{C}v_n \;\underset{Y}{\overset{X}{\rightleftarrows}}\; \mathbb{C}v_{n+2} \;\underset{Y}{\overset{X}{\rightleftarrows}}\; \dots$$

Also, note that we changed normalization for the v_n's; now we do not have a natural place to start, like a highest weight vector, so it is best to make $\pi(X)$ and $\pi(Y)$ as symmetric as possible.

Proposition 1.3.11. *Assume λ is not an integer of parity $\epsilon + 1$. Then $V_{\lambda,\epsilon}$ is irreducible.*

Proof. Let $U \subset V$ be a nonzero $\mathfrak{g}$-invariant subspace. Since U is invariant under W, it must contain an eigenvector v_n of W. Namely, from any linear combination x of two or more v_n's which lies in U, we can obtain a shorter one by combining x and $\pi(W)x$. If we now act on v_n by X and Y, we get all v_k, as the scalars in the formulas defining the action can never be zero. Hence $U = V$.

If $\lambda = k - 1$ where $k \geq 1$ is an integer of parity ϵ, then $V_{\lambda,\epsilon}$ contains two irreducible submodules, one with weights $k, k+2, k+4, \dots$ and the other with weights $-k, -k-2, -k-4, \dots$. If $k > 1$, these are called the *discrete series* representations, as they occur discretely in the decomposition of the representation $L^2(G)$. The quotient of $V_{\lambda,\epsilon}$ by the sum of these two submodules is an irreducible module of dimension $k-1$. For $k = 1$ the two submodules are called the limits of discrete series, and their sum is all of $V_{0,1}$. Irreducibility of all these modules is proved just like in the above proposition.

If $\lambda = k + 1$, where $k \leq -2$ is an integer of parity ϵ, then we get the same subquotients as for $-k-1$, but the finite dimensional representation is now a submodule, and the quotient decomposes into the sum of two discrete series representations. Also, if λ is as in Proposition 1.3.11, $V_{\lambda,\epsilon}$ is isomorphic to $V_{-\lambda,\epsilon}$. Namely, one can define $\varphi : V_{-\lambda,\epsilon} \to V_{\lambda,\epsilon}$ by sending v_ϵ into the analogous vector v'_ϵ and then adjust scalars μ_n so that $\phi(v_n) = \mu_n v'_n$ defines a $\mathfrak{g}$-morphism.

This means that it is enough to choose only one representative for each pair $\lambda, -\lambda$.

1.3.12. The Casimir operator. For a $(\mathfrak{g}, K)$-module (π, V) we define the Casimir operator $\Omega = 4\pi(X)\pi(Y) + \pi(W)^2 - 2\pi(W)$ on V. (See 1.4.6.) It is easy to check that Ω commutes with the representation π, hence it must act as a scalar if π is irreducible. This is a variant of Schur's Lemma; see 1.4.5 below. An easy calculation shows that Ω acts by the scalar $\lambda^2 - 1$ on $V_{\lambda,\epsilon}$.

Theorem 1.3.13. *Any irreducible $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-module (π, V) is isomorphic to a subquotient of some $V_{\lambda,\epsilon}$.*

Proof. Let $\lambda \in \mathbb{C}$ be such that Ω acts on V by the scalar $\lambda^2 - 1$. Clearly, the K-types of V are either all even or all odd - let ϵ be 0 or 1 accordingly.

On the K-type V_k of V, i.e., the eigenspace of $\pi(W)$ with eigenvalue k, the operator $\pi(X)\pi(Y)$ acts as

$$\pi(X)\pi(Y) = \frac{1}{4}(\Omega - \pi(W)^2 + 2\pi(W)) = \frac{1}{4}\lambda^2 - (k-1)^2.$$

The operator $\pi(Y)\pi(X) = \pi(X)\pi(Y) - \pi(W)$ then also acts by a scalar on each V_k. This means that if we take any nonzero v_n in V_n, then $\pi(X)^k v_n$ and $\pi(Y)^k v_n$, $k \in \mathbb{Z}_+$, span a nonzero submodule of V and hence all of V.

If $\lambda^2 \neq (k-1)^2$ for all K-types V_k of V, i.e., λ is not an integer of parity $\epsilon + 1$, then $\pi(X)^k v_n$ is different from zero for all k. Namely, assuming k is the smallest positive integer such that $\pi(X)^k v_n = 0$, we get a contradiction since $\pi(Y)\pi(X)^k v_n$ is a nonzero multiple of $\pi(X)^{k-1} v_n$. Similarly, we see that $\pi(Y)^k v_n \neq 0$ for all k. One can now construct an isomorphism of V onto $V_{\lambda,\epsilon}$ similar to $\varphi : V_{-\lambda,\epsilon} \to V_{\lambda,\epsilon}$ mentioned above.

If $\lambda^2 = (k-1)^2$ for some $k \geq 1$ of parity ϵ, then there are three possibilities for our fixed n such that $v_n \neq 0$: $n \geq k$, $n \leq -k$ or $-k < n < k$. Examining each of these cases separately, using similar reasoning as above, one concludes the following. If $n \geq k$ then V is isomorphic to the lowest weight discrete series representation, if $n \leq -k$ then V is isomorphic to the highest weight discrete series representation, and if $-k < n < k$ then V is isomorphic to the k-dimensional irreducible representation.

Let us remark here that for finite-dimensional modules, all that we said about finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules (Example 1.2.4) obviously also holds with W, X and Y in place of h, respectively e, respectively f.

1.4 Infinitesimal characters

Recall that a representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra morphism from $\mathfrak{g}$ into the Lie algebra End (V) of endomorphisms of V. Note that End (V) is actually an associative algebra, which is turned into a Lie algebra by defining $[a,b] = ab - ba$; this can be done for any associative algebra. In fact, it is possible to construct an associative algebra $U(\mathfrak{g})$ containing $\mathfrak{g}$, so that representations of $\mathfrak{g}$ extend to morphisms $U(\mathfrak{g}) \to$ End (V) of associative algebras. The algebra $U(\mathfrak{g})$ is called the universal enveloping algebra of $\mathfrak{g}$.

1.4.1. Universal enveloping algebra. The construction goes as follows: consider first the tensor algebra $T(\mathfrak{g})$ of the vector space $\mathfrak{g}$. This is an associative algebra with 1 generated by the monomials $X_1 \otimes \cdots \otimes X_n$ with n running over positive integers and X_i running over $\mathfrak{g}$. The only relations correspond to linearity in each variable; thus one can think of $T(\mathfrak{g})$ also as a free associative algebra on a basis of $\mathfrak{g}$.

Then one defines the *universal enveloping algebra* of $\mathfrak{g}$ as

$$U(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where I is the two-sided ideal of $T(\mathfrak{g})$ generated by elements $X \otimes Y - Y \otimes X - [X,Y]$, $X, Y \in \mathfrak{g}$. It is easy to see that $U(\mathfrak{g})$ satisfies a universal property with respect to maps of $\mathfrak{g}$ into associative algebras. Namely, let $\iota : \mathfrak{g} \to U(\mathfrak{g})$ be the obvious morphism mapping elements of $\mathfrak{g}$ to corresponding linear monomials. Then for any Lie algebra morphism $\phi : \mathfrak{g} \to A$, where A is an associative algebra considered as a Lie algebra, there is a unique associative algebra morphism $\tilde{\phi} : U(\mathfrak{g}) \to A$ such that $\tilde{\phi} \circ \iota = \phi$. This universal property determines $U(\mathfrak{g})$ up to isomorphism.

Loosely speaking, one can think of $U(\mathfrak{g})$ as "noncommutative polynomials over $\mathfrak{g}$", with the commutation laws given by the bracket of $\mathfrak{g}$. If we think of elements of $\mathfrak{g}$ as left invariant vector fields on G, then $U(\mathfrak{g})$ consists of left invariant differential operators on G.

In particular, we see that representations of $\mathfrak{g}$ (i.e., $\mathfrak{g}$-modules) are the same thing as $U(\mathfrak{g})$-modules. This enables one to study representations by applying various constructions from the associative algebra setting. For example, there is a well known notion of "extension of scalars": let $B \subset A$ be associative algebras and let V be a B-module. One can consider A as a right B-module for the right multiplication and form the vector space $A \otimes_B V$. This vector space is an A-module for the left multiplication in the first factor. So we get a functor from B-modules to A-modules. Another functor like this is obtained by considering $\operatorname{Hom}_B(A, V)$; now the Hom is taken with respect to the left multiplication action of B on A (and the given action on V), and the (left!) A-action on the space $\operatorname{Hom}_B(A, V)$ is given by right multiplication on A.

1.4.2. Verma modules. A particular example of the above situation is obtained for $A = U(\mathfrak{g})$ and $B = U(\mathfrak{b})$, where $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ is a triangular decomposition and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is the corresponding Borel subalgebra (a Borel subalgebra of $\mathfrak{g}$ is by definition a maximal solvable subalgebra). Let $\mathbb{C}_\lambda$ be a one dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts by $\lambda \in \mathfrak{h}^*$ and $\mathfrak{n}$ acts as 0. Then

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

is a highest weight $\mathfrak{g}$-module via left multiplication, called a *Verma module*. It can also be viewed as $U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$, but then only the actions of $\mathfrak{n}^-$ and $\mathfrak{h}$ are obvious, and to understand the action of the elements of $\mathfrak{n}$, one has to commute them through the first factor.

Verma modules are universal objects in the category of highest weight $\mathfrak{g}$-modules: if V is any $\mathfrak{g}$-module with a heighest vector x_λ of weight λ (i.e., x_λ is annihilated by $\mathfrak{n}$, and $\mathfrak{h}$ acts on x_λ via λ), then there is a unique $\mathfrak{g}$-morphism $M(\lambda) \to V$ mapping $1 \otimes 1$ to x_λ. It is easy to see that each $M(\lambda)$ has a unique maximal submodule, and these facts lead to a classification of irreducible highest weight $\mathfrak{g}$-modules as unique irreducible quotients of Verma modules. In particular, one can in this way obtain the finite-dimensional representations, as mentioned in 1.2.12.

1.4.3. Filtration by degree. The main tool for studying $U(\mathfrak{g})$ is the *filtration by degree*. In fact, the tensor algebra $T(\mathfrak{g})$ is a graded algebra, if we set the degree of a monomial $X_1 \otimes \cdots \otimes X_n$ ($X_i \in \mathfrak{g}$) to be equal to n. The ideal I is not homogeneous, so this grading of $T(\mathfrak{g})$ does not induce a grading on $U(\mathfrak{g}) = T(\mathfrak{g})/I$. It does however induce a filtration of $U(\mathfrak{g})$:

$$F_nU(\mathfrak{g}) = \text{ span } \{X_1 \dots X_k \big| k \le n, X_1, \dots, X_k \in \mathfrak{g}\}.$$

Here and in future we denote by $X_1 \dots X_k$ the image of the monomial $X_1 \otimes \dots \otimes X_k$ under the projection $T(\mathfrak{g}) \to U(\mathfrak{g})$. The main properties of $U(\mathfrak{g})$ are given by the following theorem; its proof can be found for example in [Hum].

Theorem 1.4.4. (Poincaré-Birkhoff-Witt.) *The graded algebra associated to the above filtration of $U(\mathfrak{g})$ is the symmetric algebra $S(\mathfrak{g})$. Furthermore, if $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$, then the monomials*

$$X_1^{i_1} \dots X_n^{i_n}, \qquad i_j \in \mathbb{Z}_+$$

form a basis of $U(\mathfrak{g})$.

1.4.5. The center of $U(\mathfrak{g})$. A further advantage of introducing the algebra $U(\mathfrak{g})$ is the fact that as opposed to $\mathfrak{g}$ which has no center if $\mathfrak{g}$ is semisimple, $U(\mathfrak{g})$ has a relatively large center $Z(\mathfrak{g})$. $Z(\mathfrak{g})$ has a nice structure of a finitely generated polynomial algebra. For example, if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, there are $n-1$ generators and their degrees are $2, 3, \dots, n$

The importance of the center follows from a simple observation that is often used in linear algebra: if two operators commute, then an eigenspace of one of them is invariant under the other. This means that we can reduce representations by taking a joint eigenspace of $Z(\mathfrak{g})$.

Furthermore, if X is an irreducible $(\mathfrak{g}, K)$-module, then every element of $Z(\mathfrak{g})$ acts on X by a scalar. For finite-dimensional X this follows from Schur's lemma 1.2.10. For infinite-dimensional X the same argument is applied to a fixed K-type in X: since $Z(\mathfrak{g})$ commutes with K, it preserves each K-isotypic component $X(\delta)$ of X. On the finite dimensional space $X(\delta)$, $z \in Z(\mathfrak{g})$ has an eigenvalue. The corresponding eigenspace in X is therefore a nonzero submodule and hence must be all of X.

All the scalars coming from the action of $Z(\mathfrak{g})$ on X put together give a homomorphism

$$\chi_X : Z(\mathfrak{g}) \to \mathbb{C}$$

of algebras, which is called the *infinitesimal character* of X. We are going to describe such homomorphisms in more concrete terms below.

1.4.6. Casimir element. There is an element of $Z(\mathfrak{g})$ that can easily be written down. It is the simplest and most important element of $Z(\mathfrak{g})$ called the *Casimir element*. Its degree is two - the lowest possible if $\mathfrak{g}$ is semisimple. The definition involves the Killing form B from 1.2.6. Let (X_i) be a basis of $\mathfrak{g}$, orthonormal with respect to B. Then the Casimir element is

$$\Omega = \sum_i X_i^2.$$

An easy calculation shows that Ω is invariant under any linear transformation T of $\mathfrak{g}$ orthogonal with respect to B, i.e., such that $B(T(X), T(Y)) = B(X, Y)$

for all $X, Y \in \mathfrak{g}$. (Here T acts on $U(\mathfrak{g})$ by sending a monomial $Y_1 \dots Y_r$ into $T(Y_1) \dots T(Y_r)$.) In particular, Ω does not depend on the choice of a basis (X_i).

Moreover, since every automorphism φ of $\mathfrak{g}$ is orthogonal with respect to B, Ω is invariant under φ. In particular, taking $\varphi = \mathrm{e}^{t \operatorname{ad} Z}$ for $Z \in \mathfrak{g}$ and differentiating with respect to $t \in \mathbb{R}$, we see that Ω commutes with $\mathfrak{g}$ and hence it is in $Z(\mathfrak{g})$.

It is also possible to write Ω as $\sum e_i f_i$ where (e_i) and (f_i) are dual bases with respect to the Killing form. In examples like $\mathfrak{sl}(n, \mathbb{C})$, one can replace B by the proportional trace form, and get the same Ω up to a scalar.

A particular choice of a basis is obtained if we choose orthonormal bases W_k for $\mathfrak{k}_0$ and Z_i for $\mathfrak{p}_0$, so that

$$B(W_k, W_l) = -\delta_{kl}; \qquad B(Z_i, Z_j) = \delta_{ij}.$$

The Casimir element is then

$$\Omega = -\sum_k W_k^2 + \sum_i Z_i^2.$$

One can calculate the action of Ω on a highest weight vector v_λ of weight $\lambda \in \mathfrak{h}^*$ in a $\mathfrak{g}$-module. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Namely, choose a system of positive roots Δ^+ of $\mathfrak{g}$ with respect to $\mathfrak{h}$, let e_α, $\alpha \in \Delta^+$ be root vectors, and let $f_\alpha \in \mathfrak{g}_{-\alpha}$ be the dual vectors with respect to the Killing form. If $h_1, \dots, h_r$ is an orthonormal basis for $\mathfrak{h}$ with respect to B, we can write

$$\Omega = \sum_{i=1}^r h_i^2 + \sum_{\alpha \in \Delta^+} (e_\alpha f_\alpha + f_\alpha e_\alpha).$$

If we write $e_\alpha f_\alpha = f_\alpha e_\alpha + h_\alpha$ where h_α denotes $[e_\alpha, f_\alpha]$, then since each e_α annihilates v_λ, we see that

$$\Omega v_\lambda = \left(\sum_{i=1}^r \lambda(h_i)^2 + \sum_{\alpha \in \Delta^+} \lambda(h_\alpha) \right) v_\lambda.$$

Since $\sum_{i=1}^r \lambda(h_i)^2 = ||\lambda||^2$ and since $\sum_{\alpha \in \Delta^+} \lambda(h_\alpha) = \langle \lambda, 2\rho \rangle$, where ρ as usual denotes the half sum of positive roots, we conclude that

$$\Omega v_\lambda = ||\lambda||^2 + 2\langle \lambda, \rho \rangle = ||\lambda + \rho||^2 - ||\rho||^2.$$

1.4.7. Harish-Chandra homomorphism. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition of 1.2.9 and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Borel subalgebra of $\mathfrak{g}$. Let us build a Poincaré-Birkhoff-Witt basis from bases of $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}$. In particular, we see that

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \left(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n} \right),$$

with the first summand spanned by monomials with all factors in $\mathfrak{h}$ and the second summand spanned by monomials that contain a factor which is not in $\mathfrak{h}$. Let $\mu_\mathfrak{b}$ be the projection of $U(\mathfrak{g})$ to the first summand in this decomposition. We claim that the restriction of $\mu_\mathfrak{b}$ to $Z(\mathfrak{g})$ is an algebra homomorphism. To see this, note that any element of $Z(\mathfrak{g})$ must have weight 0 with respect to $\mathfrak{h}$ and hence must consist of monomials of weight 0. Therefore each of these monomials is either in $U(\mathfrak{h})$, or contains both factors in $\mathfrak{n}$ (of positive weight) and in $\mathfrak{n}^-$ (of negative weight). In other words,

$$Z(\mathfrak{g}) \cap \big(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}\big) = Z(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n} = Z(\mathfrak{g}) \cap \mathfrak{n}^- U(\mathfrak{g}).$$

It is now clear that this last space is a two-sided ideal in $Z(\mathfrak{g})$ and thus $\mu_\mathfrak{b} : Z(\mathfrak{g}) \to U(\mathfrak{h})$ is indeed an algebra homomorphism. Notice also that $z \in Z(\mathfrak{g})$ acts as $\mu_\mathfrak{b}(z)$ on any highest weight vector in a representation of $\mathfrak{g}$.

Note that the homomorphism $\mu_\mathfrak{b}$ depends on the choice of $\mathfrak{n}$, that is, on the choice of a positive root system for $\mathfrak{g}$. It turns out that this dependency can be eliminated by composing $\mu_\mathfrak{b}$ with the so called *ρ-shift*. Denote by ρ the half sum of all positive roots, and consider the automorphism s_ρ of $U(\mathfrak{h}) = S(\mathfrak{h})$ defined on the generators $X \in \mathfrak{h}$ by

$$s_\rho(X) = X - \rho(X) \cdot 1.$$

The homomorphism $\gamma_\mathfrak{b} = s_\rho \circ \mu_\mathfrak{b} : Z(\mathfrak{g}) \to S(\mathfrak{h})$ is called the *Harish-Chandra homomorphism.* One shows that $\gamma_\mathfrak{b}$ is independent of the choice of a positive root system, that it is one-to-one, and that the image is equal to $S(\mathfrak{h})^W$, the Weyl group invariants in $S(\mathfrak{h})$. A proof of these facts can be found e.g. in [KV], Section IV.7. See also [Hum], Section 23.3.

1.4.8. Characters of $Z(\mathfrak{g})$. Using the Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$, one can describe all characters of $Z(\mathfrak{g})$, i.e., algebra homomorphisms from $Z(\mathfrak{g})$ into $\mathbb{C}$, in a nice way. First, we can identify $S(\mathfrak{h})$ with the algebra $P(\mathfrak{h}^*)$ of polynomials on $\mathfrak{h}^*$, and recall that any algebra homomorphism from $P(\mathfrak{h}^*)$ into $\mathbb{C}$ is given by evaluation at some $\lambda \in \mathfrak{h}^*$. With a little more work, it follows that the homomorphisms from $Z(\mathfrak{g})$ into $\mathbb{C}$ correspond to W-orbits $W\lambda$ in $\mathfrak{h}^*$. See [KV], Section IV.8, or [Hum], Section 23.3

So, infinitesimal characters are parametrized by the space $\mathfrak{h}^*/W$. They are important parameters for classifying irreducible $(\mathfrak{g}, K)$-modules. It turns out that for every fixed infinitesimal character, there are only finitely many irreducible $(\mathfrak{g}, K)$-modules with this infinitesimal character.

1.5 Tensor products of representations

We have already mentioned tensor products of finite dimensional modules in 1.2.12. The definition for general $(\mathfrak{g}, K)$-modules is analogous.

Definition 1.5.1. If V and W are $(\mathfrak{g}, K)$-modules with actions π_V and π_W, then the vector space $V \otimes W$ is a $(\mathfrak{g}, K)$-module, with the actions π of $(\mathfrak{g}, K)$ defined by

$$\pi(X)(v \otimes w) = (\pi_V(X)v) \otimes w + v \otimes (\pi_W(X)w), \qquad X \in \mathfrak{g};$$
$$\pi(k)(v \otimes w) = (\pi_V(k)v) \otimes (\pi_W(k)w), \qquad k \in K$$

for $v \in V$ and $w \in W$.

In general, assuming V and W are irreducible, it is hard to analyze $V \otimes W$ even when both V and W are finite-dimensional. If V and W are both infinite-dimensional, $V \otimes W$ is usually not of finite length. The situation is better if at least one of the factors is finite-dimensional. This is the setting of the important translation principle: to "translate" V of infinitesimal character $\lambda \in \mathfrak{h}^*$, we can tensor it with a finite-dimensional W of highest weight μ, and consider the component with (generalized) infinitesimal character $\lambda + \mu$. We will formulate some of the relevant statements in 1.5.6 below; before that, let us consider some $\mathfrak{sl}(2)$-examples for illustration.

1.5.2. Tensor products of finite dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules. Let V and W be the finite dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules with highest weights n and k respectively, and assume that $n \geq k$ (if not, use the obvious fact $V \otimes W \cong W \otimes V$). We denote the bases for V and W as in 1.2.4 by v_i, respectively w_j.

The module $V \otimes W$ is a direct sum of irreducibles, and we can find these irreducibles by examining the weights of $V \otimes W$ and their multiplicities. It is clear that all the weights that appear are $n+k, n+k-2, n+k-4, \dots, -n-k$. The multiplicity of a weight of the form $n+k-2r$ for $0 \leq r \leq k$ is $r+1$. Namely, the corresponding weight vectors are $v_n \otimes w_{k-2r}, v_{n-2} \otimes w_{k-2r+2}, \dots, v_{n-2r} \otimes w_k$. Symmetrically, a weight of the form $-n-k+2r$ for $0 \leq r \leq k$ also has multiplicity $r+1$. The remaining weights are between $-n+k$ and $n-k$, so of the form $n-k-2r$ for $1 \leq r \leq k$. These all have multiplicity $k+1$: the corresponding weight vectors are $v_{n-2k-2r+2s} \otimes w_{k-2s}$ for $0 \leq s \leq k$. The point here is that $n-2k-2r+2s$ is a weight of V for every choice of r and s.

We thus see that $V \otimes W$ is the direct sum of representations with highest weights $n+k, n+k-2, \dots, n-k$.

1.5.3. Tensor products of irreducible highest weight $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-modules. As we know from 1.3.10 – 1.3.13, there are two kinds of irreducible highest weight $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-modules: the finite-dimensional ones, and the highest weight discrete series (or limits of discrete series) representations. We already worked out the case when both factors are finite-dimensional. Suppose that V is an irreducible (infinite-dimensional) module with highest weight $-n$ and that W is an irreducible finite-dimensional module with highest weight k, where n and k are positive integers. It is clear that the weights of $V \otimes W$ are $-n+k, -n+k-2, \dots$. The multiplicity of the weight $-n+k-2r$ is $r+1$ for $0 \leq r \leq k$ and $k+1$ for $r > k$; this is seen just as in 1.5.2 above.

This time it is however in general not sufficient to calculate the multiplicity of each weight, as $V \otimes W$ is not necessarily a direct sum of irreducibles. The situation is easy to handle if $-n+k<0$, so that all the weights of $V \otimes W$ are negative. In this case, $V \otimes W$ is the direct sum of irreducible modules of highest weights $-n+k, -n+k-2, \dots, -n-k$. To see this, one first observes that $\pi(Y)$ is injective on $V \otimes W$ - this is true for any n and k. Second, if z is a vector in $V \otimes W$ of highest weight $\lambda < 0$, then we see using 1.3 that the space spanned by all $\pi(Y)^s z$, s a positive integer, is an irreducible submodule of $V \otimes W$. Namely, this space is invariant under $\pi(X)$ (and obviously also under $\pi(Y)$ and $\pi(W)$), and the kernel of $\pi(X)$ on this space is just $\mathbb{C}z$, which implies there are no nontrivial submodules. We will therefore be done with proving the above assertion, if we can produce a highest weight vector z_r for each of the weights $-n+k-2r$, $0 \leq r \leq k$, in such a way that for every r, the vectors $v_r, \pi(Y)v_{r-1}, \dots, \pi(Y)^r v_0$ are linearly independent. Namely, the injectivity of $\pi(Y)$ will then imply that the sum of submodules generated by these vectors exhausts $V \otimes W$. The reader is invited to complete these calculations; if the general case is too hard, one can try to handle the case $k=3$.

The case $-n+k \geq 0$ is also left as an exercise, and so is the case of two infinite-dimensional highest weight modules.

1.5.4. Tensor products of highest weight modules and lowest weight modules. This is one of the bad cases we mentioned above. Namely, it is quite clear that if V is irreducible of highest weight $-n$ and W is irreducible of lowest weight k, where n and k are positive integers, then every weight of parity $-n+k$ appears in $V \otimes W$, and all the multiplicities are infinite. So $V \otimes W$ certainly cannot have finite length. The reader is invited to study this interesting example further.

1.5.5. Weights of the tensor product of finite-dimensional modules. Let us get back to the general setting of modules over a reductive Lie algebra $\mathfrak{g}$. Assume V and W are finite dimensional irreducible $\mathfrak{g}$-modules, and let λ denote the highest weight of V. Then every highest weight of $V \otimes W$ is of the form $\lambda+\nu$, where ν is some weight of W. For a proof of this, see [V1], Proposition 3.2.12.

In general, it is difficult to see which of these possible weights actually appear, and with what multiplicity. The only obviously appearing weight is $\lambda+\nu$ with ν being the highest weight of W. This weight of $V \otimes W$ clearly has multiplicity one. We already noted and used this in 1.2.12. The famous PRV conjecture, proved by S. Kumar, asserts that $\lambda+\nu$ also appears in $V \otimes W$ whenever ν is an extremal weight of W.

1.5.6. Translation principle. Let us now briefly describe a very important application of tensoring representations, the so called translation functors. The basis for their construction is the following result of Kostant. Assume that V is a $Z(\mathfrak{g})$-finite $\mathfrak{g}$-module, that is, the annihilator of V in $Z(\mathfrak{g})$ has finite codimension in $Z(\mathfrak{g})$. Let F be a finite-dimensional $\mathfrak{g}$-module. Then the $\mathfrak{g}$-module $V \otimes F$ is $Z(\mathfrak{g})$-finite.

In particular, let $\lambda \in \mathfrak{h}^*$, and assume that V has *generalized infinitesimal character* $\chi_\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$. In other words, there is a positive integer N such that $(z-\chi_\lambda(z))^N$ annihilates V for every $z \in Z(\mathfrak{g})$. Then V is $Z(\mathfrak{g})$-finite, and if μ is an extremal weight of F one can consider the largest direct summand of $V \otimes F$ with generalized infinitesimal character $\lambda + \mu$. This direct summand is denoted by $\psi_\lambda^{\lambda+\mu} V$ and called the translate of V by μ. It is easy to check that in this way we get a functor $\psi_\lambda^{\lambda+\mu}$. The most basic result about this functor is the following.

Assume that λ and $\lambda + \mu$ are both dominant, i.e., lie in the closure of the dominant Weyl chamber. Moreover, assume that $\lambda + \mu$ is at least as singular as λ, i.e., if α is a root orthogonal to λ, then α is also orthogonal to $\lambda + \mu$. Under these assumptions, if V is irreducible, then $\psi_\lambda^{\lambda+\mu} V$ is irreducible or zero. If $\lambda + \mu$ is in fact equally singular as λ, i.e., $\lambda + \mu$ and λ are orthogonal to precisely the same roots, then $\psi_\lambda^{\lambda+\mu} V$ is irreducible (and non-zero).

The dominance condition in the above statement can be replaced by the weaker condition of integral dominance. For many more details including the proofs, one can study Chapter VII of [KV].

2

Clifford algebras and spinors

In this chapter we study real and complex Clifford algebras and their representations - the spin modules. This setting is essential for the definition of Dirac operators. We will also discuss the construction of Spin groups, which are certain subgroups of the groups of units in Clifford algebras.

2.1 Real Clifford algebras

Let V be an n-dimensional real vector space with an inner product $(\,|\,)$. The Clifford algebra $C(V)$ over V is defined similarly as the universal enveloping algebra of a Lie algebra: it is generated by V, but instead of requiring that the commutators are given by the brackets, one requires the anticommutators to be given by the inner product.

We are going to give three equivalent descriptions of $C(V)$: by a universal property, as a quotient of the tensor algebra of V, and a very concrete description using a basis.

2.1.1. Definition by a universal property. The *Clifford algebra* $C(V)$ over V is an associative real algebra with unity, together with a canonical map $i : V \to C(V)$, such that the following universal property holds. Let A be any associative real algebra with unity and let $\phi : V \to A$ be a linear map such that

$$\phi(v)^2 = -(v|v) \qquad v \in V \tag{2.1}$$

in A. Then there is a unique algebra homomorphism $\tilde{\phi} : C(V) \to A$ extending ϕ, i.e., such that $\tilde{\phi} \circ i = \phi$.

Applying the condition $\phi(v)^2 = -(v|v)$ to v, w and $v+w$, one immediately sees that this condition is equivalent to the seemingly stronger condition

$$\phi(v)\phi(w) + \phi(w)\phi(v) = -2(v|w), \qquad v, w \in V. \tag{2.2}$$

As usual, the universal property determines $C(V)$ up to isomorphism, if we can show existence, i.e., construct $C(V)$.

2.1.2. Construction using the tensor algebra. Let $T(V)$ be the tensor algebra over V (see 1.4.1). Consider the ideal I in $T(V)$ generated by all $v \otimes v + (v|v)$ for $v \in V$. Equivalently, I can be generated by all $v \otimes w + w \otimes v + 2(v|w)$ for $v, w \in V$. Then the quotient algebra

$$C(V) = T(V)/I$$

satisfies the universal property of 2.1.1, and therefore is the Clifford algebra of V. This fact is quite obvious, since $T(V)$ satisfies a universal property for linear maps from V into associative algebras with unity, and the ideal I exactly matches the condition 2.1.

It is clear now that $C(V)$ is generated by all $v \in V$, subject to the relations (2.1) or equivalently (2.2). (Here and in future we identify $v \in V$ with its image $i(v)$ under the canonical morphism i; this is justified by Theorem 2.1.5 below.)

Moreover, we can choose any orthonormal basis Z_i of V with respect to $(\,|\,)$ as a set of generators of $C(V)$. The relations then become

$$Z_i Z_j = -Z_j Z_i, \qquad i \neq j; \qquad\qquad Z_i^2 = -1. \tag{2.3}$$

It is thus clear that the set

$$Z_{i_1} Z_{i_2} \dots Z_{i_k}, \qquad 1 \leq i_1 < i_2 < \dots < i_k \leq n = \dim V \tag{2.4}$$

(together with element 1 regarded as an "empty product") spans $C(V)$.

2.1.3. $\mathbb{Z}_2$-grading. Although the $\mathbb{Z}$-grading of $T(V)$ does not induce a $\mathbb{Z}$-grading of $C(V)$, it does induce a $\mathbb{Z}_2$-grading. That is, one can decompose $C(V)$ as $C^0(V) \oplus C^1(V)$, where $C^0(V)$ (respectively $C^1(V)$) is the span of all products of an even (respectively odd) number of elements of V. The elements of $C^0(V)$ (respectively $C^1(V)$) will be called even (respectively odd).

The summands $C^0(V)$ and $C^1(V)$ are disjoint (namely no nonzero element can be simultaneously even and odd), and they indeed define a $\mathbb{Z}_2$-grading: a product of two even or two odd elements is even, while the product of an even element and an odd element is odd. All this follows immediately from the fact that the relations (2.3) identify monomials whose degree in $T(V)$ is either equal, or differs by 2.

Let us define $\kappa : C(V) \to C(V)$ by setting $\kappa(v) = -v$ for $v \in V$, noting that $\kappa(v)^2 = (-v)^2 = v^2 = -1$, so κ extends to $C(V)$ by the universal property. It is clear that κ is an automorphism of $C(V)$ and that it is equal to the identity on $C^0(V)$ and to the minus identity on $C^1(V)$. Thus we could have alternatively defined $C^0(V)$ and $C^1(V)$ as the ± 1-eigenspaces of κ. We will refer to κ as the *parity automorphism*.

We can now describe the construction corresponding to taking orthogonal direct sums of inner product spaces: it is the graded tensor product of Clifford algebras. If A and B are $\mathbb{Z}_2$-graded algebras, their *graded tensor product*, $A \bar{\otimes} B$

is equal to $A \otimes B$ as a vector space, and the multiplication on $A \bar{\otimes} B$ is defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg b \deg a'} aa' \otimes bb'$$

for homogeneous $a, a' \in A$ and $b, b' \in B$.

Theorem 2.1.4. *Let $V = U \oplus W$ be an orthogonal decomposition. Then $C(V) \cong C(U) \bar{\otimes} C(W)$.*

Proof. We can choose an orthonormal basis $Z_1, \ldots, Z_n$ for V as a union of orthonormal bases $Z_1, \ldots, Z_k$ for U, respectively $Z_{k+1}, \ldots, Z_n$ for W. The required isomorphism is then given on generators by

$$Z_{i_1} \ldots Z_{i_r} Z_{j_1} \ldots Z_{j_s} \leftrightarrow Z_{i_1} \ldots Z_{i_r} \otimes Z_{j_1} \ldots Z_{j_s},$$

for $1 \leq i_1 < \cdots < i_r \leq k < j_1 < \cdots < j_s \leq n$. This clearly satisfies the relations in both directions.

The following consequence is an analogue of the Poincaré-Birkhoff-Witt Theorem 1.4.4 for $C(V)$.

Theorem 2.1.5. *The elements* (2.4) *form a basis of the real vector space $C(V)$. In particular, the canonical morphism $i : V \to C(V)$ is one-to-one, and the dimension of $C(V)$ is $2^{\dim V}$.*

Proof. Since $V = \mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_n$, Theorem 2.1.4 enables us to reduce the statement of Theorem 2.1.5 to the one-dimensional case, which is obvious.

We now see that we could have defined $C(\mathbb{R}^n)$ very explicitly, as a 2^n-dimensional vector space with basis (2.4), where Z_i is the standard basis for $\mathbb{R}^n$, and with multiplication determined on the basis elements by (2.3). This description is good for explicit computations.

Examples 2.1.6. It is quite obvious that $C(\mathbb{R}^1) \cong \mathbb{C}$, with $Z_1 \leftrightarrow i$. It takes a small computation to see that $C(\mathbb{R}^2)$is isomorphic to the quaternion algebra $\mathbb{H}$. The space $\mathbb{H}$ has a basis $1, i, j, k$ over $\mathbb{R}$, where 1 is the unity, and i, j, k multiply by the rules

$$i^2 = j^2 = k^2 = -1; \qquad ij = -ji = k; \qquad jk = -kj = i; \qquad ki = -ik = j.$$

The identification $C(\mathbb{R}^2) \cong \mathbb{H}$ is then $Z_1 \leftrightarrow i$, $Z_2 \leftrightarrow j$ and $Z_1 Z_2 \leftrightarrow k$.

For $n \geq 3$, $C(\mathbb{R}^n)$ is not a division algebra, as follows immediately from the fact $(Z_1 Z_2 Z_3)^2 = 1$ which implies

$$(1 - Z_1 Z_2 Z_3)(1 + Z_1 Z_2 Z_3) = 0,$$

so $C(\mathbb{R}^n)$ has zero divisors. In fact, for $n = 3$ it is easily checked that $\frac{1}{2}(1 \pm Z_1 Z_2 Z_3)$ are central idempotents which add up to 1, and break up $C(\mathbb{R}^3)$ into a direct product of two copies of $\mathbb{H}$. It is possible to describe all $C(\mathbb{R}^n)$ in terms of matrix algebras over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$; see e.g. [LM], Chapter I, §4.

2.1.7. Filtration by degree. The $\mathbb{Z}$-grading of the tensor algebra $T(V)$ does not induce a $\mathbb{Z}$-grading of $C(V)$, since the ideal I is not homogeneous; e.g. Z_1^2+1 is in I, but the homogeneous components of Z_1^2+1, Z_1^2 and 1, clearly can not be in I. However, the associated filtration does descend to $C(V)$; explicitly,

$$F_pC(V) = \text{ span }\{v_1 \dots v_k \big| k \leq p, v_1, \dots, v_k \in V\}.$$

In particular, $F_0C(V) = \mathbb{R} = \mathbb{R}1$ and $F_1C(V) = \mathbb{R} \oplus V$. This filtration is analogous to the filtration of $U(\mathfrak{g})$ defined in 1.4.3.

The graded algebra corresponding to the above filtration is the exterior algebra $\bigwedge(V)$. This is obvious from Theorem 2.1.5 and the well known analogous result for the exterior algebras. Note also that taking the top degree terms of the relations (2.3) gives the defining relations for $\bigwedge(V)$. One can also turn things around and approach Theorem 2.1.5 starting from the above filtration and showing directly that the corresponding graded algebra is $\bigwedge(V)$; for this approach, see e.g. [LM], Proposition 1.2.

2.1.8. Chevalley map. As is well known, the canonical projection $T(V) \to \bigwedge(V)$ has a linear right inverse given by linearly embedding $\bigwedge(V)$ into the tensor algebra $T(V)$ as the skew-symmetric tensors:

$$v_1 \wedge \dots \wedge v_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{ sgn } (\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)},$$

where S_k denotes the group of permutations on k letters, and sgn $(\sigma) \in \{\pm 1\}$ is the sign of a permutation σ. The *Chevalley map* j is obtained by composing this "skew symmetrization" map with the canonical projection $T(V) \to C(V)$. Using an orthonormal basis Z_i of V, this map is determined on the corresponding basis of $\bigwedge(V)$ simply by

$$Z_{i_1} \wedge \dots \wedge Z_{i_k} \longmapsto Z_{i_1} \dots Z_{i_k} \qquad (\text{and } 1 \mapsto 1),$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. This map is clearly a linear isomorphism, and for each $p \geq 0$ it gives a right inverse to the canonical projection $F_pC(V) \to \text{Gr }_pC(V) = \bigwedge^p(V)$. It is often referred to as the Chevalley identification, or, as in [AM], the quantization map. We will say that the elements of $C(V)$ which are in the image of $\bigwedge^k(V)$ under j are of pure degree k. Of course, the obtained grading of $C(V)$ does not make $C(V)$ into a graded algebra.

2.1.9. Embedding $\mathfrak{so}(V)$ into $C(V)$. Let us consider the image $j(\bigwedge^2(V)) \subset C(V)$ under the Chevalley map. This space is a Lie subalgebra of $C(V)$, if we consider $C(V)$ to be a Lie algebra in the usual way, by setting $[a, b] = ab - ba$, $a, b \in C(V)$. Indeed, if Z_i is an orthonormal basis for V, then Z_iZ_j for $i < j$ form a basis for $j(\bigwedge^2(V))$. Obviously, $[Z_iZ_j, Z_kZ_l] = 0$ if $\{i, j\}$ and $\{k, l\}$ are either disjoint or equal. If $\{i, j\}$ and $\{k, l\}$ have exactly one element in common, we can assume (by skew symmetry) that $j = k$. Then

$$[Z_iZ_j, Z_jZ_l] = Z_iZ_j^2Z_l - Z_jZ_lZ_iZ_j = -Z_iZ_l + Z_lZ_i = -2Z_iZ_l.$$

On the other hand, $\bigwedge^2(V)$ is linearly isomorphic to the Lie algebra $\mathfrak{so}(V)$, with Z_iZ_j corresponding to the operator with matrix $E_{ij} - E_{ji}$ in basis Z_i. A small computation of matrix commutators shows that apart from the factor of -2, the rule is the same as above, i.e., that

$$E_{ij} - E_{ji} \longleftrightarrow -\frac{1}{2}Z_iZ_j$$

is an isomorphism of the Lie algebras $\mathfrak{so}(V)$ and $j(\bigwedge^2(V))$.

The copy of $\mathfrak{so}(V)$ we have just identified in $C(V)$ will be the Lie algebra of the groups $\mathrm{Pin}\,(V)$ and $\mathrm{Spin}\,(V)$ introduced below. In fact, since we will see that these groups are double covers of $O(V)$ respectively $SO(V)$, this will give another proof of $j(\bigwedge^2(V)) \cong \mathfrak{so}(V)$. Nevertheless, it is nice to have a direct algebraic proof of this fact, and also the map we have constructed will play an important role later on.

2.1.10. The group Pin (V). Let us begin by observing that if $v \in V$ has length 1, then it is invertible in $C(V)$; its inverse is $-v$, another vector of length 1. We can therefore consider the conjugation

$$x \mapsto vxv^{-1} = -vxv, \qquad x \in C(V).$$

Restricting our attention to $x \in V$, let $x = \lambda v + w$ where $w \perp v$. Then $vw = -wv$ and thus

$$-vxv = -\lambda v^3 - vwv = \lambda v - w = -(-\lambda v + w);$$

i.e., the conjugation by v preserves $V \subset C(V)$, and the operation it induces on V is minus the reflection with respect to $v^\perp$. To eliminate the minus sign, instead of conjugation by v one can consider the twisted conjugation:

$$x \mapsto \kappa(v)xv^{-1} = vxv, \qquad x \in C(V).$$

This twisted conjugation again preserves V, and the induced transformation on V is now exactly the reflection with respect to $v^\perp$. Moreover, if we denote by $\mathrm{Pin}\,(V)$ the subgroup of the group of units in $C(V)$ generated by all $v \in V$ of length 1, then we get a homomorphism ρ from $\mathrm{Pin}\,(V)$ into the orthogonal group $O(V)$, defined by

$$\rho(u)x = \kappa(u)xu^{-1}, \qquad x \in V,$$

for $u \in \mathrm{Pin}\,(V)$. The point is that ρ is a group homomorphism because κ is an algebra homomorphism.

It is well known that the group $O(V)$ is generated by hyperplane reflections, and hence ρ is onto. Let us identify the kernel of ρ.

Proposition 2.1.11. *The kernel of ρ consists of the scalars 1 and -1.*

Proof. Let $u \in \operatorname{Pin}(V)$ be such that $\rho(u)$ is the identity operator on V, i.e., such that

$$\kappa(u)x = xu, \qquad x \in V.$$

Write $u = u_0 + u_1$ with u_0 even and u_1 odd. Then $\kappa(u) = u_0 - u_1$ and it follows

$$u_0 x = x u_0; \qquad u_1 x = -x u_1, \qquad x \in V. \tag{2.5}$$

Setting $x = Z_i$ and writing u_0 and u_1 in terms of the basis (2.4), we see that (2.5) can be true only if neither u_0 nor u_1 contain any terms with Z_i. Since this holds for any i, we see that u must be a constant.

We will be done if we prove that the only constants contained in $\operatorname{Pin}(V)$ are 1 and -1. This can be proved by using another piece of structure that we describe in the following.

2.1.12. The principal antiautomorphism of $C(V)$. We denote by α the unique antiautomorphism of $C(V)$ equal to the identity on V. In terms of the basis (2.4), α is given by $\alpha(1) = 1$ and

$$\alpha(Z_{i_1} \dots Z_{i_k}) = Z_{i_k} \dots Z_{i_1}.$$

This implies that α is the scalar $(-1)^{\frac{k(k-1)}{2}}$ on the elements of pure degree k (i.e., on the image of $\bigwedge^k(V)$ under the Chevalley map). In particular, it follows that α^2 is the identity, i.e., that α is an antiinvolution. Note also that α clearly commutes with the sign automorphism κ, as both are scalars on each pure degree element.

It is clear that for $v \in V$ of length 1, $v\alpha(v) = v^2 = -1$. Hence for any $u \in \operatorname{Pin}(V)$, $u\alpha(u) = \pm 1$. More precisely, $u\alpha(u)$ is -1 if u is odd and 1 if u is even.

In particular, if u is a (real) scalar, then $\alpha(u) = u$ and thus $u^2 = 1$, which is only possible if $u = \pm 1$. This finishes the proof of Proposition 2.1.11.

2.1.13. Pin group as a Lie group. Since $C(V)$ is a finite-dimensional vector space, we can consider it as a topological vector space in a canonical way. This topology can be defined by a suitable norm, for example the one coming from the scalar product for which the basis (2.4) is orthonormal. It is clear that the multiplication is not only continuous but also smooth with respect to the usual differentiable structure on the vector space $C(V)$.

The group of units $C(V)^\times$ is open in $C(V)$. There are several ways to see this; for example, one can embed $C(V)$ into the algebra $\operatorname{End}_{\mathbb{R}}(C(V))$, by identifying $a \in C(V)$ with the operator on $C(V)$ given by the left multiplication by a. Then since $C(V)$ is finite-dimensional, $u \in C(V)$ is in $C(V)^\times$ if and only if u is invertible in $\operatorname{End}_{\mathbb{R}}(C(V))$, and the last condition is equivalent to the open condition $\det u \neq 0$.

So $C(V)^\times$ is a Lie group in the obvious way. Consider the relative topology on the subgroup $\mathrm{Pin}\,(V)$. Since $\rho : \mathrm{Pin}\,(V) \to O(V)$ is continuous, onto, and has finite kernel $\{\pm 1\}$, and since $O(V)$ is compact, it follows immediately that $\mathrm{Pin}\,(V)$ is compact. In particular, it is a closed subgroup of $C(V)^\times$ and hence a Lie group.

We claim that the Lie algebra of $\mathrm{Pin}\,(V)$ is exactly $j(\bigwedge^2(V))$ considered in 2.1.9. To see this, let Z_i as before denote an orthonormal basis of V, and consider the curve

$$\varphi(t) = \cos t + \sin t Z_i Z_j = (\cos t Z_i + \sin t Z_j)(-Z_i),$$

for a fixed pair $i \neq j$. Since $\cos t Z_i + \sin t Z_j$ and $-Z_i$ are unit vectors in V, $\varphi(t) \in \mathrm{Pin}\,(V)$ for all t. Since $\varphi(0) = 1$, $\varphi'(0) = Z_i Z_j$, this shows that the Lie algebra of $\mathrm{Pin}\,(V)$ contains the span of all $Z_i Z_j$, and this is the Lie algebra $j(\bigwedge^2(V))$. On the other hand, the existence of the double covering $\rho : \mathrm{Pin}\,(V) \to O(V)$ shows that the Lie algebra of $\mathrm{Pin}\,(V)$ is isomorphic to $\mathfrak{o}(V) = \mathfrak{so}(V)$, hence their dimensions are equal and it follows that the above containment must be equality.

Clearly, $\mathrm{Pin}\,(V)$ can not be connected since it maps onto the disconnected group $O(V)$. We will see below that it has exactly two connected components, just like $O(V)$.

2.1.14. The group Spin (V). Since the group $\mathrm{Pin}\,(V)$ is generated by odd elements $v \in V$, it can be decomposed as $(\mathrm{Pin}\,(V) \cap C^0(V)) \cup (\mathrm{Pin}\,(V) \cap C^1(V))$. The even part is a subgroup called the spin group and denoted by $\mathrm{Spin}\,(V)$. It is a subgroup of index 2, hence it is normal. It is clearly also compact. Its elements are products of an even number of $v \in V$ of length 1. Hence $\rho(u)$, $u \in \mathrm{Spin}\,(V)$ are characterized as products of an even number of hyperplane reflections. In other words, $\mathrm{Spin}\,(V) = \rho^{-1}(SO(V))$. Since 1 and -1 are even, the map $\rho : \mathrm{Spin}\,(V) \to SO(V)$ is again a double covering.

To see that $\mathrm{Spin}\,(V)$ is connected (and hence that $\mathrm{Pin}\,(V)$ has two connected components), it is enough to connect 1 and -1 by a path within $\mathrm{Spin}\,(V)$. This path will then give a path between u and $-u$ for any $u \in \mathrm{Spin}\,(V)$. So if $u_1, u_2 \in \mathrm{Spin}\,(V)$, then we first construct a path between $\rho(u_1)$ and $\rho(u_2)$, lift it to $\mathrm{Spin}\,(V)$ to get a path from u_1 to either u_2 or $-u_2$, and finally continue it from $-u_2$ to u_2 if necessary.

A path between 1 and -1 can be obtained by taking a path $v(t)$ from $v \in S^{n-1} \subset V$ to $-v$, and multiplying this last path by v.

Finally, one can also show that $\mathrm{Spin}\,(V)$ is simply connected, or equivalently that $SO(V)$ has fundamental group $\mathbb{Z}_2$. We omit the proof of this fact and refer the reader to e.g. [FH], Proposition 23.1.

2.1.15. The case of indefinite form. If B is any symmetric bilinear form on a real vector space V, we can replace the inner product in the definition of $C(V)$ by B and thus get a Clifford algebra $C(V;B)$. In particular, if B is a nondegenerate form with signature (p,q), we will denote the resulting Clifford algebra by $C(p,q)$.

It is quite clear that everything we said in 2.1.1 – 2.1.8 remains unchanged in the present situation, except that the examples now look different. For example, for a negative definite form on $\mathbb{R}^1$, we do not get $\mathbb{C}$, but rather the algebra $\mathbb{R}[X]/(X^2-1) \cong \mathbb{R}\times\mathbb{R}$, with quite different algebraic properties. Note that in case $B=0$, the Clifford algebra is equal to the exterior algebra $\bigwedge(V)$.

It is easy to see that just like in 2.1.9 one gets an embedding of the Lie algebra $\mathfrak{so}(p,q)$ into the Clifford algebra $C(p,q)$. The groups $\mathrm{Pin}\,(p,q)$ and $\mathrm{Spin}\,(p,q)$ are defined similarly as in 2.1.10 – 2.1.14; $\mathrm{Pin}\,(p,q)$ is the group generated by $v\in V$ of norm ±1, and $\mathrm{Spin}\,(p,q)$ is the subgroup consisting of even elements. The groups $\mathrm{Pin}\,(p,q)$ and $\mathrm{Spin}\,(p,q)$ are double coverings of the indefinite orthogonal groups $O(p,q)$ respectively $SO(p,q)$. Of course, none of these groups is compact if p and q are both different from 0.

2.2 Complex Clifford algebras and spin modules

Let V be a complex vector space with a symmetric bilinear form B. The Clifford algebra $C(V)$ is defined in complete analogy with the real case:

Definition 2.2.1. The (complex) Clifford algebra $C(V)$ is an associative complex algebra with unity, with a canonical linear map $i: V\to C(V)$, such that the following universal property holds. Let A be any associative complex algebra with unity and let $\phi: V\to A$ be a linear map such that

$$\phi(v)^2 = -B(v,v) \qquad v\in V$$

in A. Then there is a unique algebra homomorphism $\tilde\phi: C(V)\to A$ extending ϕ, i.e., such that $\tilde\phi\circ i=\phi$.

As in the real case, the above condition can be replaced by the seemingly stronger condition analogous to (2.2). As always, the universal property determines $C(V)$ up to isomorphism, and the construction is

$$C(V) = T(V)/I$$

where $T(V)$ is the (complex) tensor algebra of V and I is the ideal generated by all $v\otimes v+B(v,v)$, $v\in V$. (Equivalently, I is generated by all $v\otimes w+w\otimes v+2B(v,w)$.)

Most of the things we said about the real Clifford algebras in Section 2.1 hold also for complex Clifford algebras in exactly the same way. That is, $C(V)$ can be filtered by degree, the associated graded algebra is $\bigwedge(V)$, and there is an analogue of the Chevalley map $j:\bigwedge(V)\to C(V)$. One can construct a basis of $C(V)$ analogous to (2.4), starting from an orthonormal basis Z_i of V with respect to B. The algebra $C(V)$ is $\mathbb{Z}_2$-graded and has a sign automorphism

κ, like in 2.1.3. It also has a principal antiautomorphism α as in 2.1.12. The complex Lie algebra $\mathfrak{so}(V) \cong \mathfrak{so}(n,\mathbb{C})$ embeds into $C(V)$ in analogy with 2.1.9. One can also construct the complex groups $\mathrm{Pin}\,(V) = \mathrm{Pin}\,(n,\mathbb{C})$ and $\mathrm{Spin}\,(V) = \mathrm{Spin}\,(n,\mathbb{C})$ and the double coverings $\mathrm{Pin}\,(V) \to O(V)$ and $\mathrm{Spin}\,(V) \to SO(V)$ as in 2.1.10 – 2.1.14: the group $\mathrm{Pin}\,(V)$ is the subgroup of $C(V)^x$ generated by elements $v \in V$ with $B(v,v) = \pm 1$, while $\mathrm{Spin}\,(V)$ is the subgroup of $\mathrm{Pin}\,(V)$ consisting of even elements. In this case, one can not vary the choice of B as in 2.1.14, as there is only one nondegenerate symmetric bilinear form on $\mathbb{C}^n$ up to isomorphism.

Note that if we start from a real vector space V_0 with a symmetric bilinear form B and then complexify V_0 and B to a complex vector space V, then the Clifford algebra $C(V)$ is the complexification of the Clifford algebra $C(V_0)$. (The complex Pin and Spin groups are also complexifications of their real analogues.)

2.2.2. Spin module for $\dim V$ even. Assume that $\dim V = 2n$ and that the form B is nondegenerate on V. Let U and U^* be a pair of complementary n-dimensional isotropic subspaces, dual to each other under B. To see the existence of U and U^*, one can construct them explicitly starting from an orthonormal basis $Z_1, \dots, Z_{2n}$ of V with respect to B. Namely, if we set

$$u_j = \frac{Z_{2j-1} + iZ_{2j}}{\sqrt{2}}; \qquad u_j^* = \frac{Z_{2j-1} - iZ_{2j}}{\sqrt{2}}, \qquad j = 1, \dots, n,$$

then it is trivial to check that the subspaces U and U^* of V spanned by u_j's respectively u_j^*'s are isotropic and complementary, and moreover, $B(u_j, u_k^*) = \delta_{jk}$.

One can now define a *spin module* $S = S(V)$. It is equal to $\bigwedge(U)$ as a vector space, and $C(V)$ acts on it as follows: $u \in U$ and $u^* \in U^*$ act on $u_1 \wedge \dots \wedge u_k \in \bigwedge(U)$ by

$$u \cdot (u_1 \wedge \dots \wedge u_k) = u \wedge u_1 \wedge \dots \wedge u_k;$$
$$u^* \cdot (u_1 \wedge \dots \wedge u_k) = \textstyle\sum_i (-1)^i 2B(u^*, u_i) u_1 \wedge \dots \widehat{u_i} \dots \wedge u_k,$$

where the hat on u_i indicates that u_i is omitted. This defines the action of V on S, and this action extends to all of $C(V)$. To see this, one can simply check that the relations are satisfied. Let us instead embed S into the algebra $C(V)$ as a left ideal, in such a way that the above action corresponds to left multiplication.

Denote by u^*_{top} any nonzero element in $\bigwedge^{\mathrm{top}}(U^*)$, viewed as an element of $C(V)$ via the Chevalley map. Then since $u^* u^*_{\mathrm{top}} = 0$ in $C(V)$ for any $u^* \in U^*$, and since $C(U) = \bigwedge(U)$ as B is 0 on U, we see that the left ideal of $C(V)$ generated by u^*_{top} can be identified with $\bigwedge(U) u^*_{\mathrm{top}}$, which is isomorphic to S in the obvious way. To understand the action of V by left multiplication on this ideal, note that $u \in U$ simply Clifford multiplies or equivalently wedge multiplies on the left, while $u^* \in U^*$ has to be commuted through $\bigwedge(U)$ to reach u^*_{top} , where it finally gets killed. This gives exactly the action defined above on S.

The action we defined looks even simpler when described in a basis: choose a basis u_i of U and let u_i^* be the dual basis of U^* with respect to B. The basis u_i induces a basis of S consisting of monomials $u_{i_1} \wedge \cdots \wedge u_{i_k}$ with indices $i_1, \ldots, i_k$ increasing. Then the action of u_i sends a basic monomial $u_{i_1} \wedge \cdots \wedge u_{i_k}$ to 0 if i is among the indices $i_1, \ldots, i_k$, and to the monomial $u_i \wedge u_{i_1} \wedge \cdots \wedge u_{i_k}$ (which is basic up to sign) if i is not among the indices $i_1, \ldots, i_k$. The action of u_i^* sends the same basic monomial to 0 if i is not among the indices $i_1, \ldots, i_k$, and to twice the contracted monomial $u_{i_1} \wedge \ldots \widehat{u_{i_j}} \cdots \wedge u_{i_k}$ with the appropriate sign if $i = i_j$.

Lemma 2.2.3. *The $C(V)$-module $S = S(V)$ constructed above is irreducible.*

Proof. This is most easily seen using bases u_i, u_i^* and $u_{i_1} \wedge \cdots \wedge u_{i_k}$ for U respectively U^* respectively S like in the last paragraph above.

Let $x \in S$ be any nonzero element. Suppose that x is of degree $k > 0$ with respect to the standard grading of $S = \bigwedge(U)$. Write x in terms of the basis and suppose the coefficient of a basic monomial $u_{i_1} \wedge \cdots \wedge u_{i_k}$ is $\lambda \neq 0$. Then $u_{i_k}^* \ldots u_{i_1}^* x = (-2)^k \lambda$, as the term $\lambda u_{i_1} \wedge \cdots \wedge u_{i_k}$ gets contracted to a constant and all other terms of x are annihilated by $u_{i_k}^* \ldots u_{i_1}^*$. Hence the submodule generated by x contains a nonzero constant; this is of course true also if x is of degree 0.

It is clear that the element 1 generates S. Hence also x generates S. Since x was an arbitrary nonzero element of S, this implies that S is irreducible.

Similar ideas can be used to prove

Lemma 2.2.4. *The algebra $C(V)$ is isomorphic to the algebra* End (S) *of all endomorphisms of the vector space $S = S(V)$.*

Proof. For any subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, denote by u_I the basic element $u_{i_1} \wedge \cdots \wedge u_{i_k}$ of S. We will denote by the same symbol u_I the corresponding element $u_{i_1} \ldots u_{i_k}$ of $C(V)$.

We have to show that for any I and J there is an element of $C(V)$ sending $u_I \in S$ to $u_J \in S$ and all other basic elements $u_{I'}$ of S to 0. This will mean that the homomorphism $C(V) \to \mathrm{End}\,(S)$ given by the action is onto, and since both spaces have the same dimension 2^{2n}, it then has to be an isomorphism.

To see this, note first that obviously

$$p_1 = \frac{1}{(-2)^n} u_n^* \ldots u_1^* u_1 \ldots u_n \in C(V) \tag{2.6}$$

sends $1 = u_\emptyset$ to itself, and all other basic elements u_I to 0. It follows that $u_J p_1 \in C(V)$ sends 1 to u_J and all $u_I \neq 1$ to 0. Moreover, for $I = \{i_1, \ldots, i_k\}$, let

$$u_I^* = \frac{1}{(-2)^k} u_{i_k}^* \ldots u_{i_1}^* \in C(V).$$

Then clearly u_I^* maps $u_I \in S$ to 1, and all other $u_{I'}$ either to 0 (when I' does not contain I) or to $u_{I' \setminus I}$ up to sign (when I' contains I). Consequently, $p_1 u_I^*$ maps u_I to 1 and all other $u_{I'}$ to 0. Finally, our element mapping u_I to u_J and other $u_{I'}$ to 0 is

$$u_J p_1 p_1 u_I^* = u_J p_1 u_I^* \in C(V).$$

(Namely, an easy calculation shows that $p_1^2 = p_1$.)

Corollary 2.2.5. *The algebra $C(V)$ is simple.*

Proof. It is quite well known that the algebra End W is simple for any vector space W. We give a proof of this fact for convenience of the reader.

Let I be a nonzero ideal of End (W), and let $x \in I$ be nonzero. Fixing a basis of W and passing to matrices, we can write

$$x = \sum_{i,j} \alpha_{ij} E_{ij},$$

where as usual E_{ij} denote the matrices with all entries 0 except the ij entry which is 1. Using the obvious identity

$$E_{kl} E_{rs} = \delta_{lr} E_{ks},$$

we see that $E_{ii} x E_{jj} = \alpha_{ij} E_{ij}$. So, picking i and j such that $\alpha_{ij} \neq 0$, we conclude that $E_{ij} \in I$. But then also $E_{kl} = E_{ki} E_{ij} E_{jl} \in I$ for any k and l, so I is all of End (W).

It is well known that W is up to isomorphism the only irreducible module for the algebra End (W). We sketch a proof of this fact for $C(V)$ below, since it fits nicely into the other calculations of this section.

Proposition 2.2.6. *Let $C(V)$ be a complex Clifford algebra, and let S be a spin module for $C(V)$. Then S is the only irreducible $C(V)$-module up to isomorphism.*

Proof. Let S' be any irreducible $C(V)$-module. We will again make use of the element $p_1 \in C(V)$ defined by (2.6). Since $p_1^2 = p_1$, it can only have eigenvalues 0 and 1 on S'. Moreover, since $C(V)$ is simple, S' cannot have nonzero annihilator in $C(V)$ and hence p_1 is not identically 0 on S'.

Let us take some nonzero $x \in S'$ such that $p_1 x = x$. Then all u_i^* annihilate x, since $u_i^* p_1 = \pm \frac{1}{2^n} u_n^* \dots (u_i^*)^2 \dots u_1^* u_1 \dots u_n = 0$. Now define $\phi : S \to S'$ by

$$\phi(u_I) = u_I x, \qquad I \subset \{1, \dots, n\}.$$

To see that ϕ is a homomorphism, it is enough to check that

$$\phi(u u_I) = u \phi(u_I), \qquad \phi(u^* u_I) = u^* \phi(u_I)$$

for all $u \in U$, $u^* \in U^*$ and $I \subset \{1, \dots, n\}$. The first of these two equalities is obvious, while the other follows from the fact that all u^* annihilate x.

Since ϕ is obviously nonzero, it must be an isomorphism by irreducibility of S and S'. (This is a variant of Schur's lemma 1.2.10; it follows from the fact that $\operatorname{Ker} \phi$ and $\operatorname{Im} \phi$ are submodules of S respectively S'.)

2.2.7. Spin modules for $\dim V$ odd. Let now V be a $2n+1$-dimensional complex vector space with a symmetric bilinear form B. As before, choose an orthonormal basis $Z_1, \dots, Z_{2n+1}$ of V with respect to B. Let us denote by $\bar{V}$ the span of $Z_1, \dots, Z_{2n}$; so $V = \bar{V} \oplus \mathbb{C}Z_{2n+1}$.

Let S be a spin module for $C(\bar{V})$. We want to make S into a $C(V)$-module, i.e., define an action of Z_{2n+1} on S, in such a way that the relations of $C(V)$ are satisfied. First, since $Z_{2n+1}^2 = -1$, any action of Z_{2n+1} will have two eigenspaces, corresponding to the eigenvalues i and $-i$. Furthermore, since $Z_i Z_{2n+1} = -Z_{2n+1} Z_i$ for $i = 1, \dots, 2n$, each of the Z_i's for $i \leq 2n$ should interchange the two eigenspaces of Z_{2n+1}. An obvious choice for the two eigenspaces of Z_{2n+1} is thus $S^{\pm}$, the even and odd part of S with respect to the natural grading of $S = \bigwedge(U)$. This gives us two choices: Z_{2n+1} can act as i on S^+ and as $-i$ on S^-, or as $-i$ on S^+ and as i on S^-.

It is clear from the above analysis that for each of the two choices we are getting a $C(V)$-module structure on S. In either case, the $C(V)$-module S is irreducible, as it is already irreducible for $C(\bar{V})$. Moreover, the two $C(V)$-module structures are not isomorphic: any $C(\bar{V})$-automorphism of S must be a scalar by Schur's Lemma and hence can not intertwine the two actions of Z_{2n+1}.

Let us now assume that M is any irreducible $C(V)$-module. The key observation is that in our present case the center of $C(V)$ does not consist only of scalars, but also contains the element $Z_{\text{top}} = Z_1 \dots Z_{2n+1}$. Namely, a direct calculation shows that Z_{top} commutes with each Z_i. It now follows from Schur's Lemma that Z_{top} must act on M by a scalar. Since a scalar operator leaves every subspace invariant, and since obviously

$$C(V) = (\mathbb{C} \oplus \mathbb{C}Z_{\text{top}})C(\bar{V}),$$

it follows that M is irreducible as a $C(\bar{V})$-module. Hence $M \cong S$ as $C(\bar{V})$-modules.

To determine the possible actions of Z_{top}, note that

$$(Z_{\text{top}})^2 = Z_1 \dots Z_{2n+1} Z_1 \dots Z_{2n+1} = (-1)^{(2n+1)+2n+\dots+2+1} = (-1)^{n+1}.$$

So if n is even, Z_{top} acts as i or as $-i$, and if n is odd, Z_{top} acts as 1 or as -1. In any case, we are getting exactly two inequivalent irreducible $C(V)$-modules, hence they must be the two we constructed above.

In fact, it is not very difficult to relate the actions of Z_{top} and Z_{2n+1} explicitly; we leave this as an exercise for the reader.

Denoting the two spin modules we have constructed, one can identify $C(V)$ with End $S_1 \oplus$ End S_2. This decomposition can be obtained using the orthogonal central idempotents $\frac{1}{2}(1 \pm i^{n+1} Z_{\text{top}})$. The details are left to the reader.

2.2.8. Graded spin modules. Since the algebra $C(V)$ is $\mathbb{Z}_2$-graded, it also makes sense to study the $\mathbb{Z}_2$-graded modules over $C(V)$. In case dim V is even, the spin module S is clearly $\mathbb{Z}_2$-graded. One can however change the grading of S by exchanging the even and odd parts. One shows that the two graded modules obtained in this way are not isomorphic (as graded modules), and that they are the only irreducible graded $C(V)$-modules up to isomorphism.

In case $\dim V = 2n+1$, the spin modules we constructed are not graded, since the odd element Z_{2n+1} preseves the odd and even parts instead of exchanging them. To obtain a graded module, one can embed V into a $(2n+2)$-dimensional space $\tilde{V} = V \oplus \mathbb{C} Z_{2n+2}$, consider the spin module $S(\tilde{V})$ for $C(\tilde{V})$, and restrict it to $C(V)$ embedded into $C(\tilde{V}) = C(V) \bar{\otimes} C(\mathbb{C} Z_{2n+2})$ as $C(V) \otimes 1$. This module is irreducible in the category of graded $C(V)$-modules, but as an ungraded module it splits into the direct sum of S_1 and S_2, the irreducible ungraded modules for $C(V)$. One shows that $S(\tilde{V})$ is the unique irreducible graded $C(V)$-module.

A good approach to proving the above facts (taken from [LM]) is to consider any n-dimensional space W embedded into an $n+1$-dimensional space $\tilde{W} = W \oplus \mathbb{C} Z_{n+1}$, and to consider $C(W)$ embedded into $C(\tilde{W})$ as the even part, $C^0(\tilde{W})$, via the map defined on generators by $Z_i \mapsto Z_{n+1} Z_i$. Then the functor

$$M = M^0 \oplus M^1 \longmapsto M^0$$

from the category of graded $C(\tilde{W})$-modules into the category of ungraded $C(W) \cong C^0(\tilde{W})$-modules is an equivalence of categories. Namely, $M \mapsto M^0$ has an inverse, the functor

$$M \longmapsto C(\tilde{W}) \otimes_{C(W)} M,$$

with the grading of $C(\tilde{W}) \otimes_{C(W)} M$ coming from the first factor.

Let us also mention that one can define tensor structures on the category of graded modules over a Clifford algebra $C(V)$. Namely, $C(V)$ has a family of graded coproducts constructed in [Pan3].

2.3 Spin representations of Lie groups and algebras

Since the Clifford algebras contain many Lie groups and algebras, like for example the spin groups and their subgroups, as well as their Lie algebras, we can restrict the spin modules we have constructed to these groups and algebras. We will only consider the Lie algebras; since the spin groups are simply connected, there is no problem in passing to the groups if required.

2.3.1. Spin representation of $\mathfrak{so}(V)$. Let V be a finite-dimensional complex vector space with a nondegenerate symmetric bilinear form B. Complexifying the construction of 2.1.9, we identify the Lie algebra $\mathfrak{so}(V)$ with the space of degree 2 elements in the Clifford algebra $C(V)$, that is, with the image of $\bigwedge^2(V)$ under the Chevalley map. As in 2.2.2 and 2.2.7, we fix a pair U, U^* of maximal (n-dimensional) isotropic subspaces of V, dual under B, with dual bases u_i and u_i^*. If $\dim V = 2n$, then $V = U \oplus U^*$, while if $\dim V = 2n+1$ we pick an element Z complementary to $U \oplus U^*$, with $B(Z, Z) = 1$.

We choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{so}(V)$ as the span of the elements

$$h_i = u_i^* u_i + 1, \qquad i = 1, \dots, n.$$

It is obvious that h_i commute with each other; moreover $\operatorname{ad} h_i$ can be simultaneously diagonalized on $\mathfrak{so}(V)$, as we can check by commuting h_i with the elements of the obvious basis of $\mathfrak{so}(V)$, consisting of elements $u_i u_j$, $u_i u_j^*$ and $u_i^* u_j^*$. As a result, we get that the roots of $\mathfrak{so}(V)$ with respect to $\mathfrak{h}$ are as follows:

1. α_{ij}, $i \neq j$, sending h_i to -2, h_j to 2, and other h_k to 0; the corresponding root vector is $u_i^* u_j$;
2. β_{ij}, $i < j$, sending h_i and h_j to 2, and other h_k to 0; the corresponding root vector is $u_i u_j$;
3. $-\beta_{ij}$, $i < j$; the corresponding root vector is $u_i^* u_j^*$;
4. in case V is odd-dimensional, there are also roots γ_i sending h_i to 2 and other h_k to 0, with the corresponding root vector $u_i Z$, and $-\gamma_i$ with the corresponding root vector $u_i^* Z$.

We choose the roots α_{ij} with $i < j$, β_{ij}, and γ_i in case V is odd-dimensional, to be positive. Note that $\alpha_{ji} = -\alpha_{ij}$. The corresponding simple roots are then $\alpha_{i\,i+1}$ for $i = 1, \dots, n-1$, and in addition β_{12} if V is even-dimensional, respectively γ_n if V is odd-dimensional.

It is an easy exercise to check all of the above assertions. Another exercise is to see how to pass to the more usual interpretation of $\mathfrak{so}(V)$ as the space of linear operators on V which are skew-symmetric with respect to B.

Let us now consider the spin module $S = \bigwedge(U)$ for $C(V)$ as a module for $\mathfrak{so}(V)$ by restricting the action. In case V is odd-dimensional, S carries two different actions of $C(V)$, however they restrict to isomorphic representations of $\mathfrak{so}(V)$. Namely, it is easy to check that $\epsilon : S \to S$ given as id on the even part S^+ of S, and as $-\operatorname{id}$ on the odd part S^- of S, intertwines the two actions of $\mathfrak{so}(V)$.

It is obvious that all the standard basic monomials in S,

$$u_I, \qquad I \subseteq \{1, \dots, n\} \tag{2.7}$$

are weight vectors for $\mathfrak{h}$. Here as usual, $u_I = u_{i_1} \wedge \cdots \wedge u_{i_r}$ if $I = \{i_1, \dots, i_r\}$ with $1 \leq i_1 < \cdots < i_r \leq n$, and $u_\emptyset = 1$.

In fact, it is obvious from the definition of the $C(V)$-action on S that h_k fixes u_I if $k \in I$, and acts on it as multiplication by -1 if $k \notin I$. So we see that all the weights of the $\mathfrak{so}(V)$-module S have multiplicity one, and they are just all possible n-tuples with entries ± 1. In particular, all the highest weight vectors must be among our basic elements (2.7).

It is clear that the basic element $u_{\text{top}} = u_1 \wedge \cdots \wedge u_n$ is a highest weight vector, as it is annihilated by all the positive root vectors, $u_i^* u_j$ for $i < j$, $u_i u_j$ for $i < j$, and also $u_i Z$ in case $\dim V$ is odd. (Note that in fact this vector is also annihilated by the negative root vectors $u_i^* u_j$ for $i > j$.) In case $\dim V$ is odd, no other basic monomial can be a highest weight vector, since if the monomial does not contain u_i, it is not annihilated by $u_i Z$. So S is irreducible in this case. If $\dim V$ is even, then there is another highest weight vector, $u_2 \wedge u_3 \wedge \cdots \wedge u_n$. Clearly, this last vector generates S^-, the odd part of S (i.e., the span of elements of odd degree), while u_{top} generates S^+, the even part of S.

We have proved the following proposition.

Proposition 2.3.2. *If* $\dim V$ *is even, the restriction of the spin module S for $C(V)$ to $\mathfrak{so}(V)$ decomposes into a direct sum of two irreducible submodules, S^+ and S^-.*

If $\dim V$ *is odd, the two actions of $C(V)$ on the spin module S restrict to the same irreducible action of $\mathfrak{so}(V)$.*

2.3.3. Quadratic Lie algebras. A *quadratic Lie algebra* is a Lie algebra $\mathfrak{g}$ with a nondegenerate invariant symmetric bilinear form B. A quadratic subalgebra of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{r} \subset \mathfrak{g}$, such that the restriction of B to $\mathfrak{r} \times \mathfrak{r}$ is nondegenerate. We will be interested in cases when $\mathfrak{g}$ and $\mathfrak{r}$ are both reductive and complex. If $\mathfrak{g}$ is reductive, then it is always quadratic; indeed, if $\mathfrak{g}$ is semisimple, we can take B to be the Killing form, and if $\mathfrak{g}$ is only reductive, B can be extended over the center by any nondegenerate symmetric bilinear form. Of course, not all reductive subalgebras will be quadratic subalgebras; for example, $\mathbb{C}X$ with X nilpotent is not a quadratic subalgebra of $\mathfrak{g}$, but it is abelian and thus reductive.

If $\mathfrak{r}$ is a quadratic subalgebra of $\mathfrak{g}$, then it is clear that

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s},$$

where $\mathfrak{s} = \mathfrak{r}^\perp$ is the orthogonal complement to $\mathfrak{r}$ with respect to B. Moreover, the restriction of B to $\mathfrak{s} \times \mathfrak{s}$ is nondegenerate. Furthermore, the invariance of B immediately implies that

$$[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{s}.$$

The proofs of the above facts are easy and left to the reader.

By the invariance of B, the adjoint action of $\mathfrak{r}$ on $\mathfrak{s}$ defines a map $\operatorname{ad} : \mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$. This map is clearly a Lie algebra homomorphism. Composing it with the embedding of the Lie algebra $\mathfrak{so}(\mathfrak{s})$ into the Clifford algebra $C(\mathfrak{s})$ from 2.1.9, we get a Lie algebra map

$$\alpha : \mathfrak{r} \to C(\mathfrak{s}).$$

We note that Kostant denotes this map by ν_* in his papers.

If we pick an orthonormal basis Z_i for $\mathfrak{s}$, then the embedding $\mathfrak{so}(\mathfrak{s}) \hookrightarrow C(\mathfrak{s})$ is explicitly given as

$$E_{ij} - E_{ji} \mapsto -\frac{1}{2} Z_i Z_j.$$

Here E_{ij} is (as usual) the matrix in the basis Z_i having all entries equal to 0 except the ij entry which is equal to 1. Since the matrix entries of ad X, $X \in \mathfrak{r}$, in the basis Z_i are

$$(\operatorname{ad} X)_{ij} = B(\operatorname{ad} X(Z_j), Z_i) = B([X, Z_j], Z_i) = -B(X, [Z_i, Z_j]),$$

we get an explicit formula for α in this basis:

$$\alpha(X) = \frac{1}{2} \sum_{i<j} B(X, [Z_i, Z_j]) Z_i Z_j, \qquad X \in \mathfrak{r}. \tag{2.8}$$

Since the above sum over $i < j$ is twice the same sum over $i \neq j$, and since

$$\sum_{i \neq j} B([X, Z_j], Z_i) Z_i = \sum_i B([X, Z_j], Z_i) Z_i = [X, Z_j],$$

the map α can also be written as

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j. \tag{2.9}$$

If b_i is any basis of $\mathfrak{s}$, and if d_i is the dual basis with respect to B, i.e., $B(d_i, b_j) = \delta_{ij}$, then we can substitute $Z_j = \sum_i B(d_i, Z_j) b_i = \sum_k B(d_k, Z_j) b_k$ into (2.9), and after a short calculation obtain

$$\alpha(X) = -\frac{1}{4} \sum_j [X, d_j] b_j. \tag{2.10}$$

Finally, substituting $[X, d_j] = \sum_i B([X, d_j], d_i) b_i$ into (2.10) leads to

$$\alpha(X) = \frac{1}{4} \sum_{i,j} B(X, [d_i, d_j]) b_i b_j = \frac{1}{2} \sum_{i<j} B(X, [d_i, d_j])(b_i b_j + B(b_i, b_j)) \tag{2.11}$$

The extra constant in the last expression when compared with (2.8) comes from $b_i b_j = -b_j b_i - 2B(b_i, b_j)$; this is not just $-b_j b_i$, unless the basis b_i is orthogonal.

2.3.4. The spin module for $C(\mathfrak{s})$ as an $\mathfrak{r}$-module. In view of the map $\alpha : \mathfrak{r} \to C(\mathfrak{s})$ defined above, any $C(\mathfrak{s})$-module can be viewed as an $\mathfrak{r}$-module.

To describe the structure of the spin module(s) for $C(\mathfrak{s})$ restricted to $\mathfrak{r}$, we need some more notation related to the decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{r}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Since the restrictions of B to $\mathfrak{h}$ and $\mathfrak{t}$ are nondegenerate, $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{s}$ is orthogonal to $\mathfrak{t}$. Let Δ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let Δ_0 be the subset of those roots that vanish on $\mathfrak{t}$. Let $H \in \mathfrak{t}$ be an element which is maximally regular, in the sense that it is not annihilated by any roots outside of Δ_0. We can choose H so that it is also hyperbolic, i.e., the operator $\operatorname{ad} H$ on $\mathfrak{g}$ has real eigenvalues. It is now clear that the centralizer of H in $\mathfrak{g}$ is equal to $\mathfrak{t} \oplus \mathfrak{s}^0$, where $\mathfrak{s}^0$ is the subspace of $\mathfrak{s}$ spanned by $\mathfrak{a}$ and all $\mathfrak{g}_\delta$, $\delta \in \Delta_0$. The same subalgebra is also the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. Moreover, H defines a parabolic subalgebra of $\mathfrak{g}$, with a Levi subalgebra equal to $\mathfrak{t} \oplus \mathfrak{s}^0$, and with nilradical equal to the span of those root spaces $\mathfrak{g}_\delta$ for which $\delta(H) > 0$.

We will call any $\mathfrak{t}$-weight of $\mathfrak{g}$ positive if it is positive on the element H; clearly, these consist of the restrictions to $\mathfrak{t}$ of the roots positive on H. In particular, the nonzero $\mathfrak{t}$-weights of $\mathfrak{r}$ are the roots of $\mathfrak{r}$ with respect to $\mathfrak{t}$, and we have defined a positive root system $\Delta^+(\mathfrak{r}, \mathfrak{t})$. The $\mathfrak{t}$-weights of $\mathfrak{s}$ also get divided into positive and negative ones; denote the positive ones by β_j, repeated according to multiplicity. (Of course, some of β_j can also appear in $\Delta^+(\mathfrak{r}, \mathfrak{t})$, but this does not increase the multiplicity which is meant only relative to $\mathfrak{s}$.) By invariance of B and nondegeneracy of $\mathfrak{s}$, it easily follows that the weight spaces in $\mathfrak{s}$ with respect to the weights β_j and $-\beta_j$ must be nondegenerately paired. In particular, negative weights are exactly the $-\beta_j$, and the $-\beta_j$-weight space has the same dimension as the β_j-weight space, for every j.

We choose a basic element u_i for every positive root α_i, and denote by u_i^* the dual element with respect to B, which lies in the $-\alpha_i$-root space. Similarly, v_j will be β_j-weight vectors in $\mathfrak{s}$, with dual elements v_j^*. Inside $\mathfrak{s}^0$, we choose a pair of dual maximal isotropic spaces W and W^*, with dual bases w_k and w_k^*, and if $\dim \mathfrak{s}^0$ is odd, we also choose a vector Z orthogonal to $W \oplus W^*$, with $B(Z,Z) = 1$.

We can now choose a basis b_i for $\mathfrak{s}$, consisting of v_j, v_j^*, w_k, w_k^*, and possibly Z. The dual basis d_i then consists of v_j^*, v_j, w_k^*, w_k, and possibly Z. Since $\mathfrak{t}$ commutes with $\mathfrak{s}_0$, the formula (2.10) applied to $X \in \mathfrak{t}$ gives

$$\alpha(X) = -\frac{1}{4}\sum_j [X, v_j]v_j^* + [X, v_j^*]v_j = -\frac{1}{4}\sum_j \beta_j(X)v_jv_j^* - \beta_j(X)v_j^*v_j.$$

Since $v_jv_j^* = -v_j^*v_j - 2$, this implies

$$\alpha(X) = \frac{1}{2}\sum_j \beta_j(X)(v_j^*v_j + 1), \qquad X \in \mathfrak{t}. \tag{2.12}$$

A spin module S for $C(\mathfrak{s})$ can be constructed as the exterior algebra over the isotropic subspace of $\mathfrak{s}$ spanned by the vectors v_j and w_k. In other words,

$S = \bigwedge(V) \otimes \bigwedge(W)$, where V and W are the spaces spanned by the vectors v_j respectively w_k. The action of $\mathfrak{t}$ on $\bigwedge(W)$ is zero. Also, each of the standard monomials

$$v_I, \qquad I \subseteq \{1, \dots, \dim V\}$$

(where as usual $v_I = v_{i_1} \wedge \cdots \wedge v_{i_r}$ if $I = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$, and $v_\emptyset = 1$) is a weight vector for $\mathfrak{t}$, of weight

$$\frac{1}{2}\sum_{k\in I}\beta_k - \frac{1}{2}\sum_{k\notin I}\beta_k. \tag{2.13}$$

This follows immediately from (2.12). Each of these weight vectors can be combined with any element of $\bigwedge(W)$ to get a vector of the same weight. Therefore, each of the weights has multiplicity at least $\dim W = \left[\frac{1}{2}\dim \mathfrak{s}^0\right]$. These multiplicities can further increase if some of the weights (2.13) are equal. This can happen even in the equal rank case; for example, if $\mathfrak{r}$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, then β_j are simply the positive roots, and enumerating them so that $\beta_1 + \beta_2 = \beta_3$, we see that $v_1 \wedge v_2$ and v_3 are both of weight zero.

If $\mathfrak{r}$ has the same rank as $\mathfrak{g}$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, the space $\mathfrak{s}^0$ disappears, and the α_i and β_j are (all) positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

Getting back to general $\mathfrak{r}$, it is clear that the weight

$$\frac{1}{2}\sum_{j=1}^{\dim V}\beta_j \tag{2.14}$$

of the vector $v_{\,\mathrm{top}}$ is the highest of all $\mathfrak{t}$-weights of S, in the sense that it has the largest possible value on our fixed element $H \in \mathfrak{t}$. This implies that this is a highest weight of S for $\mathfrak{r}$ with respect to our choice α_i of positive roots for $(\mathfrak{r}, \mathfrak{t})$ corresponding to H. In other words, the vector $v_{\,\mathrm{top}}$ is annihilated by all $\alpha(u_i)$. Moreover, since α maps $\mathfrak{r}$ into the even part of $C(\mathfrak{s})$, it follows that in case $\mathfrak{s}$ is even-dimensional, the decomposition $S = S^+ \oplus S^-$ of Proposition 2.3.2 is $\mathfrak{r}$-invariant. It follows that the $\mathfrak{so}(\mathfrak{s})$-highest weight vector of S^-, $v_2 \wedge \cdots \wedge v_{\dim V}$ is also a highest weight vector for $\mathfrak{r}$. The corresponding highest weight is

$$-\frac{1}{2}\beta_1 + \frac{1}{2}\sum_{j=2}^{\dim V}\beta_j. \tag{2.15}$$

If we change H in such a way that the positive roots for $(\mathfrak{r}, \mathfrak{t})$, the α_i, stay the same, but the positive weights of $\mathfrak{s}$, the β_j, change, and thus also the space V changes, we get another highest weight of S for $\mathfrak{r}$, given by (2.14) for the new β_j. In case $\mathfrak{s}$ is even-dimensional, we get in addition a highest weight analogous to (2.15). This will be made more explicit in 2.3.6 below in case $\mathfrak{r}$ is symmetric. The results we have obtained in the general case are summarized in the following proposition.

Proposition 2.3.5. *Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ as above, and let S be a spin module for $C(\mathfrak{s})$ viewed as an $\mathfrak{r}$-module. Then all weights of S are of the form (2.13). Among these weights, the weight (2.14) is always a highest weight of an $\mathfrak{r}$-submodule of S. In case* $\dim \mathfrak{s}$ *is odd, the weight (2.15) is another such highest weight.*

2.3.6. The case when $\mathfrak{r}$ is symmetric. Let us now assume that $\mathfrak{r}$ is a *symmetric subalgebra* of $\mathfrak{g}$, i.e., there is an involution σ of $\mathfrak{g}$ such that $\mathfrak{r}$ is the fixed point set of σ. We further assume that σ is orthogonal with respect to B. In the examples we are interested in, B is essentially the Killing form, which is invariant under all automorphisms of $\mathfrak{g}$ and hence σ is orthogonal. It follows that $\mathfrak{s}$ is exactly the (-1)-eigenspace of σ. It is then clear that $[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{r}$.

The main example of this situation comes from a real reductive G with Cartan involution Θ such that the fixed points of Θ form a maximal compact subgroup K of G. If $\mathfrak{g}$ is the complexified Lie algebra of G and θ is the complexified differential of Θ, then the complexified Lie algebra $\mathfrak{k}$ of K is a symmetric subalgebra of $\mathfrak{g}$, corresponding to the involution θ. In fact, every symmetric subalgebra is of this form for suitably chosen G; we will therefore change notation and denote σ by θ, $\mathfrak{r}$ by $\mathfrak{k}$ and $\mathfrak{s}$ by $\mathfrak{p}$ in the following.

As in 2.3.4, let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{k}$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{p}$. Such a Cartan subalgebra is called *fundamental*, or maximally compact. We claim that the set Δ_0 of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ vanishing on $\mathfrak{t}$ is empty. Indeed, if a root δ vanishes on $\mathfrak{t}$, then $\theta\delta = -\delta$; here we view θ as operating on $\mathfrak{h}^*$ by duality:

$$(\theta\lambda)(X) = \lambda(\theta X), \qquad \lambda \in \mathfrak{h}^*,\ X \in \mathfrak{h}.$$

Therefore, if we choose a nonzero element x_δ spanning the root space $\mathfrak{g}_\delta$, then θx_δ spans $\mathfrak{g}_{-\delta}$. Then $x_\delta + \theta x_\delta$ is in $\mathfrak{k}$, and it commutes with $\mathfrak{t}$ since both δ and $-\delta$ vanish on $\mathfrak{t}$. On the other hand, $x_\delta + \theta x_\delta$ is linearly independent from $\mathfrak{h}$, hence can not be in $\mathfrak{t}$, and this is a contradiction with the fact that $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{k}$.

This means that our hyperbolic element H of $\mathfrak{t}$ is now $\mathfrak{g}$-regular, its centralizer is $\mathfrak{h}$, and it defines a system of positive roots Δ^+: a root δ is positive if and only if $\delta(H) > 0$. This positive root system is θ-stable, i.e., a root δ is positive if and only if $\theta\delta$ is positive. This is immediate from the fact that $\theta H = H$ (note also that for any root δ, $\theta\delta$ is a root). The Borel subalgebra corresponding to Δ^+ (spanned by $\mathfrak{h}$ and the positive root spaces) is thus also θ-stable.

We choose the $\mathfrak{t}$-weights α_i and β_j as in 2.3.4. Clearly, α_i form a positive root system for $\mathfrak{k}$ with respect to $\mathfrak{t}$.

A root δ is called *imaginary* if $\delta\big|_{\mathfrak{a}} = 0$, or equivalently, $\theta\delta = \delta$. For other choices of θ-stable Cartan subalgebras, one also has the notion of *real roots*, the ones that vanish on $\mathfrak{t}$, or equivalently such that $\theta\delta = -\delta$. In our present situation however, real roots can not exist, as shown above. Finally, if $\mathfrak{k}$ has smaller rank than $\mathfrak{g}$, there are also the *complex roots*, which restrict nontrivially to both $\mathfrak{t}$ and $\mathfrak{a}$, or equivalently, satisfy $\theta\delta \neq \pm\delta$. Note that δ and $\theta\delta$ always have the same restriction to $\mathfrak{t}$.

An imaginary root δ is called *compact* if $\mathfrak{g}_\delta \subset \mathfrak{k}$ and *noncompact* if $\mathfrak{g}_\delta \subset \mathfrak{p}$. We can now conclude the following about our weights α_i and β_j:

1. α_i consist of
   - compact imaginary roots;
   - the restriction $\delta\big|_{\mathfrak{t}} = \theta\delta\big|_{\mathfrak{t}}$ for every pair δ, $\theta\delta$ of positive complex roots; namely, if we choose $x_\delta \in \mathfrak{g}_\delta$, then $\theta x_\delta \in \theta\mathfrak{g}_\delta = \mathfrak{g}_{\theta\delta}$, and $x_\delta + \theta x_\delta \in \mathfrak{k}$ is of weight $\delta\big|_{\mathfrak{t}} = \theta\delta\big|_{\mathfrak{t}}$.
2. β_j consist of
   - noncompact imaginary roots;
   - the restriction $\delta\big|_{\mathfrak{t}} = \theta\delta\big|_{\mathfrak{t}}$ for every pair δ, $\theta\delta$ of positive complex roots; namely, with notation as above, $x_\delta - \theta x_\delta \in \mathfrak{p}$ is of weight $\delta\big|_{\mathfrak{t}} = \theta\delta\big|_{\mathfrak{t}}$.

Note that α_i are of multiplicity one, since they are exactly the roots of $\mathfrak{k}$ with respect to $\mathfrak{t}$. It follows then that β_j are also of multiplicity one. All this can be found with more details and in greater generality in [KV], Section IV.4.

For every choice of a θ-stable positive root system as above containing a fixed positive root system α_i for $(\mathfrak{k}, \mathfrak{t})$, we get a highest weight vector v_{top} for the $\mathfrak{k}$-action on the spin module $S(\mathfrak{p})$, of weight (2.14). In our present situation, the weight (2.14) is equal to $\rho_\mathfrak{g} - \rho_\mathfrak{k}$, where as usual $\rho_\mathfrak{g}$ and $\rho_\mathfrak{k}$ denote the half sums of positive roots for $(\mathfrak{g}, \mathfrak{h})$ respectively $(\mathfrak{k}, \mathfrak{t})$. This follows easily from the above relations of α_i and β_j to the roots of $\mathfrak{g}$ and $\mathfrak{k}$. Moreover, each of these highest weights clearly has multiplicity exactly $\left[\frac{1}{2} \dim \mathfrak{a}\right]$.

If $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a θ-stable Borel subalgebra corresponding to our choice of positive roots, then the $\mathfrak{t}$-weight $\rho_\mathfrak{g} - \rho_\mathfrak{k}$ is often denoted by $\rho(\mathfrak{n} \cap \mathfrak{p})$, or by ρ_n. In case $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, this is the half sum of noncompact positive roots.

We want to show that in this way we have obtained all the highest weights of the $\mathfrak{k}$-module $S(\mathfrak{p})$. The point of the approach we take (from [W], 9.2.7 and 9.3) is that we can calculate the action of the Casimir element $\Omega_\mathfrak{k}$ of the center $Z(\mathfrak{k})$ of the universal enveloping algebra $U(\mathfrak{k})$, and see that it acts on $S(\mathfrak{p})$ by the scalar $||\rho_\mathfrak{g}||^2 - ||\rho_\mathfrak{k}||^2$. This is done in Proposition 2.3.7 below. Once we prove this, we can finish the argument as follows. Let λ be any $\mathfrak{k}$-highest weight of S. Then $\Omega_\mathfrak{k}$ acts on the corresponding highest weight vector by the scalar $||\lambda + \rho_\mathfrak{k}||^2 - ||\rho_\mathfrak{k}||^2$ (see 1.4.6). It follows that

$$||\lambda + \rho_\mathfrak{k}|| = ||\rho_\mathfrak{g}||.$$

On the other hand, we know from (2.13) that λ, being a weight of S, must be equal to $\rho_\mathfrak{g} - \rho_\mathfrak{k}$ minus a sum of some of the β_j's. Since all β_j are restrictions to $\mathfrak{t}$ of positive roots for $(\mathfrak{g}, \mathfrak{h})$, we obtain a sum μ of distinct positive roots such that $||\rho_\mathfrak{g} - \mu|| \geq ||\rho_\mathfrak{g}||$. Since $\rho_\mathfrak{g} - \mu$ is a weight of the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight $\rho_\mathfrak{g}$, this shows that it must be an extremal weight, and thus it is a highest weight for another choice of positive roots for $(\mathfrak{g}, \mathfrak{h})$. Hence $\rho_\mathfrak{g} - \mu$ is $\rho'_\mathfrak{g}$, the half sum of the positive roots for this other positive root system. Moreover, $||\rho_{\mathfrak{g}} - \mu|| = ||\rho_{\mathfrak{g}}||$, hence μ is in $\mathfrak{t}^*$ and it follows that this new positive root system is θ-stable and compatible with our fixed positive root system for $(\mathfrak{k}, \mathfrak{t})$. In other words, λ is one of the highest weights we already described above.

The following proposition will imply that the Casimir element $\Omega_{\mathfrak{k}}$ really acts on $S(\mathfrak{p})$ by the scalar $||\rho_{\mathfrak{g}}||^2 - ||\rho_{\mathfrak{k}}||^2$. We will use this result again in Section 3.1 to calculate the square of the Dirac operator.

Proposition 2.3.7. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and let $\alpha : U(\mathfrak{k}) \to C(\mathfrak{p})$ be the map defined in 2.3.3. Then the image of the Casimir element $\Omega_{\mathfrak{k}}$ under α is the scalar $||\rho_{\mathfrak{g}}||^2 - ||\rho_{\mathfrak{k}}||^2$.*

Proof. Let W_k be an orthonormal basis of $\mathfrak{k}$ with respect to $-B$, and let Z_i be an orthonormal basis of $\mathfrak{p}$ with respect to B. Then $\Omega_{\mathfrak{k}} = -\sum_k W_k^2$ and $\alpha(\Omega_{\mathfrak{k}}) = -\sum_k \alpha(W_k)^2$. It follows from (2.8) that

$$\alpha(X) = \frac{1}{4}\sum_{i,j} B(X, [Z_i, Z_j])Z_i Z_j, \qquad X \in \mathfrak{k}.$$

Thus

$$\sum_k \alpha(W_k)^2 = \frac{1}{16}\sum_{k,i,j,r,s} B(W_k, [Z_i, Z_j])B(W_k, [Z_r, Z_s])Z_i Z_j Z_r Z_s.$$

Since $\sum_k B(W_k, [Z_i, Z_j])B(W_k, [Z_r, Z_s]) = -B([Z_i, Z_j], [Z_r, Z_s])$, this can be rewritten as

$$\sum_k \alpha(W_k)^2 = \sum_{i,j,r,s} R_{ijrs} Z_i Z_j Z_r Z_s,$$

where R_{ijrs} denotes the scalar $-\frac{1}{16}B([Z_i, Z_j], [Z_r, Z_s])$.

The scalars R_{ijrs} are easily seen to satisfy the conditions of Lemma 2.3.8 below; condition (2.18) follows from invariance of B and the Jacobi identity. It follows that

$$\sum_k \alpha(W_k)^2 = 2\sum_{i,j} R_{ijji} = \frac{1}{8}\sum_{i,j} B([Z_i, Z_j], [Z_i, Z_j]).$$

In particular, $\alpha(\Omega_{\mathfrak{k}})$ is a constant. Rather than calculating directly what this constant is, we can note that it follows that $\Omega_{\mathfrak{k}}$ acts by a constant on the spin module $S(\mathfrak{p})$. On the other hand, we know from 2.3.6 that one of the highest weight of $S(\mathfrak{p})$ for $\mathfrak{k}$ is $\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$ (relative to the choice of positive roots made in 2.3.6). Therefore the action of $\Omega_{\mathfrak{k}}$ on the corresponding highest weight space is $||(\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}) + \rho_{\mathfrak{k}}||^2 - ||\rho_{\mathfrak{k}}||^2 = ||\rho_{\mathfrak{g}}||^2 - ||\rho_{\mathfrak{k}}||^2$ (see 1.4.6). Since we already know that $\Omega_{\mathfrak{k}}$ acts on $S(\mathfrak{p})$ by the scalar $\alpha(\Omega_{\mathfrak{k}})$, this proves the proposition.

Lemma 2.3.8. *Suppose that* R_{ijrs}, $i, j, r, s \in \{1, \dots, \dim \mathfrak{p}\}$ *are complex constants satisfying*

$$R_{ijrs} = R_{rsij}; \tag{2.16}$$

$$R_{ijrs} = -R_{jirs}; \tag{2.17}$$

$$R_{ijrs} + R_{rijs} + R_{jris} = 0. \tag{2.18}$$

Then

$$\sum_{i,j,r,s} R_{ijrs} Z_i Z_j Z_r Z_s = 2(\sum_{i,j} R_{ijji}) \cdot 1.$$

Proof. Let S denote the left hand side of the asserted equality and let R denote the sum on the right hand side; so we are to prove that $S = 2R \cdot 1$. The idea is to rewrite S with indices j and r interchanged, and then use the relation $Z_j Z_r + Z_r Z_j = -2\delta_{jr}$. However, this does not immediately work, as $R_{irjs} \neq R_{ijrs}$. Rather, by (2.18) and (2.17), $R_{irjs} = R_{ijrs} + R_{jris}$. So we get

$$\begin{aligned} 2S &= \sum_{i,j,r,s} R_{ijrs} Z_i Z_j Z_r Z_s + R_{irjs} Z_i Z_r Z_j Z_s \\ &= \sum_{i,j,r,s} R_{ijrs} Z_i (Z_j Z_r + Z_r Z_j) Z_s + R_{jris} Z_i Z_r Z_j Z_s \\ &= -2 \sum_{i,j,s} R_{ijjs} Z_i Z_s + S', \end{aligned}$$

where S' denotes the sum $\sum_{i,j,r,s} R_{jris} Z_i Z_r Z_j Z_s$. By (2.16) and (2.17), $R_{ijjs} = R_{sjji}$, hence

$$2 \sum_{i,j,s} R_{ijjs} Z_i Z_s = \sum_{i,j,s} R_{ijjs} (Z_i Z_s + Z_s Z_i) = -2 \sum_{i,j} R_{ijji} = -2R.$$

Thus we have obtained

$$2S = 2R + S'. \tag{2.19}$$

Now we apply a very similar reasoning to S' as we did to S; this time we interchange indices i and r. As before, $R_{jris} = R_{jirs} + R_{irjs}$, and hence

$$\begin{aligned} 2S' &= \sum_{i,j,r,s} R_{jris} Z_i Z_r Z_j Z_s + R_{jirs} Z_r Z_i Z_j Z_s \\ &= \sum_{i,j,r,s} R_{jirs} (Z_i Z_r + Z_r Z_i) Z_j Z_s + R_{irjs} Z_i Z_r Z_j Z_s \\ &= -2 \sum_{i,j,s} R_{jiis} Z_j Z_s + S = 2R + S \end{aligned}$$

Together with (2.19), this gives $S = 2R$ as asserted.

2.3.9. Unitary structure on the spin module. Let V_0 be a real vector space with inner product $(\,,\,)$ and let V be the complexification of V_0. We denote the conjugation of V with respect to V_0 by a bar, the unique extension of $(\,,\,)$ to a bilinear form on V by B, and the unique Hermitian inner product on V extending $(\,,\,)$ again by $(\,,\,)$. Thus clearly

$$(v,w) = B(v,\bar{w}), \qquad v,w \in V.$$

We choose complementary maximal isotropic subspaces U and U^* of V as in 2.2.2, starting from an orthonormal basis Z_i of V_0. Thus $U^* = \bar{U}$. Furthermore, $V = U \oplus \bar{U}$ if $\dim V$ is even, and if $\dim V = 2n+1$ is odd, then $V = U \oplus \bar{U} \oplus \mathbb{C}Z_{2n+1}$.

The restriction of $(\,,\,)$ to U is a Hermitian inner product on U. We multiply this inner product by 2 and then extend it to the spin module $S = \bigwedge(U)$ by using the determinant. In other words, $\bigwedge^i(U)$ is orthogonal to $\bigwedge^j(U)$ if $i \neq j$, while

$$\langle u_1 \wedge \cdots \wedge u_k, u_1' \wedge \cdots \wedge u_k' \rangle = \det\,(2(u_i, u_j')), \tag{2.20}$$

for $u_1, \ldots, u_k, u_1', \ldots, u_k' \in U$. Finally, we define $\langle 1,1 \rangle = 1$. It is easy to check that this defines a Hermitian inner product $\langle\,,\,\rangle$ on S. The point of introducing the factor two is to avoid this factor appearing elsewhere. If $u_1, \ldots, u_n$ denotes the basis of U from 2.2.2, then the monomials $u_{i_1} \wedge \cdots \wedge u_{i_r}$ together with 1 form an orthogonal basis of S, with the squared norm of $u_{i_1} \wedge \cdots \wedge u_{i_r}$ being 2^r. Namely, note that $(u_i, u_i) = B(u_i, \bar{u}_i) = B(u_i, u_i^*) = 1$.

Let us find the adjoint of $u \in U$ with respect to $\langle\,,\,\rangle$, where u is understood as an element of $C(V)$ acting on S. Let us assume $u = u_k$ is a basis element. Denoting the adjoint of u_k by u_k^{adj}, we first note that since u_k is of degree 1, u_k^{adj} must be of degree -1 with respect to the grading of S. Furthermore, we have

$$\langle u_k \wedge u_{i_1} \wedge \cdots \wedge u_{i_r}, u_{j_1} \wedge \cdots \wedge u_{j_{r+1}} \rangle = \langle u_{i_1} \wedge \cdots \wedge u_{i_r}, u_k^{\mathrm{adj}}\,(u_{j_1} \wedge \cdots \wedge u_{j_{r+1}}) \rangle.$$

The left hand side of this equality is clearly zero unless $\{k, i_i, \ldots, i_r\} = \{j_1, \ldots, j_{r+1}\}$, in which case it is $\pm 2^{r+1}$. Comparing with the action of the dual basis element u_k^* from 2.2.2, we see that

$$u_k^{\mathrm{adj}} = -u_k^* = -\bar{u}_k.$$

It now follows that $(u_k^*)^{\mathrm{adj}} = (-u_k^{\mathrm{adj}})^{\mathrm{adj}} = -u_k = -\overline{u_k^*}$, so that $v^{\mathrm{adj}} = -\bar{v}$ for every $v \in U \oplus U^*$. If V is odd-dimensional, then it is trivial to check that this is also true for Z_{2n+1}, so we get:

Proposition 2.3.10. *Let S be a spin module for $C(V)$ with the above inner product. Then the adjoint of any $v \in V$ acting on S is $v^{\mathrm{adj}} = -\bar{v}$. In particular, all elements of the real form V_0 of V act on S by skew-symmetric operators.*

One can also start from a complex V with a Hermitian inner product $(\,,\,)$. Then a real form V_0 can be obtained as the real span of an orthonormal basis. Similarly, if we are given a non-degenerate bilinear form B on V, we can again obtain a real form V_0 as the real span of an orthonormal basis with respect to B. Of course, such V_0 is highly non-unique in each of the two cases.

Let us now consider the following situation. Let $\mathfrak{g}_0$ be a real reductive Lie algebra with complexification $\mathfrak{g}$, and let θ denote a Cartan involution on $\mathfrak{g}_0$ and $\mathfrak{g}$. As usual, the corresponding Cartan decompositions are denoted by $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be a decomposition as in 2.3.3. Let $\mathfrak{g}_c = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ be the compact real form of $\mathfrak{g}$ corresponding to the conjugation

$$X \mapsto \theta \bar{X}, \qquad X \in \mathfrak{g},$$

where the bar denotes conjugation with respect to $\mathfrak{g}_0$. Since B is negative definite on $\mathfrak{g}_c$, we can start with the inner product $-B$ on $\mathfrak{g}_c$ and extend it to a Hermitian inner product

$$(X, Y) = -B(X, \theta \bar{Y})$$

on $\mathfrak{g}$. We restrict this inner product to $\mathfrak{s}$ and consider the corresponding inner product on the spin module S for $C(\mathfrak{s})$ as in 2.3.9. The conclusion of Proposition 2.3.10 then becomes

Corollary 2.3.11. *With notation as above, the adjoint of any $X \in \mathfrak{s}$ on the spin module S for $C(\mathfrak{s})$ is $X^{\mathrm{adj}} = -\theta \bar{X}$.*

3

Dirac operators in the algebraic setting

Dirac operators were introduced into representation theory by Parthasarathy [Par] as a tool to construct the discrete series representations. The final results, which applied to all discrete series, were obtained by Atiyah and Schmid in [AS]. In this chapter we study an algebraic version of Parthasarathy's Dirac operator, due to Vogan. In particular, we explain the notion of Dirac cohomology of Harish-Chandra modules, and Vogan's conjecture which predicts the infinitesimal character of modules with nonzero Dirac cohomology [V3]. We present a proof of this conjecture following [HP1].

3.1 Dirac operators

Dirac defined his operator in [D] as a square root of the D'Alembert wave operator, which is an analogue of the Laplace operator on the Minkowski space $\mathbb{R}^{3,1}$. The main point for his and all subsequent applications was the fact that taking the square root gives more eigenvalues. In other words, if $D^2 = \varDelta$, then the eigenspace decomposition for D is finer than the eigenspace decomposition for $\varDelta$, as two opposite eigenvalues for D square to the same eigenvalue for $\varDelta$. We first illustrate the definition of Dirac operators on $\mathbb{R}^n$ and then move on to the Lie algebra setting we actually need.

3.1.1. Dirac operator on $\mathbb{R}^n$. Let us consider the differential operator

$$\varDelta = -\sum_i \frac{\partial^2}{\partial x_i^2}$$

on $\mathbb{R}^n$. If we try to find a square root of this operator of the form

$$D = \sum_i e_i \frac{\partial}{\partial x_i},$$

then $D^2 = \varDelta$ leads to equations

$$e_i^2 = -1 \qquad \text{and} \qquad e_i e_j + e_j e_i = 0$$

for the coefficients e_i. If we insist on having only real or complex coefficients, then this is of course impossible. If we however allow e_i to be in the Clifford algebra $C(\mathbb{R}^n)$, then we can take e_i simply to be the vectors of the standard basis of $\mathbb{R}^n \subset C(\mathbb{R}^n)$.

Note that we can also view $\frac{\partial}{\partial x_i}$ as corresponding to e_i under the obvious identification of left invariant vector fields on $\mathbb{R}^n$ with $\mathbb{R}^n$ itself. Thus we can define the Dirac operator as an element

$$D = \sum_i e_i \otimes e_i$$

of the algebra $\mathbb{D}(\mathbb{R}^n) \otimes C(\mathbb{R}^n)$ of differential operators on $\mathbb{R}^n$ with coefficients in the Clifford algebra $C(\mathbb{R}^n)$. Such operators operate on functions from $\mathbb{R}^n$ into some module for $C(\mathbb{R}^n)$ – for example a spin module S, or the Clifford algebra itself.

It is quite obvious that all this can be extended with only minor changes to the setting of an indefinite form B. Assuming B is diagonal in the standard basis, the natural operator Δ would then be

$$\Delta = -\sum_i B(e_i, e_i) \frac{\partial^2}{\partial x_i^2},$$

the Clifford algebra would also be defined with respect B, and D would still be given as $\sum_i e_i \otimes e_i$.

We are now getting back to the setting of a connected real reductive Lie group G with a maximal compact subgroup K and a corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Here as usual $\mathfrak{g}_0$ respectively $\mathfrak{k}_0$ denote the Lie algebras of G respectively K, and the complexifications are denoted by ommiting the subscript 0; thus $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. As before, B will denote a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$, which is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$; if $\mathfrak{g}$ is semisimple, one can take B to be the Killing form (or the trace form) of 1.2.6, and if $\mathfrak{g}$ is reductive, one can easily extend this form to all of $\mathfrak{g}$.

We consider the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ (see 1.4.1) and $C(\mathfrak{p})$ is the (complex) Clifford algebra of $\mathfrak{p}$ with respect to B (see Definition 2.2.1).

Definition 3.1.2. The *Dirac operator* D is an element of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined as

$$D = \sum_i Z_i \otimes Z_i,$$

where Z_i is an orthonormal basis of $\mathfrak{p}$ with respect to B.

This operator was introduced and studied by Parthasarathy [Par] in the geometric setting of the symmetric space G/K. The algebraic version is due to Vogan [V3]

Lemma 3.1.3. *The operator D does not depend on the choice of an orthonormal basis Z_i of $\mathfrak{p}$. Furthermore, it is K-invariant for the action of K on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ given as the tensor product of adjoint actions on the factors.*

Proof. Let T be an orthogonal operator on $\mathfrak{p}$ with matrix (T_{ij}) in the basis Z_i. Then

$$\sum_i TZ_i \otimes TZ_i = \sum_{i,j,k} T_{ji} Z_j \otimes T_{ki} Z_k = \sum_{j,k} (\sum_i T_{ji} T_{ki}) Z_j \otimes Z_k.$$

Since T is orthogonal, $\sum_i T_{ji} T_{ki} = \delta_{jk}$, and the above sum is equal to D.

This immediately implies both claims of the lemma, since any orthonormal basis of $\mathfrak{p}$ can be obtained from Z_i via an orthogonal transformation, and since operators $\operatorname{Ad}(k)$, $k \in K$ are orthogonal on $\mathfrak{p}$.

The main reason why the Dirac operator D is useful is the fact that its square is nice and simple. In order to formulate the result, we need some more notation.

3.1.4. Diagonal embedding of $\mathfrak{k}$ into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Recall from 2.3.3 that there is a homomorphism of Lie algebras

$$\alpha : \mathfrak{k} \to C(\mathfrak{p}),$$

given by the formula

$$\alpha(X) = \frac{1}{2} \sum_{i<j} B(X, [Z_i, Z_j]) Z_i Z_j, \qquad X \in \mathfrak{k}. \tag{3.1}$$

Here Z_i is an orthonormal basis of $\mathfrak{p}$ with respect to B.

Using α we can embed the Lie algebra $\mathfrak{k}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \longmapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{k})$; moreover it is still one-to-one on the level of $U(\mathfrak{k})$:

Lemma 3.1.5. *The map $\alpha : U(\mathfrak{k}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined above is one to one.*

Proof. We define a filtration of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ as the usual filtration by degree of $U(\mathfrak{g})$ from 1.4.3, tensored with the trivial filtration of $C(\mathfrak{p})$, i.e.,

$$F_p(U(\mathfrak{g}) \otimes C(\mathfrak{p})) = (F_p U(\mathfrak{g})) \otimes C(\mathfrak{p}).$$

This filtration will play a major role in Section 3.3. If we equip $U(\mathfrak{k})$ with an analogous filtration by degree, we see that α is compatible with filtrations; in fact, if $u \in U(\mathfrak{k})$ is a PBW monomial relative to some ordered basis of $\mathfrak{k}$, then $\alpha(u)$ is equal to $u \otimes 1$ up to terms of lower filtration degree. This immediately implies the statement of the lemma.

We denote the image of $\mathfrak{k}$ by $\mathfrak{k}_\Delta$, and then the image of $U(\mathfrak{k})$ is the enveloping algebra $U(\mathfrak{k}_\Delta)$ of $\mathfrak{k}_\Delta$. In particular, the image of the center $Z(\mathfrak{k})$ of $U(\mathfrak{k})$ is the center $Z(\mathfrak{k}_\Delta)$ of $U(\mathfrak{k}_\Delta)$.

We denote by $\Omega_\mathfrak{g}$ and $\Omega_\mathfrak{k}$ the Casimir elements for $\mathfrak{g}$ respectively $\mathfrak{k}$. The image of $\Omega_\mathfrak{k}$ under Δ is denoted by $\Omega_{\mathfrak{k}_\Delta}$; this is the Casimir element for $\mathfrak{k}_\Delta$.

We denote by $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ the fundamental Cartan subalgebra of $\mathfrak{g}$, i.e., $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, while $\mathfrak{a} \subset \mathfrak{p}$ (see 2.3.4 and 2.3.6). As usual, we denote by $\rho_\mathfrak{g}$ the half sum of positive roots for $(\mathfrak{g}, \mathfrak{h})$. Of course, $\rho_\mathfrak{g}$ depends on the choice of a positive root system, but its norm $||\rho_\mathfrak{g}||$ (with respect to the usual inner product on $\mathfrak{h}^*$, induced by the Killing form) does not. Analogously, $\rho_\mathfrak{k}$ denotes the half sum of positive roots for $(\mathfrak{k}, \mathfrak{t})$, and $||\rho_\mathfrak{k}||$ is independent of the choice of positive roots.

The following formula for D^2 was first obtained in [Par], Section 3. We adopt the approach of [W], 9.2.7 and 9.3.

Proposition 3.1.6. *The square of the Dirac operator D is given by*

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C1 \otimes 1,$$

where C is the constant $||\rho_\mathfrak{k}||^2 - ||\rho_\mathfrak{g}||^2$.

Proof. Let Z_i be an orthonormal basis of $\mathfrak{p}$ with respect to B and let W_k be on orthonormal basis of $\mathfrak{k}$ with respect to $-B$. Then using the relations $Z_i Z_j = -Z_j Z_i$, $i \neq j$, and $Z_i^2 = -1$ is $C(\mathfrak{p})$, we see

$$\begin{aligned} D^2 &= (\sum_i Z_i \otimes Z_i)(\sum_j Z_j \otimes Z_j) = \sum_{i,j} Z_i Z_j \otimes Z_i Z_j \\ &= \sum_i Z_i^2 \otimes Z_i^2 + \sum_{i<j} (Z_i Z_j - Z_j Z_i) \otimes Z_i Z_j \\ &= -\sum_i Z_i^2 \otimes 1 + \sum_{i<j} [Z_i, Z_j] \otimes Z_i Z_j. \end{aligned}$$

On the other hand,

$$\Omega_\mathfrak{g} \otimes 1 = -\sum_k W_k^2 \otimes 1 + \sum_i Z_i^2 \otimes 1,$$

and

$$\begin{aligned} \Omega_{\mathfrak{k}_\Delta} &= -\sum_k (W_k \otimes 1 + 1 \otimes \alpha(W_k))^2 \\ &= -\sum_k W_k^2 \otimes 1 - 2\sum_k W_k \otimes \alpha(W_k) - \sum_k 1 \otimes \alpha(W_k)^2. \end{aligned}$$

So we see it suffices to prove

$$\sum_{i<j}[Z_i, Z_j] \otimes Z_i Z_j = -2 \sum_k W_k \otimes \alpha(W_k) \tag{3.2}$$

and

$$\sum_k \alpha(W_k)^2 = ||\rho_{\mathfrak{k}}||^2 - ||\rho_{\mathfrak{g}}||^2. \tag{3.3}$$

The equation (3.3) was proved in Proposition 2.3.7. To prove (3.2), we use (3.1) to write

$$\begin{aligned}-2\sum_k W_k \otimes \alpha(W_k) &= -\sum_k \sum_{i<j} B(W_k, [Z_i, Z_j])W_k \otimes Z_i Z_j \\ &= \sum_{i<j}[Z_i, Z_j] \otimes Z_i Z_j;\end{aligned}$$

namely, as $[Z_i, Z_j] \in \mathfrak{k}$, $[Z_i, Z_j] = -\sum_k B(W_k, [Z_i, Z_j])W_k$. This finishes the proof of the proposition.

3.2 Dirac cohomology and Vogan's conjecture

As in the previous section, G is a connected real reductive group with a maximal compact subgroup K, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the corresponding Cartan decomposition of the Lie algebra of G, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the complexified Cartan decomposition.

Let X be a $(\mathfrak{g}, K)$-module. We would like to get an action of the Dirac operator D, but in order for the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ to act, we need to replace X by $X \otimes S$, where S is the spin module for $C(\mathfrak{p})$. It is clear that $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ acts on $X \otimes S$; the definition of the action is

$$(u \otimes a)(x \otimes s) = ux \otimes as,$$

for $u \in U(\mathfrak{g})$, $a \in C(\mathfrak{p})$, $x \in X$ and $s \in S$. The group K however does not act on S. We do have a map $K \to SO(\mathfrak{p}_0)$ defined by the adjoint action, however it is not the group $SO(\mathfrak{p}_0)$ that acts on S, but rather its double cover, the group $\operatorname{Spin}(\mathfrak{p}_0)$. What we need is a corresponding double cover of K:

3.2.1. The spin double cover of K. We define the *spin double cover* $\tilde{K}$ of K by the following pullback diagram:

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & \operatorname{Spin}(\mathfrak{p}_0) \\ \downarrow & & \downarrow p \\ K & \xrightarrow{\operatorname{Ad}} & SO(\mathfrak{p}_0) \end{array}$$

In other words, $\tilde{K}$ is the subgroup of $K \times \operatorname{Spin}(\mathfrak{p}_0)$ consisting of all pairs (k, s) such that $\operatorname{Ad}(k) = p(s)$, where $\operatorname{Ad} : K \to SO(\mathfrak{p}_0)$ is defined by the adjoint action, and $p :$ Spin $(\mathfrak{p}_0) \to SO(\mathfrak{p}_0)$ is the double covering map of 2.1.14. The arrows from $\tilde{K}$ are the restrictions of the projections from $K \times$ Spin $(\mathfrak{p}_0)$ onto each the factors.

It is a standard fact that in this situation $\tilde{K} \to K$ is a double covering; it may be split or not; for example, if K is simply connected then the covering must be split.

It is now clear that we can make $\tilde{K}$ act on $X \otimes S$: the action on X is through K while the action on S is through Spin $(\mathfrak{p}_0)$.

Moreover, we can define an action of $\tilde{K}$ on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ using the adjoint action of K. The differential of this action is the adjoint action of the Lie algebra $\mathfrak{k}_0$ of $\tilde{K}$ on both factors $U(\mathfrak{g})$ and $C(\mathfrak{p})$. This last action can be viewed as the action of the diagonal copy $\mathfrak{k}_{0\Delta}$ of $\mathfrak{k}_0$ by commutators in the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Thus we can define the notion of a $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \tilde{K})$-module: it is a module for $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ and for $\tilde{K}$, such that the differential of the $\tilde{K}$-action coincides with the action of the diagonal algebra $\mathfrak{k}_{0\Delta}$, and such that the action of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is $\tilde{K}$-equivariant. Then we have the following simple fact:

Lemma 3.2.2. *If X is a $(\mathfrak{g}, K)$ module and if S is a spin module for $C(\mathfrak{p})$, then $X \otimes S$ is a $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \tilde{K})$-module.*

Proof. The differentiated version of the pullback diagram defining $\tilde{K}$ is

$$\begin{array}{ccc} \mathfrak{k}_0 & \longrightarrow & \mathfrak{so}(\mathfrak{p}_0) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \mathfrak{k}_0 & \xrightarrow{\mathrm{ad}} & \mathfrak{so}(\mathfrak{p}_0). \end{array}$$

It follows that the spin action of $\mathfrak{k}_0$ on S is given exactly by the map $\alpha : \mathfrak{k}_0 \to C(\mathfrak{p})$ from 3.1.4. This implies that the differential of the $\tilde{K}$-action coincides with the action of the diagonal algebra $\mathfrak{k}_{0\Delta}$, and it is obvious that the action of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is $\tilde{K}$-equivariant.

In particular, the Dirac operator D acts on $X \otimes S$, and D commutes with the action of $\tilde{K}$ by Lemma 3.1.3.

Definition 3.2.3. The *Dirac cohomology* of a $(\mathfrak{g}, K)$-module X is the $\tilde{K}$-module

$$H_D(X) = \text{Ker } (D) \big/ \text{Im } (D) \cap \text{Ker } (D),$$

where D is considered as an operator on $X \otimes S$.

Another way to define Dirac cohomology is as follows. Consider the $\tilde{K}$-submodule Ker (D^2) of $X \otimes S$. On this space, D defines a differential, and the Dirac cohomology of X is the cohomology of this differential. By Proposition 3.1.6, if X has infinitesimal character, then the subspace Ker (D^2) of $X \otimes S$

consists of the $\tilde{K}$-isotypic components with value of the Casimir element $\Omega_{\mathfrak{k}_\Delta}$ equal to a fixed constant. Since the Casimir element for $\mathfrak{k}$ has value $||\mu+\rho_{\mathfrak{k}}||^2-||\rho_{\mathfrak{k}}||^2$ on the $\tilde{K}$-type with highest weight μ, and since the possible μ form a lattice in $\mathfrak{t}^*$, it follows that Ker (D^2) is finite-dimensional for any admissible X with infinitesimal character. In particular, if X is irreducible, then the Dirac cohomology of X is finite-dimensional.

Remark 3.2.4. Assume that X is a unitary $(\mathfrak{g},K)$-module, and let $\langle\, ,\rangle_X$ be the corresponding inner product. On the other hand, there is an inner product $\langle\, ,\rangle_S$ on S, such that all $Z\in\mathfrak{p}_0$ are skew Hermitian with respect to $\langle\, ,\rangle_S$. To construct $\langle\, ,\rangle_S$, recall the setting of 2.2.2: U and U^* are maximal isotropic subspaces of $V=\mathfrak{p}$, u_i and u_i^* are dual bases of U respectively U^*, and we can assume that u_i and u_i^* are obtained from an orthonormal basis Z_i of $\mathfrak{p}_0$ as in 2.2.2. Now $\langle\, ,\rangle_S$ is the inner product such that the elements

$$\frac{1}{\sqrt{2}^k}u_{i_1}\wedge\cdots\wedge u_{i_k},\qquad 1\leq i_1<\cdots<i_k\leq\dim U$$

form an orthonormal basis of S. Equivalently, we define $\langle u_i,u_j\rangle=2\delta_{ij}$ and then extend this inner product to all of $S=\bigwedge(U)$ in the usual way, using the determinant.

By an easy calculation, the adjoint of the operator u_i on S (i.e., the left multiplication by u_i) is the operator $-u_i^*$. It thus follows that the Z_j are skew Hemitian, since they can be expressed from the u_j and u_j^* as

$$Z_{2j-1}=\frac{1}{\sqrt{2}}(u_j+u_j^*),\qquad Z_{2j}=-\frac{i}{\sqrt{2}}(u_j-u_j^*).$$

In case $\mathfrak{p}$ is odd dimensional, the extra basis element Z has eigenvalues $\pm i$, hence it is also skew Hermitian.

Let us denote by $\langle\, ,\rangle$ the tensor product of the two inner products $\langle\, ,\rangle_X$ and $\langle\, ,\rangle_S$. Then $\langle\, ,\rangle$ is an inner product on $X\otimes S$, and

$$\langle x\otimes s,x'\otimes s'\rangle=\langle x,x'\rangle_X\langle s,s'\rangle_S$$

for all $x,x'\in X$ and $s,s'\in S$. Since our orthonormal basis Z_i of $\mathfrak{p}$ can be chosen to be in $\mathfrak{p}_0$, it follows that each Z_i is skew Hermitian with respect to $\langle\, ,\rangle_X$ and with respect to $\langle\, ,\rangle_S$, so every $Z_i\otimes Z_i$ is Hermitian with respect to $\langle\, ,\rangle$. Hence D is Hermitian (i.e., self-adjoint) with respect to $\langle\, ,\rangle$.

Similarly, if X is finite-dimensional, we can consider the so called admissible inner product $\langle\, ,\rangle_X$ on X; with respect to this inner product, all elements from $\mathfrak{k}_0$ are skew Hermitian, while all elements of $\mathfrak{p}_0$ are Hermitian. (This is the inner product giving unitarity of X with respect to the compact form $\mathfrak{k}_0\oplus i\mathfrak{p}_0$ of $\mathfrak{g}$.) It follows that D is skew Hermitian in this case.

In both these cases, Ker (D) and Im (D) intersect trivially, and the Dirac cohomology of X is simply Ker $(D)=$ Ker (D^2).

To formulate Vogan's conjecture from [V3], note that if $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$, then we can view $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$: if μ is a functional on $\mathfrak{t}$, we extend it to $\mathfrak{h}$ by setting $\mu\big|_{\mathfrak{a}} = 0$. Moreover, we fix a choice of positive roots for $(\mathfrak{k}, \mathfrak{t})$.

Theorem 3.2.5 (Vogan's conjecture). *Let X be an irreducible $(\mathfrak{g}, K)$-module. Assume that the Dirac cohomology of X contains a $\tilde{K}$-type E_μ of highest weight $\mu \in \mathfrak{t}^* \subset \mathfrak{h}^*$. Then the infinitesimal character of X is $\Lambda = \mu + \rho_\mathfrak{k}$.*

Note that the infinitesimal character of E_μ is exactly $\mu + \rho_\mathfrak{k}$, so the theorem says that the $\mathfrak{k}$-infinitesimal character of $H_D(X)$ is the same as the $\mathfrak{g}$-infinitesimal character of X under the identification $\mathfrak{t}^* \subset \mathfrak{h}^*$ explained above.

For unitary modules X, the Dirac cohomology is just the kernel of D on $X \otimes S$, so it follows

Corollary 3.2.6 (Vogan's conjecture). *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module. Assume that the kernel of the Dirac operator D on $X \otimes S$ contains a $\tilde{K}$-type E_μ of highest weight $\mu \in \mathfrak{t}^* \subset \mathfrak{h}^*$. Then the infinitesimal character of X is $\Lambda = \mu + \rho_\mathfrak{k}$.*

Vogan has shown in [V3] how Theorem 3.2.5 follows from the following two results that he conjectured. The first result is about the structure of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Theorem 3.2.7 (Vogan's conjecture). *For any $z \in Z(\mathfrak{g})$, there is a unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$ and there are some $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ such that*

$$z \otimes 1 = \zeta(z) + Da + bD.$$

The second result complements Theorem 3.2.7 by describing $\zeta(z)$ explicitly.

Theorem 3.2.8 (Vogan's conjecture). *The map $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta) \cong Z(\mathfrak{k})$ is a homomorphism of algebras, and it fits into the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{k}) \\ \downarrow & & \downarrow \\ S(\mathfrak{h})^W & \xrightarrow{\mathrm{Res}} & S(\mathfrak{t})^{W_K} \end{array}$$

Here the vertical arrows are the Harish-Chandra isomorphisms, and the map Res *corresponds to the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$ under the identifications $S(\mathfrak{h})^W = P(\mathfrak{h}^*)^W$ and $S(\mathfrak{t})^{W_K} = P(\mathfrak{t}^*)^{W_K}$. As before, we view $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$ by extending functionals from $\mathfrak{t}$ to $\mathfrak{h}$, letting them act by 0 on $\mathfrak{a}$. Finally, W and W_K are the Weyl groups of $(\mathfrak{g}, \mathfrak{h})$ respectively $(\mathfrak{k}, \mathfrak{t})$.*

Proof that Theorem 3.2.7 and Theorem 3.2.8 imply Theorem 3.2.5. Let $x \in (X \otimes S)(\gamma)$ be nonzero, in Ker (D), but not in Im (D), where γ is some $\tilde{K}$-type. Then $z \otimes 1$ acts on x by the scalar $\Lambda(z)$, where Λ is the infinitesimal character of X. On the other hand, since x is of $\tilde{K}$-type γ, $\zeta(z)$ acts on x by the scalar $(\gamma + \rho_{\mathfrak{k}})(\zeta(z))$ (that is, by the $\mathfrak{k}$-infinitesimal character of γ applied to $\zeta(z)$).

Since by Theorem 3.2.7, $(z \otimes 1 - \zeta(z))x = Dax + bDx = Dax$, and $x \notin$ Im (D), it follows that $(z \otimes 1 - \zeta(z))x = 0$. Hence $\Lambda(z) = (\gamma + \rho_{\mathfrak{k}})(\zeta(z))$. In view of Theorem 3.2.8, this means precisely that Λ is the extension of $\gamma + \rho_{\mathfrak{k}}$ to $\mathfrak{h}$, given as 0 on $\mathfrak{a}$.

It therefore remains to prove Theorem 3.2.7 and Theorem 3.2.8. This will occupy the next two sections.

3.3 A differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$

Since the algebra $C(\mathfrak{p})$ has a $\mathbb{Z}_2$-grading (see 2.1.3), the same is true for the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, if we proclaim elements of $U(\mathfrak{g})$ to be all even. In other words, $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is an associative *superalgebra*. The grading is obviously preserved by the adjoint action of K. For each homogeneous (i.e., even or odd) element $a \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, we will denote by ϵ_a its sign with respect to the $\mathbb{Z}_2$-grading. In other words, $\epsilon_a = 1$ if a is even and $\epsilon_a = -1$ if a is odd.

We denote by d the operator from $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ to itself given by *supercommuting* with the Dirac operator D:

$$d(a) = Da - \epsilon_a aD, \qquad a \in U(\mathfrak{g}) \otimes C(\mathfrak{p}) \text{ homogeneous.}$$

Since D is of degree 1, we see that for any homogeneous $a \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$,

$$\begin{aligned} d^2(a) &= d(Da - \epsilon_a aD) \\ &= D^2a - \epsilon_{Da} DaD - \epsilon_a \left(DaD - \epsilon_{aD} aD^2\right) \\ &= D^2a - aD^2. \end{aligned}$$

So $d^2a = 0$ if a is in the centralizer of D^2 in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Lemma 3.3.1. *The operator d on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is K-equivariant and defines a differential on the algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. Moreover, d is odd, i.e., if a is even, $d(a)$ is odd, and if a is odd, $d(a)$ is even.*

Proof. Since D is K-invariant by Lemma 3.1.3, it follows that

$$\begin{aligned} d(\operatorname{Ad}(k)a) &= D \operatorname{Ad}(k)a - \epsilon_{\operatorname{Ad}(k)a} \operatorname{Ad}(k)aD \\ &= \operatorname{Ad}(k)(Da - \epsilon_a aD) = \operatorname{Ad}(k)d(a) \end{aligned}$$

for any homogeneous $a \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$, so d is K-equivariant.

By Proposition 3.1.6, $D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C1 \otimes 1$, where C is a constant. Clearly, $-\Omega_{\mathfrak{g}} \otimes 1 + C1 \otimes 1$ is central in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. So commuting with D^2 is equivalent to commuting with $\Omega_{\mathfrak{k}_\Delta}$. Since all elements in $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ commute with the Lie algebra $\mathfrak{k}_\Delta$ of K, they commute with D^2 and so $d^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. The last statement is obvious since D is clearly odd.

The main result of [HP1] is the following theorem.

Theorem 3.3.2. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ defined by supercommuting with the Dirac operator D as above. Then*

$$\operatorname{Ker} d = Z(\mathfrak{k}_\Delta) \oplus \operatorname{Im} d.$$

In particular, the cohomology of d is isomorphic to $Z(\mathfrak{k}_\Delta)$.

The proof of Theorem 3.3.2 will occupy the rest of this section. Before proceeding, let us show that the main claim of Vogan's conjecture, Theorem 3.2.7 now follows.

3.3.3. Proof that Theorem 3.3.2 implies Theorem 3.2.7. Since $Z(\mathfrak{g}) \otimes 1$ is central in $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, in particular it commutes with the Dirac operator D. Also, $Z(\mathfrak{g}) \otimes 1$ is even with respect to the $\mathbb{Z}_2$-grading, and so it is in the kernel of d. So for every $z \in Z(\mathfrak{g})$, Theorem 3.3.2 implies that we can write

$$z \otimes 1 = \zeta(z) + d(a),$$

for some odd $a \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. Since $d(a) = Da + aD$, we conclude that the statement of Theorem 3.2.7 holds with $b = a$.

We were assuming here that the elements of $Z(\mathfrak{g})$ are all K-invariant. This is certainly true if K is connected, but it is also true under milder assumptions, for example if the group G is in the so-called Harish-Chandra class.

The main idea for the proof of Theorem 3.3.2 is a standard one: we introduce a filtration of the algebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, consider the associated graded algebra, prove an analogue of the theorem in the graded setting, and then come back to the filtered setting by an induction on degree. (We will be lucky in that we will not need a spectral sequence to come back.)

We consider the filtration of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, which we already used in the proof of Lemma 3.1.5: the standard filtration of $U(\mathfrak{g})$ by degree, tensored with the trivial filtration of $C(\mathfrak{p})$. The associated graded algebra associated to this filtration is $S(\mathfrak{g}) \otimes C(\mathfrak{p})$. Moreover, the filtration is K-invariant, so it also defines a filtration of $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, by

$$F_p((U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K) = (F_p(U(\mathfrak{g}) \otimes C(\mathfrak{p})))^K,$$

with associated graded algebra $(S(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. It is also clear that this filtration is compatible with the $\mathbb{Z}_2$-grading.

Since obviously the Dirac operator D is in $F_1(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, the differential d raises the filtration degree by 1. We denote the corresponding graded differential by $\bar{d}$. Then $\bar{d}$ maps each $\operatorname{Gr}_p(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K = (S^p(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ into $\operatorname{Gr}_{p+1}(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, and it is given on $\operatorname{Gr}_p(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ by

$$\bar{d}(\bar{a}) = \bar{D}\bar{a} - \epsilon_a \bar{a}\bar{D}, \tag{3.4}$$

where we denote by $\bar{a}$ the image of $a \in F_p(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ in $\operatorname{Gr}_p(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ and by $\bar{D}$ the image of D in $\operatorname{Gr}_1(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.

Note that d is actually defined on all of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, although it is not a differential there, and it still raises the filtration degree by one. Then $\bar{d} : \operatorname{Gr}_p U(\mathfrak{g}) \otimes C(\mathfrak{p}) \to \operatorname{Gr}_{p+1}(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is also defined by (3.4).

Lemma 3.3.4. *Upon decomposing $S(\mathfrak{g})$ as $S(\mathfrak{k}) \otimes S(\mathfrak{p})$, and identifying $C(\mathfrak{p})$ with $\bigwedge(\mathfrak{p})$ via the Chevalley map of 2.1.8, the operator $\bar{d} : S(\mathfrak{g}) \otimes C(\mathfrak{p}) \to S(\mathfrak{g}) \otimes C(\mathfrak{p})$ is equal to*

$$(-2) \operatorname{id} \otimes d_{\mathfrak{p}} : S(\mathfrak{k}) \otimes (S(\mathfrak{p}) \otimes \textstyle\bigwedge(\mathfrak{p})) \to S(\mathfrak{k}) \otimes (S(\mathfrak{p}) \otimes \bigwedge(\mathfrak{p})).$$

Here $d_{\mathfrak{p}} : S(\mathfrak{p}) \otimes \bigwedge(\mathfrak{p}) \to S(\mathfrak{p}) \otimes \bigwedge(\mathfrak{p})$ denotes the Koszul differential, given by

$$d_p(s \otimes X_1 \wedge \dots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i-1} s X_i \otimes X_1 \wedge \dots \widehat{X}_i \wedge \dots \wedge X_k \tag{3.5}$$

for $s \in S(\mathfrak{p})$ and $X_1, \dots, X_k \in \mathfrak{p}$.

Proof. Let Z_i be an orthonormal basis of $\mathfrak{p}$. If $\bar{a}$ is an element of $S(\mathfrak{g}) \otimes C(\mathfrak{p})$ of the form $\bar{a} = s \otimes Z_{i_1} \dots Z_{i_k}$, then $\bar{d}(\bar{a}) = \bar{D}\bar{a} - (-1)^k \bar{a}\bar{D}$ is

$$\bar{d}(\bar{a}) = \sum_i \left(Z_i s \otimes Z_i Z_{i_1} \dots Z_{i_k} - (-1)^k s Z_i \otimes Z_{i_1} \dots Z_{i_k} Z_i \right). \tag{3.6}$$

There are two kinds of summands in this sum: i can be equal to one of $i_1, \dots, i_k$, or can be different from all of them. If i is different from all i_j, then $Z_{i_1} \dots Z_{i_k} Z_i = (-1)^k Z_i Z_{i_1} \dots Z_{i_k}$ and hence the i-th summand in (3.6) is 0. On the other hand, for $i = i_j$,

$$Z_i Z_{i_1} \dots Z_{i_k} = (-1)^{j-1} Z_{i_1} \dots (Z_{i_j})^2 \dots Z_{i_k} = (-1)^j Z_{i_1} \dots \widehat{Z_{i_j}} \dots Z_{i_k},$$

and similarly $Z_{i_1} \dots Z_{i_k} Z_i = (-1)^{k-j-1} Z_{i_1} \dots \widehat{Z_{i_j}} \dots Z_{i_k}$. So the i_j-th summand of (3.6) is

$$\begin{aligned} &\left((-1)^j - (-1)^k (-1)^{k-j-1}\right) s Z_{i_j} \otimes Z_{i_1} \dots \widehat{Z_{i_j}} \dots Z_{i_k} \\ &\qquad = -2(-1)^{j-1} s Z_{i_j} \otimes Z_{i_1} \dots \widehat{Z_{i_j}} \dots Z_{i_k}. \end{aligned}$$

This proves the lemma, since the Chevalley map identifies $Z_{j_1} \dots Z_{j_r}$ in $C(\mathfrak{p})$ with $Z_{j_1} \wedge \dots \wedge Z_{j_r}$ in $\bigwedge(\mathfrak{p})$.

The following facts about Koszul differentials are very well known. We will however prove them for convenience of the reader. The proof we present is slightly more elegant than the usual ones because it replaces computations with a little of superalgebra language. Along the way, we will introduce some notions from the superalgebra language. We took this proof from [GS].

Proposition 3.3.5. *Let V be a vector space and let the Koszul differential d_V on $S(V)\otimes\bigwedge(V)$ be defined by a formula analogous to (3.5). Then d_V is a differential, i.e., $d_V^2=0$. Moreover,*

$$\mathrm{Ker}\, d_V = \mathbb{C}1\otimes 1 \oplus \ \mathrm{Im}\, d_V.$$

In particular, the cohomology of d_V is isomorphic to $\mathbb{C}1\otimes 1$.

Proof. Consider the $\mathbb{Z}_2$-grading of the algebra $S(V)\otimes\bigwedge(V)$ induced by the grading of $\bigwedge(V)$. It is clear that $S(V)\otimes\bigwedge(V)$ is generated by $V\otimes 1\oplus 1\otimes V$, with the only relations (except for linearity in each variable) being the commutation relations: elements of $V\otimes 1$ commute with each other and with elements of $1\otimes V$, while the elements of $1\otimes V$ anticommute with each other.

An operator L on $S(V)\otimes\bigwedge(V)$ is called even if it preserves the $\mathbb{Z}_2$-grading and odd if it maps even elements to odd and vice versa. Furthermore, L is called an even (respectively odd) derivation if it is even (respectively odd) and

$$L(xy)=(Lx)y+(-1)^{\deg L\deg x}xL(y),$$

for any homogeneous $x,y\in S(V)\otimes\bigwedge(V)$. Here $\deg L$ is 0 if L is even and 1 if L is odd. As usual, any derivation annihilates the unity $1\otimes 1$.

It is clear that an even derivation is uniquely determined on the generators, where it can be defined by any two linear maps $V\otimes 1\to V\otimes 1$ and $1\otimes V\to 1\otimes V$. Likewise, an odd derivation is uniquely determined on the generators, where it can be defined by any two linear maps interchanging $V\otimes 1$ and $1\otimes V$.

It is now clear from the definition that d_V is an odd derivation defined on the generators by

$$d_V(v\otimes 1)=0,\qquad d_V(1\otimes v)=v\otimes 1,\qquad\qquad v\in V.$$

Since the even derivations d_V^2 and 0 obviously agree on the generators, it follows $d_V^2=0$.

To check the other claim of the proposition, define an odd derivation h on $S(V)\otimes\bigwedge(V)$ by setting

$$h(v\otimes 1)=1\otimes v,\qquad h(1\otimes v)=0,\qquad\qquad v\in V.$$

Note that $h^2=0$; in fact, upon identifying $S(V)$ with polynomials on V^*, h is exactly the de Rham differential on differential forms on V^* with polynomial coefficients.

Finally, let us note that the (total) degree operator $\deg$ on $S(V)\otimes\bigwedge(V)$, which multiplies any monomial by its degree, is an even derivation defined by

$$\deg\,(v \otimes 1) = v \otimes 1, \qquad \deg\,(1 \otimes v) = 1 \otimes v, \qquad\qquad v \in V.$$

Then it is obvious that the even derivations $hd_V + d_V h$ and $\deg$ agree on the generators, hence

$$hd_V + d_V h = \deg\,.$$

In other words, h is a homotopy of $\deg$ and 0. In particular, if $a \in S(V) \otimes \bigwedge(V)$ is homogeneous of degree different from 0, then $d_V a = 0$ implies $a = \frac{1}{\deg a} d_V h(a) \in \operatorname{Im} d_V$. Moreover, d_V annihilates $\mathbb{C}1 \otimes 1$, and $\mathbb{C}1 \otimes 1$ can not be in $\operatorname{Im} d_V$ since d_V maps any homogeneous element either into an element of the same (total) degree, or into 0.

Note that one can use the same calculation to prove the analogue of the proposition for the de Rham differential h; now d_V is a homotopy of h and 0, and it follows that the cohomology of of h is also $\mathbb{C}1 \otimes 1$.

Getting back to our d given as supercommuting with the Dirac operator D, we see that in fact $\bar{d}$ defines a differential on the whole algebra $\operatorname{Gr} U(\mathfrak{g}) \otimes C(\mathfrak{p}) = S(\mathfrak{g}) \otimes C(\mathfrak{p})$, even before passing to K-invariants, and that

$$\operatorname{Ker} \bar{d} = S(\mathfrak{k}) \otimes 1 \otimes 1 \,\oplus\, \operatorname{Im} \bar{d} \quad \subset S(\mathfrak{k}) \otimes S(\mathfrak{p}) \otimes C(\mathfrak{p}). \tag{3.7}$$

Since $\bar{d}$ is K-equivariant, the kernel of $\bar{d}$ on $(S(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is the same as the K-invariants in the kernel of $\bar{d}$ on $S(\mathfrak{g}) \otimes C(\mathfrak{p})$. Analogously, the image of $\bar{d}$ on $(S(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is the same as the K-invariants in the image of $\bar{d}$ on $S(\mathfrak{g}) \otimes C(\mathfrak{p})$. Using (3.7), we can therefore conclude:

Lemma 3.3.6. *For the above defined differential $\bar{d}$ on $(S(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, we have*

$$\operatorname{Ker} \bar{d} = S(\mathfrak{k})^K \otimes 1 \otimes 1 \,\oplus\, \operatorname{Im} \bar{d}.$$

In particular, the cohomology of $\bar{d}$ on $(S(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ is isomorphic to $S(\mathfrak{k})^K \otimes 1 \otimes 1$.

3.3.7. Proof of Theorem 3.3.2. It is clear that $Z(\mathfrak{k}_\Delta)$ is contained in $\operatorname{Ker} d$, since it is even, and commutes with the Dirac operator D; D is K-invariant and thus commutes with $\mathfrak{k}_\Delta$. Moreover, $\operatorname{Im} d \subset \operatorname{Ker} d$ because d is a differential. Also, the sum $Z(\mathfrak{k}_\Delta) + \operatorname{Im} d$ is direct, because its graded version in Lemma 3.3.6 is direct. (Namely, the top term of any element of $U(\mathfrak{k}_\Delta)$ with respect to our filtration by degree is in $S(\mathfrak{k})^K \otimes 1$, and the top term of any element of $\operatorname{Im} d$ is in $\operatorname{Im} \bar{d}$.)

It remains to prove that $a \in \operatorname{Ker} d$ implies $a \in Z(\mathfrak{k}_\Delta) + \operatorname{Im} d$. We will prove this by induction on the filtration degree of a. If a is of degree -1, then $a = 0$ and there is nothing to prove. Assume that a is of degree n and that the statement holds for all a of degree $n-1$. Since $d(a) = 0$, it follows that $\bar{d}(\bar{a}) = 0$, where $\bar{a}$ denotes the image of a in $\operatorname{Gr}_n(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. Thus Lemma 3.3.6 implies that

$$\bar{a} = s \otimes 1 + \bar{d}\,\bar{b},$$

for some $s \in S(\mathfrak{k})^K$ and some $\bar{b} \in \operatorname{Gr}_{n-1}(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.

Note that by Lemma 3.1.5, there is a unique $z \in Z(\mathfrak{k}_\Delta)$ such that $s \otimes 1 = \bar{z}$. Moreover, let $b \in F_{n-1}(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ be any representative of $\bar{b}$. (For example, one can take b to be the symmetrization of $\bar{b}$.) Then

$$\overline{a - z - db} = \bar{a} - \bar{z} - \bar{d}\,\bar{b} = 0,$$

so that $a - z - db \in F_{n-1}(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. Moreover,

$$d(a - z - db) = da - dz - d^2 b = 0;$$

namely, $da = 0$ by assumption, $dz = 0$ since $Z(\mathfrak{k}_\Delta) \subset \operatorname{Ker} d$ as remarked above, and $d^2 b = 0$ since d is a differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ by Lemma 3.3.1.

So the induction hypothesis implies that $a - z - db = z' + dc$ for some $z' \in Z(\mathfrak{k}_\Delta)$ and $c \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, hence

$$a = (z + z') + d(b + c) \in Z(\mathfrak{k}_\Delta) + \operatorname{Im} d.$$

This finishes the proof of Theorem 3.3.2.

3.4 The homomorphism ζ

In this section we want to determine the map $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta) \cong Z(\mathfrak{k})$ more explicitly, i.e., prove Theorem 3.2.8. For this, we need a large enough collection of representations for which we know the infinitesimal character and a $\tilde{K}$-type in the Dirac cohomology. We will use finite-dimensional representations of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{t}^*$, i.e., λ restricts to 0 on $\mathfrak{a}$.

We start by proving that ζ is a homomorphism of algebras. For this we need a simple lemma:

Lemma 3.4.1. *The differential d on $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ introduced in the previous section is an odd derivation of the superalgebra $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$.*

Proof. We need to show that

$$d(xy) = d(x)y + \epsilon_x x d(y),$$

for any two homogeneous elements x and y of $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$. This is a straightforward calculation:

$$d(x)y + \epsilon_x x d(y) = (Dx - \epsilon_x x D)y + \epsilon_x x(Dy - \epsilon_y y D) = Dxy - \epsilon_x \epsilon_y xyD = d(xy).$$

Proposition 3.4.2. *The map $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta)$ from Theorem 3.2.7 is an algebra homomorphism.*

Proof. Let $z, z' \in Z(\mathfrak{g})$. Then by Theorem 3.3.2, one can choose odd $a, a' \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$ such that

$$z \otimes 1 = \zeta(z) + d(a), \qquad z' \otimes 1 = \zeta(z') + d(a').$$

Multiplying these two equations, we get

$$zz' \otimes 1 = \zeta(z)\zeta(z') + \zeta(z)d(a') + d(a)\zeta(z') + d(a)d(a').$$

By Lemma 3.4.1, taking into account that $d(\zeta(z)) = d(\zeta(z')) = d^2(a') = 0$, we see that this can be rewritten as

$$zz' \otimes 1 = \zeta(z)\zeta(z') + d\left(\zeta(z)a' + a\zeta(z') + ad(a')\right).$$

Since $\zeta(zz')$ is the unique element of $Z(\mathfrak{k}_\Delta)$ such that $zz' = \zeta(zz') + d(c)$ for some $c \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^K$, we see that $\zeta(zz') = \zeta(z)\zeta(z')$. So ζ is indeed a homomorphism.

Let now $V(\lambda)$ be the irreducible finite-dimensional $(\mathfrak{g}, K)$-module with highest weight λ and assume $\lambda \in \mathfrak{t}^*$. Let $\xi \in \mathfrak{t}^*$ be a highest weight for $\mathfrak{k}$. By Proposition 3.1.6 and by the formula for the action of the Casimir element from 1.4.6,

$$D^2 = -||\lambda + \rho_{\mathfrak{g}}||^2 + ||\xi + \rho_{\mathfrak{k}}||^2 \tag{3.8}$$

on the $\tilde{K}$-isotypic component $(V(\lambda) \otimes S)(\xi)$ of type ξ.

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be the θ-stable Borel subalgebra corresponding to a $\mathfrak{g}$-regular element $H \in i\mathfrak{t}_0$, so that $\mathfrak{h}$ is the 0-eigenspace of $\operatorname{ad}(H)$ while $\mathfrak{n}$ is the sum of eigenspaces of $\operatorname{ad}(H)$ with positive eigenvalues (see 2.3.4 and 2.3.6). The element H defines compatible systems of positive roots for $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$: a root is positive if it has positive value on H. Then $\rho_{\mathfrak{g}}$, the half sum of positive roots for $(\mathfrak{g}, \mathfrak{h})$, is in $\mathfrak{t}^*$, i.e., vanishes on $\mathfrak{a}$. Namely, since $\theta H = H$, the positive root system corresponding to H is θ-stable, and hence $\theta\rho_{\mathfrak{g}} = \rho_{\mathfrak{g}}$, so $\rho_{\mathfrak{g}} \in \mathfrak{t}^*$.

Obviously, $\rho_{\mathfrak{k}}$ is also in $\mathfrak{t}^*$, and so $\xi = \lambda + \rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$ is a highest weight for $\mathfrak{k}$. Moreover, since $\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$ is a highest weight of the $\mathfrak{k}$-module S (see 2.3.6), $(V(\lambda) \otimes S)(\xi) \neq 0$. On the other hand, by (3.8), $D^2 = 0$ on $(V(\lambda) \otimes S)(\xi)$. By Remark 3.2.4, D is skew Hermitian, so this means that $(V(\lambda) \otimes S)(\xi)$ is in the Dirac cohomology. For any $z \in Z(\mathfrak{g})$ we can therefore use the fact that $z = \zeta(z)$ on Dirac cohomology (by Theorem 3.2.7) to conclude

$$(\lambda + \rho_{\mathfrak{g}})(z) = (\xi + \rho_{\mathfrak{k}})(\zeta(z)) = (\lambda + \rho_{\mathfrak{g}})(\zeta(z)); \tag{3.9}$$

namely $\xi + \rho_{\mathfrak{k}} = \lambda + \rho_{\mathfrak{g}}$ by our choice of ξ.

Let $\bar{\zeta} : P(\mathfrak{h}^*)^W \to P(\mathfrak{t}^*)^{W_K}$ be the homomorphism induced by $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k})$ under the identifications via Harish-Chandra isomorphisms. To show that $\bar{\zeta}$ is the restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$, we can alternatively prove that the corresponding morphism of algebraic varieties $\tilde{\zeta} : \mathfrak{t}^*/W_K \to \mathfrak{h}^*/W$ is the inclusion map, i.e., that $\tilde{\zeta}(\mu) = \mu$ for all $\mu \in \mathfrak{t}^*/W_K$. It is enough to check this for an algebraically dense set of μ. However, (3.9) says that this is true for all $\mu = \lambda + \rho_{\mathfrak{g}}$ where $\lambda \in \mathfrak{t}^* \subset \mathfrak{h}^*$ is a highest weight for $\mathfrak{g}$. Such λ form a lattice in $\mathfrak{t}^*$, hence an algebraically dense subset. This finishes the proof of Theorem 3.2.8.

3.5 An extended Parthasarathy's Dirac inequality

We first indicate how to check if a unitarizable $(\mathfrak{g}, K)$-module X has non-zero Dirac cohomology. If X is unitarizable, then each $Z \in \mathfrak{g}_0$ acts on X by a skew-symmetric operator. It follows from Proposition 2.3.10 that D is self-adjoint on $X \otimes S$. In this case, $\operatorname{Ker} D \cap \operatorname{Im} D = 0$ and the Dirac cohomology of X is $\operatorname{Ker} D = \operatorname{Ker} D^2$.

Proposition 3.5.1. *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Assume that $X \otimes S$ contains a $\tilde{K}$-type γ, i.e., $(X \otimes S)(\gamma) \neq 0$. Then the Dirac cohomology $\operatorname{Ker} D$ contains $(X \otimes S)(\gamma)$ if and only if $||\Lambda|| = ||\gamma + \rho_c||$.*

Proof. Note that D is self-adjoint on $X \otimes S$ and Dirac cohomology $\operatorname{Ker} D = \operatorname{Ker} D^2$. The formula of D^2 implies that D^2 acts on $(X \otimes S)(\gamma)$ by the scalar

$$-(||\Lambda||^2 - ||\rho||^2) + (||\gamma + \rho_c||^2 - ||\rho_c||^2) + (||\rho_c||^2 - ||\rho||^2) = -||\Lambda||^2 + ||\gamma + \rho_c||^2.$$

It follows that $D = 0$ on $(X \otimes S)(\gamma)$ if and only if $||\Lambda|| = ||\gamma + \rho_c||$. $\square$

We note that all irreducible unitary representations with nonzero Dirac cohomology and strongly regular infinitesimal characters were described in [HP1]. They are all $A_{\mathfrak{q}}(\lambda)$-modules (cf. Chapter 6). We remark that in above theorem the condition $||\Lambda|| = ||\gamma + \rho_c||$ is equivalent to the condition that infinitesimal character Λ is cojugate to $\gamma + \rho_c$. Thus, we obtain an extended version of the *Parthasarathy's Dirac inequality*:

Theorem 3.5.2. (Extended Parthasarathy's Dirac Inequality) *Let X be an irreducible unitarizable $(\mathfrak{g}, K)$-module with infinitesimal character Λ. If $(X \otimes S)(\gamma) \neq 0$, then*

$$||\Lambda|| \leq ||\gamma + \rho_c||.$$

The equality holds if and only if some W conjugate of Λ is equal to $\gamma + \rho_c$.

Proof. Since D is self-adjoint on $X \otimes S$, D^2 acts on $(X \otimes S)(\gamma)$ as a semi-positive definite operator. It follows that

$$-(||\Lambda||^2 - ||\rho||^2) + (||\gamma + \rho_c||^2 - ||\rho_c||^2) + (||\rho_c||^2 - ||\rho||^2) = -||\Lambda||^2 + ||\gamma + \rho_c||^2 \geq 0.$$

Therefore, one has $||\Lambda|| \leq ||\gamma + \rho_c||$. The equality $||\Lambda|| = ||\gamma + \rho_c||$ holds if and only if the Dirac cohomology $\operatorname{Ker} D$ contains $(X \otimes S)(\gamma)$. As we remarked, $X \otimes S(\gamma) \subset \operatorname{Ker} D$ is equivalent to that Λ is conjugate to $\gamma + \rho_c$. Therefore, the theorem is proved. $\square$

4

A Generalized Bott-Borel-Weil Theorem

The Borel-Weil Theorem gives a geometric realization of each irreducible representation of a compact connected semisimple Lie group G. Equivalently, this is a realization of each irreducible holomorphic representation of the complexification $G_{\mathbb{C}}$ of G. The realization is in the space of holomorphic sections of a holomorphic line bundle over the flag variety of G.

The Bott-Borel-Weil Theorem, which includes the Borel-Weil Theorem as a special case, is a statement about cohomology of holomorphic line bundles over complex homogeneous space $X = G/R$, where R is a centralizer of a torus in G. This is equivalent to the determination of how the subgroup R operates on the cohomology of a certain nilpotent Lie algebra of an arbitrary irreducible representation of G. Upon replacing the complex structure by a space S of spinors, the Bott-Borel-Weil Theorem is equivalent to determination of Dirac cohomology of the irreducible G-module V_λ with highest weight λ. Then the complex structure is eliminated and the setting can be extended to any subgroup R of maximal rank. For the purpose of constructing all irreducible finite-dimensional representations on vector bundles over G/R, it is necessary for R to be of the same rank as G. A special case is when R is the centralizer of a torus; then one gets a version of the Bott-Borel-Weil Theorem.

In this chapter we first define the cubic Dirac operator due to Kostant. Then we show that the Vogan's conjecture for symmetric pairs can be extended to the cubic Dirac operators. Using this proved conjecture, we determine the Dirac cohomology of finite dimensional representations. Our calculation is independent of Weyl character formula. As a consequence of determination of Dirac cohomology, we obtain the generalized Weyl character formula of [GKRS] and a generalized Bott-Borel-Weil theorem.

4.1 Kostant Cubic Dirac Operators

Let G be a compact semisimple Lie group and R a closed subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{r}$ be the complexifications of the Lie algebras of G and R respectively.

Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be the corresponding orthogonal decomposition with respect to the Killing form B.

4.1.1. The cubic Dirac operator We choose an orthonormal basis $Z_1, \ldots, Z_n$ of $\mathfrak{s}$ with respect to the Killing form B. Kostant [Ko2] defines his *cubic Dirac operator* to be the element

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{s}),$$

where $v \in C(\mathfrak{s})$ is the image of the fundamental 3-form $\omega \in \bigwedge^3(\mathfrak{s}^*)$,

$$\omega(X, Y, Z) = \frac{1}{2} B(X, [Y, Z])$$

under the Chevalley identification $\bigwedge(\mathfrak{s}^*) \to C(\mathfrak{s})$ and the identification of $\mathfrak{s}^*$ with $\mathfrak{s}$ by the Killing form B. Explicitly,

$$v = \frac{1}{2} \sum_{1 \leq i,j,k \leq n} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k.$$

(Note that Kostant uses an exterior product in place of the Clifford product to define v. For an orthonormal basis these are however the same.)

Kostant's cubic Dirac operator reduces to the ordinary Dirac operator when $(\mathfrak{g}, \mathfrak{r})$ is a symmetric pair, since $\omega = 0$ for the symmetric pair. Note that for the non-symmetric pair $(\mathfrak{g}, \mathfrak{r})$ the square of $\sum_{i=1}^{n} Z_i \otimes Z_i$ is not similar to what it looks like for a symmetric pair. The cubic term can be viewed as a necessary modification in the non-symmetric case, so that the associated Dirac operator has a good square as in the symmetric case. With the cubic term correction, Kostant ([Ko2], Theorem 2.16) shows that

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{r}_\Delta} + C, \tag{5.1}$$

where C is the constant $||\rho_{\mathfrak{r}}||^2 - ||\rho_{\mathfrak{g}}||^2$. This is a generalization of Proposition 3.1.6. The sign is different in [Ko2]; this change comes from the fact that Kostant uses a slightly different definition of $C(\mathfrak{s})$, requiring Z_i^2 to be 1 and not -1. Over $\mathbb{C}$, there is no substantial difference between the two conventions.

4.1.2. Extension of Vogan's conjecture We can define a differential of the complex $(U(\mathfrak{g}) \otimes C(\mathfrak{s}))^R$ using Kostant's cubic Dirac operator exactly as in Chapter 3, i.e., by $d(a) = Da - \epsilon_a aD$. As before, $d^2 = 0$ on $(U(\mathfrak{g}) \otimes C(\mathfrak{s}))^R$. Since the degree of the cubic term is zero in the filtration of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ used in Chapter 3, the proof goes through without change and we get

Theorem 4.1.3. *Let d be the differential on $(U(\mathfrak{g}) \otimes C(\mathfrak{s}))^R$ defined by Kostant's cubic Dirac operator as above. Then* Ker $d =$ Im $d \oplus Z(\mathfrak{r}_\Delta)$. *In particular, the cohomology of d is isomorphic to $Z(\mathfrak{r}_\Delta)$.*

Here $\mathfrak{r}_\Delta$ is a diagonally embedded copy of $\mathfrak{r}$, in analogy with 3.1.4. Furthermore, as in Chapter 3, the projection of $Z(\mathfrak{g}) \subset \text{ Ker } d$ to $Z(\mathfrak{r}_\Delta) \cong Z(\mathfrak{r})$ obtained by Theorem 4.1.3 is a homomorphism of algebras $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r})$, given explicitly as follows. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{r}$. Embed $\mathfrak{t}^*$ into $\mathfrak{h}^*$, extending functionals from $\mathfrak{t}$ to $\mathfrak{h}$ by defining them to be zero on $\mathfrak{a}$. Then under the identifications $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)$ and $Z(\mathfrak{r}) \cong P(\mathfrak{t}^*)$ given by Harish-Chandra homomorphisms, ζ is given by restricting polynomials from $\mathfrak{h}^*$ to $\mathfrak{t}^*$. Thus all parts of Vogan's conjecture generalize fully to our present setting. All this was proved by Kostant [Ko4]. To summarize

Theorem 4.1.4. *Let $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r}) \cong Z(\mathfrak{r}_\Delta)$ be as above. Then for any $z \in Z(\mathfrak{g})$ one has*

$$z - \zeta(z) = Da + aD$$

for some $a \in U(\mathfrak{g}) \otimes C(\mathfrak{s})$.

The proof of the above theorem follows exactly the line for the case when $(\mathfrak{g}, \mathfrak{r})$ is a symmetric pair in Chapter 3. For any $\mathfrak{g}$-module V and a spin module S for $C(\mathfrak{s})$, one can consider the map

$$D \colon V \otimes S \to V \otimes S.$$

We define the *Dirac cohomology* of V to be the $\mathfrak{r}$-module

$$H_D(V) = \text{ Ker } D / \operatorname{Im} D \cap \text{ Ker } D.$$

If V is a finite-dimensional $\mathfrak{g}$-module, then D is skew self-adjoint on $V \otimes S$. (See Lemma 4.2.1.) It follows in this case that $H_D(V) = \text{ Ker } D$. For the case $\operatorname{rank} \mathfrak{g} = \operatorname{rank} \mathfrak{r}$, it is clear that any finite-dimensional $\mathfrak{g}$-module has nonzero Dirac cohomology, and it follows that the map ζ is indeed induced by the Harish-Chandra isomorphisms as above. However, in the general case when $\operatorname{rank} \mathfrak{g}$ need not be equal to $\operatorname{rank} \mathfrak{r}$, then $\rho_\mathfrak{g}$ need not be in $\mathfrak{t}^*$ and a finite-dimensional $\mathfrak{g}$-module may have zero Dirac cohomology. The detailed proof for the general case was given by Kostant in [Ko4], by constructing a sufficiently large family of highest weight modules with known infinitesimal characters and nonzero Dirac cohomology.

We note that for the case $\operatorname{rank} \mathfrak{g} = \operatorname{rank} \mathfrak{r}$ it is much easier to prove that ζ is determined by the Harish-Chandra homomorphisms as above. This is due to the fact that any finite-dimensional irreducible $\mathfrak{g}$-modules V_λ contains a nonzero $\mathfrak{r}$-module with highest weight $\lambda + \rho_\mathfrak{g} - \rho_\mathfrak{r}$ in Dirac cohomology.

It follows immediately from the above theorem that there is a connection between the infinitesimal character of a $\mathfrak{g}$-module V and the infinitesimal character of its Dirac cohomology $H_D(V)$. This is seen in exactly the same way as we saw that Theorem 3.2.5 follows from Theorem 3.2.7 and Theorem 3.2.8.

Theorem 4.1.5. *Suppose V is a $\mathfrak{g}$-module with an infinitesimal character. If an irreducible $\mathfrak{r}$-module W with infinitesimal character $\mu \in \mathfrak{t}^*$ is contained in the Dirac cohomology* $\mathrm{Ker}\ D / \mathrm{Im}\ D \cap \mathrm{Ker}\ D$ *of V, then the infinitesimal character of V is conjugate to μ.*

The extension of the Vogan's conjecture to the cubic Dirac operator setting was first obtained and pointed out to us by Kostant. He observed that the original proof of the Vogan's conjecture in [HP1] can be applied to the cubic Dirac operator as well. Moreover, Kostant pointed out that the homomorphism $\zeta\colon Z(\mathfrak{g}) \to Z(\mathfrak{r})$ makes $Z(\mathfrak{r})$ a $Z(\mathfrak{g})$-module, which has topological significance. Namely, Kostant [Ko4] has shown that from a well-known theorem of H. Cartan [Ca], which is by far the most comprehensive result on the real (or complex) cohomology of a homogeneous space, one has

Corollary 4.1.6. *There exists an isomorphism*

$$H^*(G/R, \mathbb{C}) \cong \mathrm{Tor}_*^{Z(\mathfrak{g})}(\mathbb{C}, Z(\mathfrak{r})).$$

4.2 Dirac cohomology of finite dimensional representations

Let G be a connected semisimple Lie group and R a closed subgroup of rank equal to that of G. We can always choose a Cartan involution θ so that R is θ-stable. Denote by $\mathfrak{g}_0$ and $\mathfrak{r}_0$ the Lie algebras of G and R respectively. We remove the subscripts for their complexifications. As usual, the Cartan decompositions with respect to θ are denoted by $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

The cubic Dirac operator D is defined in association with the pair of complex Lie algebras $(\mathfrak{g}, \mathfrak{r})$ and is independent of the real forms. We assume that the Killing form B of $\mathfrak{g}$ restricts to $\mathfrak{r}$ non-degenerately and let $\mathfrak{s}$ be the orthogonal complement to $\mathfrak{r}$ with respect to B. Note that just like $\mathfrak{r}$, $\mathfrak{s}$ is also the complexification of its real form $\mathfrak{s}_0 = \mathfrak{s} \cap \mathfrak{g}_0$.

We choose a maximal isotropic subspace $\mathfrak{s}^+$ of $\mathfrak{s}$. Since

$$\langle X, Y \rangle = -2B(X, \theta\bar{Y})$$

(with $^-$ denoting conjugation with respect to $\mathfrak{g}_0$) defines a positive definite hermitian form on $\mathfrak{g}$ and hence also on $\mathfrak{s}^+$, the subspace $\mathfrak{s}^- = \theta\overline{\mathfrak{s}^+}$ intersects $\mathfrak{s}^+$ trivially. Let $S = \bigwedge^{\cdot} \mathfrak{s}^+$ be the spin module for the Clifford algebra $C(\mathfrak{s})$ corresponding to this polarization. We extend the form $\langle\,,\rangle$ to all of S in the usual way, using the determinant. The explicit formula was given in (2.20).

Let V be a finite-dimensional $\mathfrak{g}$-module. We consider D as an element in $\mathrm{End}\,(V \otimes S)$. Recall that by Proposition 2.3.11 the adjoint of the operator $X \in \mathfrak{s} \subset C(\mathfrak{s})$ on S with respect to the form $\langle\,,\rangle$ is $\theta\bar{X}$. On the other hand, V is a unitary module for a compact group G_c with Lie algebra equal to the compact real form $\mathfrak{g}_c = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of $\mathfrak{g}$. Denoting the corresponding form on V

by $\langle\,,\rangle$, it follows that all operators $X \in \mathfrak{g}_c$ are skew self-adjoint with respect to $\langle\,,\rangle$. In other words, any $X \in \mathfrak{k}_0$ is skew self-adjoint, while any $X \in \mathfrak{p}_0$ is self adjoint with respect to $\langle\,,\rangle$. This implies that the adjoint of any $X \in \mathfrak{g}$ on V with respect to $\langle\,,\rangle$ is $-\theta\bar{X}$. The form $\langle\,,\rangle$ on V is often called the admissible form on V.

Let us choose bases Z_i of $\mathfrak{s}_0 \cap \mathfrak{k}_0$ and Z_r' of $\mathfrak{s}_0 \cap \mathfrak{p}_0$ orthonormal with respect to $\langle\,,\rangle$. Then the linear part of D, $\sum Z_i \otimes Z_i + \sum Z_r' \otimes Z_r'$, is skew self-adjoint with respect to the form $\langle\,,\rangle$ on $V \otimes S$ obtained by combining the forms on V and S described above. Namely, $\theta Z_i = Z_i$ and $\theta Z_r' = -Z_r'$. Hence the adjoint of $Z_i \otimes Z_i$ with respect to $\langle\,,\rangle$ is $(-\theta Z_i) \otimes \theta Z_i = -Z_i \otimes Z_i$, while the adjoint of $Z_r' \otimes Z_r'$ is $(-\theta Z_r') \otimes \theta Z_r' = -Z_r' \otimes Z_r'$. In fact, we have the following lemma.

Lemma 4.2.1. *Kostant's cubic Dirac operator is skew self-adjoint with respect to the form $\langle\,,\rangle$ on $V \otimes S$ described above.*

Proof. It remains to prove that the cubic part v of D is skew self-adjoint with respect to the form $\langle\,,\rangle$ on S. As we already noted above, by Proposition 2.3.11 the adjoint of Z_i on S is $\theta Z_i = Z_i$, while the adjoint of Z_r' is $\theta Z_r' = -Z_r'$. Moreover, the dual bases of Z_i, Z_r' with respect to B are $-Z_i$, Z_r', and so

$$
\begin{aligned}
v &= \frac{1}{2}\left(\sum_{i<j<k} B([-Z_i,-Z_j],-Z_k)Z_iZ_jZ_k + \sum_{i;r<s} B([-Z_i,Z_r'],Z_s')Z_iZ_r'Z_s'\right) \\
&= -\frac{1}{2}\left(\sum_{i<j<k} B([Z_i,Z_j],Z_k)Z_iZ_jZ_k + \sum_{i;r<s} B([Z_i,Z_r'],Z_s')Z_iZ_r'Z_s'\right)
\end{aligned}
$$

(the other terms are zero). Since the adjoint of $Z_iZ_jZ_k$ is $Z_kZ_jZ_i = -Z_iZ_jZ_k$, the adjoint of $Z_iZ_r'Z_s'$ is $(-Z_s')(-Z_r')Z_i = -Z_iZ_r'Z_s'$, and the coefficients are real, we see that v is skew self-adjoint.

Since D is skew self-adjoint, it follows that $\operatorname{Ker} D \cap \operatorname{Im} D = 0$. Thus the Dirac cohomology of V is simply $\operatorname{Ker} D$.

We now determine the Dirac cohomology of any irreducible finite dimensional $\mathfrak{g}$-module. Let $W^1 \subset W_{\mathfrak{g}}$ be the subset of Weyl group elements that map the positive Weyl chamber for $\mathfrak{g}$ into the positive Weyl chamber for $\mathfrak{r}$. Thus, the multiplication $(w,\tau) \mapsto w\tau$ with $w \in W^1$ and $\tau \in W_{\mathfrak{r}}$ gives a bijection

$$W^1 \times W_{\mathfrak{r}} \to W.$$

Let V_λ denote the irreducible (finite dimensional) representation of $\mathfrak{g}$ with highest weight λ, and let $U_{w\centerdot\lambda}$ denote the irreducible (finite dimensional) representation of $\mathfrak{r}$ with highest weight $w \centerdot \lambda$.

Theorem 4.2.2. *Let G be a compact semisimple Lie group. Let $R \subset G$ be a closed subgroup with $\operatorname{rank} R = \operatorname{rank} G$. Then the Dirac cohomology of V_λ is equal to*

$$\text{Ker } D = \text{ Ker } D^2 = \bigoplus_{w \in W^1} U_{w \centerdot \lambda},$$

and

$$\text{Ker } D^+ = \bigoplus_{w \in W^1_+} U_{w \centerdot \lambda}, \quad \text{Ker } D^- = \bigoplus_{w \in W^1_-} U_{w \centerdot \lambda},$$

where W^1_+ is the subset of W^1 of all even elements and W^1_- is the subset of W^1 of all odd elements.

Proof. If λ is a dominant weight for $\mathfrak{g}$, then $\lambda + \rho_{\mathfrak{g}}$ lies in the interior of the Weyl chamber for $\mathfrak{g}$. It follows that $w(\lambda + \rho_{\mathfrak{g}})$ lies in the interior of the Weyl chamber for $\mathfrak{r}$ for any $w \in W^1$. Thus, $w \centerdot \lambda = w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{r}}$ is a dominant weight for $\mathfrak{r}$. Clearly $w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{r}} = w\lambda + (w\rho_{\mathfrak{g}} - \rho_{\mathfrak{r}})$ is a sum of extreme weights in V and S respectively. They are both in the dominant $\mathfrak{r}$-chamber. It follows that $U_{w \centerdot \lambda}$ is an $\mathfrak{r}$-subomdule of $V \otimes S$, and it is contained in Ker D.

Conversely, if γ is the highest weight of an $\mathfrak{r}$-module contained in Ker D, one has $\gamma + \rho_{\mathfrak{r}} = w(\lambda + \rho_{\mathfrak{g}})$ for some $w \in W^1$ due to the infinitesimal character condition. It remains to show that the weight $\gamma = w\lambda + w\rho_{\mathfrak{g}}$ occurs with multiplicity one in $V_\lambda \otimes S$. Recall from Section 2.3 that any weight in S is of the form $\rho_{\mathfrak{p}} - \langle \Phi \rangle$ for a subset $\Phi \subset \Sigma^+_{\mathfrak{p}}$. If the highest weight is $\gamma = \delta + \rho_{\mathfrak{p}} + \langle \Phi \rangle$ for some weight δ in V_λ, one has

$$||\gamma + \rho_{\mathfrak{r}}||^2 = ||\delta + \rho_{\mathfrak{g}} + \langle \Phi \rangle||^2.$$

On the other hand, there exists a $w \in W^1$ such that

$$\gamma + \rho_{\mathfrak{g}} - \langle \Phi \rangle = w(\lambda + \rho_{\mathfrak{g}}).$$

If $A = \lambda - w^{-1}\delta$ and $B = \rho_{\mathfrak{g}} - w^{-1}(\rho_{\mathfrak{g}} - \langle \Phi \rangle)$, then both A and B are the sum of (possibly empty) positive roots. But $\lambda + \rho_{\mathfrak{g}} = A + B + w^{-1}(\delta + \rho_{\mathfrak{g}} - \langle \Phi \rangle)$. It follows that $A = B = 0$ and $\delta = w\lambda$ and $\rho_{\mathfrak{p}} - \langle \Phi \rangle = w\rho_{\mathfrak{g}} - \rho_{\mathfrak{r}}$. Hence, the highest weight vector v_γ occurs with multiplicity one in the weight space of $V_\lambda \otimes S$ and the proof is completed.

4.3 Characters

In this section we review the definition and some of the basic properties of characters of finite-dimensional representations of compact Lie groups. The character theory provides some deeper insights into the struture of representations.

4.3.1. Definition of characters. The *character* χ of a finite dimensional representation (π, V) of G is the map $\chi_V : G \to \mathbb{C}$ defined by

$$\chi_V(g) = \text{trace}(\pi(g)).$$

We may fix a basis of V and identify $\pi(g)$ with the corresponding matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\pi(g)$, then

$$\chi_V(g) = \lambda_1 + \cdots + \lambda_n.$$

Clearly, this computation of $\chi_V(g)$ is independent of the choice of a basis, for changing to another basis, the eigenvalues remain the same.

Proposition 4.3.2. *Let χ_V be the character of a finite dimensional representation (π, V) of a compact Lie group G. Then*

(i) $\chi_V(e)$ is the dimension of V.

(ii) $\chi_V(g) = \chi_V(hgh^{-1})$ for all $g, h \in G$.

(iii) $\chi_{V^*}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$ for all $g \in G$.

(iv) If $\chi_{V'}$ is the character of another representation (π', V'), then the character of $(\pi \oplus \pi', V \oplus V')$ is $\chi_V + \chi_{V'}$.

(v) The character of $(\pi \otimes \pi', V \otimes V')$ is $\chi_V \cdot \chi_{V'}$.

Proof. We fix a basis and may regard $\pi(g)$ as a matrix. Property (i) is true since $\chi(e) = \text{trace } I = \dim V$. Property (ii) follows from the fact that $\pi(hgh^{-1}) = \pi(h)\pi(g)\pi(h^{-1})$ and therefore

$$\text{trace } \pi(hgh^{-1}) = \text{trace } \pi(h)\pi(g)\pi(h^{-1}) = \text{trace } \pi(g).$$

To prove the property (iii), we note that $V^* \cong \overline{V}$, since V is unitary. This implies that the eigenvalues of $\pi^*(g)$ are the same as the eigenvalues of the complex conjugate of $\pi(g)$. Hence $\chi_{V^*}(g) = \overline{\chi_V(g)}$. Since the matrix for $\pi_{V^*}(g)$ is the transpose inverse of the matrix for $\pi(g)$, we have $\chi_{V^*}(g) = \chi_V(g^{-1})$. To prove the properties (iv) and (v) we fix a basis of V' and assume that $\pi'(g)$ has eigenvalues $\mu_1, \ldots, \mu_m$. Then the eigenvalues of $(\pi \oplus \pi')(g)$ are $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$, and the eigenvalues of $(\pi \otimes \pi')(g)$ are $\{\lambda_i \mu_j\}$ with $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Therefore, the character of $\pi \oplus \pi'$ evaluated at g is equal to

$$\chi_{V \oplus V'}(g) = \lambda_1 + \cdots + \lambda_n + \mu_1 + \cdots + \mu_m = \chi_V(g) + \chi_{V'}(g)$$

and the character of $\pi \otimes \pi'$ at g is equal to

$$\chi_{V \otimes V'}(g) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j = (\lambda_1 + \cdots + \lambda_n)(\mu_1 + \cdots + \mu_m) = \chi_V(g) \cdot \chi_{V'}(g).$$

A complex function $\phi\colon G \to \mathbb{C}$ which is constant on each conjugacy class is called a *class function*. It may be described as a function on the set of conjugacy classes. It follows from the property (ii) of the above proposition that characters are class functions. We define a hermitian inner product on the set of all the class functions by

$$\langle \chi, \chi' \rangle = \int_G \overline{\chi(g)} \chi'(g) dg.$$

Here the invariant integral over G is normalized so that $\int_G dg = 1$. This hermitian product defined for the characters turns out to be extremely useful for us.

Theorem 4.3.3. *Let G be a compact Lie group. Let V and W be finite dimensional representations of G. Then we have*
(i) $\int_G \chi_V(g)dg = dim\ V^G$.
(ii) $\langle \chi_V, \chi_W \rangle = dim\ Hom_G(V, W)$.
(iii) If V and W are irreducible, then $\langle \chi_V, \chi_W \rangle = \begin{cases} 1 \ \textit{if} \ \ V \cong W \\ 0 \ \textit{if} \ V \ncong W. \end{cases}$

Proof. If (π, V) is a representation of G, then the set of G-fixed points defined by

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$$

is a subspace of V. Here we write the action of the $\pi(g)$ by g for simplicity. We define a map

$$\phi\colon\ V \to V, \quad v \mapsto \int_G gv\ dg.$$

Then ϕ is G-equivariant in the sense that

$$x\phi = \phi x, \text{ for all } \ x \in G,$$

since the measure dg is G-invariant. Thus both the kernel and the image of ϕ are subrepresentations of V.

We claim that ϕ is a projection of V onto the G-invariant subspace V^G. Supposing that $v = \phi(w)$, we have

$$hv = \int_G hgwdg = \int_G g'wdg' = \phi(w) = v,$$

for any $h \in G$. If $v \in V^G$ then $\phi(v) = \int_G gv = v$. So V^G is contained in the image of ϕ and $\phi \circ \phi = \phi$. This proves the claim. It follows from the claim that

$$\dim V^G = \text{trace } \phi = \int_G \text{trace } \pi(g)dg = \int_G \chi(g)dg. \qquad (a)$$

This proves (i) of the theorem.

Let V, W be two representations of G. Note that

$$\text{Hom}(V, W)^G = \{G - \text{equivariant homomorphisms from } V \text{ to } W\}.$$

This space is usually denoted by $\text{Hom}_G(V, W)$. If both V and W are irreducible then by Schur's lemma, we have

$$\dim \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V = W \\ 0 & \text{if } V \ncong W. \end{cases} \qquad (b)$$

Note that Hom $(V, W) \cong V^* \otimes W$ is a representation of G. It follows from the properties (iii) and (v) of the Proposition 4.3.2 that its character is given by

$$\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V(g)} \cdot \chi_W(g).$$

Note that $V \cong W$ is equivalent to $\chi_V = \chi_W$. Applying the formulas (a) to this case, we obtain

$$\langle \chi_V, \chi_W \rangle = \int_G \overline{\chi_V(g)} \chi_W(g) = \dim \mathrm{Hom}_G(V, W).$$

This proves (ii) of the theorem. Then it follows from (ii) and (b) that

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W, \end{cases}$$

which proves the orthonormal relations of irreducible characters.

Corollary 4.3.4. *Any finite-dimensional representation (π, V) of G is determined by its character χ_V. More precisely, if we decompose V into a direct sum of irreducible representations*

$$V = n_1 V_1 \oplus \cdots \oplus n_r V_r,$$

then the multiplicity n_i of V_i in V is equal to $\langle \chi_V, \chi_{V_i} \rangle$ for $i = 1, \ldots, r$.

Proof. The character $\chi_V = n_1 \chi_{V_1} + \cdots + n_r \chi_{V_r}$. Then the identity $\langle \chi_V, \chi_{V_i} \rangle$ $(i = 1, \ldots, r)$ follows from (iii) of Theorem 4.3.3, which says that the characters of irreducible representations are orthonormal.

Corollary 4.3.5. *A representation (π, V) of G is irreducible if and only if its character χ_V satisfies the condition $\langle \chi_V, \chi_V \rangle = 1$.*

Proof. If $\chi_V = n_1 \chi_{V_1} + \cdots + n_r \chi_{V_r}$, then $\langle \chi_V, \chi_V \rangle = n_1^2 + \cdots + n_r^2$. This gives value 1 if and only if a single n_i is 1 and the rest are zero.

4.4 A generalized Weyl character formula

Let G be a connected compact Lie group and R be a connected closed subgroup of the same rank as G. The following *generalized Weyl character formula* first appeared in [GKRS]. This formula includes the Weyl character formula as a special case when R is a Cartan subgroup T. The proofs of this formula given in [GKRS] as well as in [Ko2] use the Weyl character formula. Our proof given here is independent of the Weyl character formula.

Theorem 4.4.1. *Let G be a connected compact Lie group. Let R be a connecetd closed subgroup of G of maximal possible rank. Then one has*

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{w \in W^1} (-1)^{l(w)} U_{w \bullet \lambda}$$

as $\tilde{R}$-modules. It follows that

$$ch(V_\lambda) = \frac{\sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \bullet \lambda})}{\sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \bullet 0})}.$$

Proof. We consider the R-equivariant homomorphism

$$D\colon V \otimes S \to V \otimes S.$$

Since $\operatorname{Ker} D \cap \operatorname{Im} D = 0$, one has the direct sum decomposition

$$V \otimes S = \operatorname{Ker} D \oplus \operatorname{Im} D.$$

Then $D^+\colon V \otimes S^+ \to V \otimes S^-$ maps $\operatorname{Im} D^-$ isomorphically onto $\operatorname{Im} D^+$. Hence, one has

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \operatorname{Ker} D^+ - \operatorname{Ker} D^-.$$

It follows that

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{w \in W^1} (-1)^{l(w)} U_{w \bullet \lambda}.$$

Then one has

$$ch(V_\lambda)(ch(S^+) - ch(S^-)) = \sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \bullet \lambda}).$$

In particular, if $\lambda = 0$ and V_λ correspond to the trivial representation, then the above identity gives

$$ch(S^+) - ch(S^-) = \sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \bullet 0}).$$

Hence, we have

$$ch(V_\lambda) = \frac{\sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \bullet \lambda})}{\sum_{w \in W^1} (-1)^{l(w)} ch(U_{w \bullet 0})}.$$

When R is a Cartan subgroup, this reduces to the *Weyl character formula.*

4.5 A generalized Bott-Borel-Weil Theorem

We retain the notation that G is a connected compact Lie group and R is a closed subgroup of maximal rank. The Borel-Weil Theorem gives a first geometric realization of each irreducible representation of G. Equivalently, this is a realization of each irreducible holomorphic representation of the complexification $G_{\mathbb{C}}$ of G. The realization is in the space of holomorphic sections of a holomorphic line bundle over the flag variety G/T, where T is a maxiaml torus. The Bott-Borel-Weil Theorem, which includes the Borel-Weil Theorem as a special case, is a statement about cohomology of holomorphic line bundles over a complex homogeneous space $X = G/R$, where R is a centralizer of a torus in G. Kostant [Ko1] showed that this is equivalent to the determination of how the subgroup R operates on the cohomology of a certain nilpotent Lie algebra of an arbitrary irreducible representation of G. Upon replacing the complex structure by a space S of spinors, the Bott-Borel-Weil Theorem is equivalent to determination of Dirac cohomology of the irreducible G-module V_λ with highest weight λ. Then the complex structure is eliminated and the setting can be extend to any subgroup R. For the purpose of constructing all irreducible finite-dimensional representations on vector bundles over G/R, it is necessary for R to be of the same rank as G. A special case is when R is the centralizer of a torus then it yields a version of the Bott-Borel-Weil Theorem.

We assume that U_μ is an irreducible representation of R (or a two fold cover $\tilde{R}$ of R) so that $S \otimes U_\mu$ is a representation of R. The Dirac operator acts on the smooth and L^2-sections on the twisted spinor bundles over G/R, if we let $Z_i \in \mathfrak{g}$ act by differentiating from the right. So we can extend D to a closed operator (still denoted by D):

$$D\colon L^2(G) \otimes_R (S \otimes U_\mu) \to L^2(G) \otimes_R (S \otimes U_\mu).$$

We write this action in another form:

$$D\colon \text{ Hom }_{\tilde{R}}(U_\mu^*, L^2(G) \otimes S) \to \text{ Hom }_{\tilde{R}}(U_\mu^*, L^2(G) \otimes S).$$

Then D is formally self-adjoint. By Peter-Weyl theorem, one has $L^2(G) \cong \oplus_{\lambda \in \widehat{G}} V_\lambda \otimes V_\lambda^*$. It follows that the closed subspace $\text{Ker } D$ decomposes as

$$\text{Ker } D = \bigoplus_{\lambda \in \widehat{G}} V_\lambda \otimes \text{ Ker } \{D\colon \text{ Hom }_{\tilde{R}}(U_\mu^*, V_\lambda^* \otimes S) \circlearrowleft\}.$$

The proved Vogan's conjecture implies $\text{Ker } D \neq 0$ if and only if there is some $\lambda \in \widehat{G}$ such that $\lambda + \rho(\mathfrak{g})$ is conjugate to $\mu + \rho(\mathfrak{r})$ by the Weyl group. Further consideration of the multiplicity results in

Theorem 4.5.1. *In the above setting, one has* $\text{Ker } D = V_{w(\mu+\rho(\mathfrak{r}))-\rho(\mathfrak{g})}$ *if there exists a* $w \in W_{\mathfrak{g}}$ *so that* $w(\mu + \rho(\mathfrak{r})) - \rho(\mathfrak{g})$ *is dominant, and* $\text{Ker } D$ *is zero if no such* w *exists.*

This theorem can be viewed as a *generalized Bott-Borel-Weil theorem* . In the case $R = T$, a maximal torus, this is a version of the *Bott-Borel-Weil theorem* .

Corollary 4.5.2. *Consider*

$$D^+ : L^2(G) \otimes_R (S^+ \otimes U_\mu) \to L^2(G) \otimes_R (S^- \otimes U_\mu)$$

and the adjoint

$$D^- : L^2(G) \otimes_R (S^- \otimes U_\mu) \to L^2(G) \otimes_R (S^+ \otimes U_\mu).$$

One has $\mathrm{Index}\, D = \dim \mathrm{Ker}\ D^+ - \dim \mathrm{Ker}\ D^- = (-1)^{l(w)} \dim V_{w(\mu+\rho(\mathfrak{r}))-\rho(\mathfrak{g})}$ *if there exists a* $w \in W_{\mathfrak{g}}$ *so that* $w(\mu+\rho(\mathfrak{r}))-\rho(\mathfrak{g})$ *is dominant and* $\mathrm{Index}\, D = 0$ *if no such* w *exists.*

5

Cohomological Induction

5.1 Overview

In this chapter we review the basic constructions involved in cohomological induction, most notably the Zuckerman and Bernstein functors. Our definitions are slightly different from the ones available in the literature. For example we do not use Hecke algebras which are basic ingredients in the definitions in [KV]. Also, we use a direct description of derived functors, including the $\mathfrak{g}$-action; this approach has its roots in [B], [W] and [DV], and it was fully developed in the setting of equivariant derived categories by D. Miličić and the second author, [MP1], [MP2], [MP3], [Pan1], [Pan2]. In particular, this will provide for a very simple treatment of the duality results.

We will work in the setting of a connected real reductive group G with a maximal compact subgroup K corresponding to a Cartan involution Θ. Whenever convenient, we will not mind assuming that G is connected. In fact, the reader may choose to think only about connected semisimple G with finite center - these contain most of the main examples. The connectedness assumption is not essential for all parts of the theory, for example the definition of Zuckerman and Bernstein functors. However, in many places it has to be present at least in some weaker form. The book [KV] includes a lot of discussion about disconnected groups. We would like to emphasize that avoiding of the Hecke algebras mentioned above has nothing to do with the connectivity assumptions.

The main idea of cohomological induction is to complement the better known and older construction of real parabolic induction by inducing from the θ-stable parabolic subalgebras (or from corresponding Levi subgroups). Let us first explain this setting. As before, we will denote by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ the Lie algebras of G and K, and by $\mathfrak{g}$ and $\mathfrak{k}$ their complexifications. The differentiated Cartan involution θ induces the Cartan decompositions $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

5.1.1. θ-stable parabolic subalgebras Let us fix a fundamental Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ of $\mathfrak{g}_0$. In other words, $\mathfrak{h}_0$ is maximally compact, i.e., $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$. Let $h \in i\mathfrak{t}_0$. Then the eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}$ are real. The corresponding θ-stable parabolic subalgebra $\mathfrak{q}$ is the sum of the nonnegative eigenspaces of $\operatorname{ad} h$. The *Levi subalgebra* $\mathfrak{l}$ is the zero eigenspace of $\operatorname{ad} h$, i.e., the centralizer of h in $\mathfrak{g}$. The sum of positive eigenspaces is the *nilradical* $\mathfrak{u}$ of $\mathfrak{q}$ and the sum of negative eigenspaces is the nilradical $\bar{\mathfrak{u}}$ of the opposite parabolic subalgebra $\bar{\mathfrak{q}}$. All these subalgebras are obviously θ stable, since h is fixed by θ. In particular, they can all be decomposed as sums of their intersections with $\mathfrak{k}$ and $\mathfrak{p}$.

Clearly, $\mathfrak{l}$ contains $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Note also that $\mathfrak{l}$ has a nontrivial center: h is a central element of $\mathfrak{l}$. Furthermore, $\mathfrak{l}$ is the complexification of a subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}_0$.

Some special cases: if h is a regular element, then $\mathfrak{l} = \mathfrak{h}$ and $\mathfrak{q}$ is a Borel subalgebra of $\mathfrak{g}$. For $h = 0$, $\mathfrak{l} = \mathfrak{q} = \mathfrak{g}$. If $(\mathfrak{g}, \mathfrak{k})$ is a hermitian pair with $\mathfrak{g}$ simple, then for h in the center of $\mathfrak{k}$ we are getting $\mathfrak{l} = \mathfrak{k}$.

The Levi subgroup L of G corresponding to $\mathfrak{q}$ is defined to be the normalizer of $\mathfrak{q}$ in G, i.e., $L = \{g \in G \,|\, \operatorname{Ad}(g)\mathfrak{q} = \mathfrak{q}\}$.

One could get real parabolic subalgebras in an analogous fashion, but starting from a maximally split Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, and choosing h to be in $\mathfrak{a}_0$.

5.1.2. Definition of cohomologically induced modules. Suppose now that Z is an $(\mathfrak{l}, L \cap K)$-module. The *cohomologically induced modules* are obtained from Z as follows. First, for technical reasons one replaces Z by the twisted module $Z^\# = Z \otimes \bigwedge^{\mathrm{top}} \mathfrak{u}$, with the action on $\bigwedge^{\mathrm{top}} \mathfrak{u}$ being the adjoint action. This is just to make some formulas later on look nicer, and can be ignored for the time being.

Now $Z^\#$ can be viewed as a $(\mathfrak{q}, L \cap K)$-module by letting $\mathfrak{u}$ act as zero. Then one constructs a *produced* $(\mathfrak{g}, L \cap K)$-module

$$\operatorname{pro}(Z^\#) = \operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#) = \operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z^\#)_{L\cap K},$$

where the subscript $L \cap K$ means we are taking the $L \cap K$-finite vectors. Here $\mathfrak{g}$ acts by right translation of the argument, and $L \cap K$ by conjugation:

$$(X\alpha)(u) = \alpha(uX); \quad (k\alpha)(u) = \pi(k)(\alpha(\operatorname{Ad}(k^{-1})u)),$$

for $\alpha \in \operatorname{pro}(Z^\#)$, $X \in \mathfrak{g}$, $k \in L \cap K$ and $u \in U(\mathfrak{g})$, with $\pi(k)$ denoting the original action on $Z^\#$.

Now one applies the (right) derived Zuckerman functors to the produced module and obtains the $(\mathfrak{g}, K)$-modules cohomologically induced from Z:

$$\mathcal{R}^i(Z) = R^i\Gamma(\operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Z^\#)).$$

The Zuckerman functor Γ roughly extracts the maximal $(\mathfrak{g}, K)$ submodule, i.e., the largest subspace where the action of $\mathfrak{k} \subset \mathfrak{g}$ exponentiates to a finite action of K. This functor is left exact and usually zero on modules we are studying here. However, the right derived functors will typically not all be 0. In good cases (under some positivity assumptions), there will be exactly one nonzero module, the one obtained from S-th derived Zuckerman functor, where $S = \dim \mathfrak{u} \cap \mathfrak{k}$. Also, if Z is an irreducible unitary module, the induced one will be such too. These facts are however not at all easy to see. We will study all the functors appearing above in detail. However, let us say a few more words now about what the Zuckerman functors should be like.

5.1.3. A rough description of Zuckerman functors. As we said, the Zuckerman functor should (roughly) extract the largest $(\mathfrak{g}, K)$-submodule from a $(\mathfrak{g}, T)$-module V, where $T = L \cap K$. So the question is if V contains some copies of irreducible (unitary, finite dimensional) K-modules V_δ, disguised as $(\mathfrak{k}, T)$-submodules. For a fixed V_δ, $\delta \in \hat{K}$, consider

$$V_\delta \otimes \operatorname{Hom}_{(\mathfrak{k},T)}(V_\delta, V);$$

this is something like a "δ-isotypic component of V". Rewrite this as

$$V_\delta \otimes \operatorname{Hom}_{\mathbb{C}}(V_\delta, V)^{(\mathfrak{k},T)} = (V_\delta^* \otimes V_\delta \otimes V)^{(\mathfrak{k},T)},$$

where now the $(\mathfrak{k}, T)$-action with respect to which the invariants are taken is on V_δ^* and on V, while the K-action is on V_δ. We now sum this over $\delta \in \hat{K}$, and recall that

$$\bigoplus_{\delta \in \hat{K}} V_\delta^* \otimes V_\delta = R(K),$$

the space of regular (smooth, left and right finite) functions on K decomposed with respect to the $K \times K$ action by the tensor product of left and right regular decomposition.

So we got a candidate for the Zuckerman functor:

$$\Gamma(V) = (R(K) \otimes V)^{(\mathfrak{k},T)}.$$

We still have to define a $\mathfrak{g}$-action on this space, and establish its various properties. For the time being, let us just mention that seeing this definition it is not hard to imagine that the derived functors will be given as $(\mathfrak{k}, T)$-cohomology of $R(K) \otimes V$.

5.1.4. Left cohomological induction. There is a "dual" (or "left" as opposed to "right") construction, usually leading to the same cohomologically induced modules, but having different homological properties, which is useful for proofs. Here we extend $Z^\#$ to a $\bar{\mathfrak{q}}$-module by letting $\bar{\mathfrak{u}}$ act as zero, and then bring in a $\mathfrak{g}$-action by setting

$$\operatorname{ind}(Z^\#) = \operatorname{ind}_{\bar{\mathfrak{q}}, L \cap K}^{\mathfrak{g}, L \cap K}(Z^\#) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^\#.$$

This is a $(\mathfrak{g}, L\cap K)$-module (finiteness is automatic in this case). Here $\mathfrak{g}$ acts by left multiplication in the first factor, and $L\cap K$ acts on both factors, on $U(\mathfrak{g})$ by Ad and on $Z^{\#}$ by the given action.

To get a $(\mathfrak{g}, K)$-module, we apply to $\mathrm{ind}\,(Z^{\#})$ the *Bernstein functor Π*, which is roughly a substitute for taking the largest quotient which is a $(\mathfrak{g}, K)$-module. As before, to get something non-zero we actually have to apply the (in this case left) derived functors of the Bernstein functor: the $(\mathfrak{g}, K)$-modules left cohomologically induced from Z are

$$\mathcal{L}_i(Z) = L^i\Pi(\mathrm{ind}\,{}^{\mathfrak{g}}_{\mathfrak{q}}(Z^{\#})).$$

The Bernstein functors are defined in a similar way as the Zuckerman functors: for a $(\mathfrak{g}, T)$-module V, $\Pi(V)$ is the space of $(\mathfrak{k}, T)$-coinvariants of $R(K)\otimes V$. The left derived functors are given by $(\mathfrak{k}, T)$-homology.

Example 5.1.5. To finish this introduction, let us see what the construction outlined above looks like in case $G = SL(2,\mathbb{R})$. This example is somewhat too small to really show all the features mentioned above, but it is still going to give us some ideas about what is going on.

For $SL(2,\mathbb{R})$, there is up to conjugacy only one θ-stable parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ with $\mathfrak{l} = \mathfrak{k} = \mathbb{C}W$, and $\mathfrak{u} = \mathbb{C}u$ where

$$W = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \qquad u = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

(note that u was denoted by X in 1.3.10). The opposite parabolic subalgebra is $\mathfrak{q}^- = \bar{\mathfrak{q}} = \mathfrak{l}\oplus\bar{\mathfrak{u}}$, where $\bar{\mathfrak{u}} = \mathbb{C}\bar{u}$ with $\bar{\ }\}$ denoting the complex conjugation with respect to $\mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R})$ ($\bar{u}$ was denoted by Y in 1.3.10).

Let us start with a character $Z^{\#}$ of K with W acting as $k\in\mathbb{Z}$. Form

$$\mathrm{pro}\,(Z^{\#}) = \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z^{\#})_K$$

where $\mathfrak{u}$ acts on $Z^{\#}$ by 0. As mentioned above, the $\mathfrak{g}$-action on $\mathrm{pro}\,(Z^{\#})$ is by right multiplication in the first variable, and the $L\cap K = K$-action is by conjugation.

Since $U(\mathfrak{g}) = U(\mathfrak{q})\otimes U(\bar{\mathfrak{u}})$ by the Poincaré-Birkhoff-Witt theorem, we can identify

$$\mathrm{pro}\,(Z^{\#}) = \mathrm{Hom}_{\mathbb{C}}(U(\bar{\mathfrak{u}}), Z^{\#})) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\bar{u}], Z^{\#})).$$

Here the action of $\bar{u}$ will just raise the degree of the variable, but to see the action of W and u, we have to commute them to the left. In particular,

$$W\alpha(\bar{u}^n) = \alpha(\bar{u}^n W) = \alpha(W\bar{u}^n - [W, \bar{u}^n]) = (k+2n)\alpha(\bar{u}^n).$$

We claim that the K-finite α are exactly those that are 0 on all but finitely many $\bar{u}^n$'s. This is a special case of the well known description of the K-finite dual. Here is an argument: let α be non-zero on infinitely many $\bar{u}^n$, say for $n \in A = \{n_1, n_2, \dots\}$, with n_i increasing. Let $\alpha_1 = (k+2n_1)\alpha - W\alpha$. Then α_1 is 0 on $\bar{u}^{n_1}$ and nonzero on all other $\bar{u}^n$, $n \in A$. Now let $\alpha_2 = (k+2n_2)\alpha_1 - W\alpha_1$; this is 0 on $\bar{u}^{n_1}$ and $\bar{u}^{n_2}$, and nonzero on all other $\bar{u}^n$, $n \in A$. Continuing like this, we get a sequence α_r in the span of the W-orbit of α, with the property that α_r is 0 on $\bar{u}^{n_1}, \dots, \bar{u}^{n_r}$, and nonzero on all other $\bar{u}^n$, $n \in A$.

It is now enough to see that α_r are linearly independent, for this means that α is not K-finite. Suppose that α_r are linearly dependent and let

$$c_1\alpha_{i_1} + \dots + c_r\alpha_{i_r} = 0, \quad i_1 < \dots < i_r$$

be the shortest linear dependence relation among them. Evaluating at $\bar{u}^{n_{i_2}}$ we get a contradiction.

Using the pairing of $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ via the Killing form, which is K-invariant, we can thus identify $\operatorname{pro}(Z^\#)$ with $\mathbb{C}[u] \otimes Z^\#$. Since the weights of $\mathbb{C}[u] \otimes Z^\#$ are $k, k+2, k+4, \dots$, it is clear that for $k > 0$ we obtained the lowest weight discrete series representation (or limit of discrete series if $k = 1$). For $k \leq 0$, there is however also a finite dimensional subquotient. This explains the need for a “positivity condition”, or rather compatibility of $\mathfrak{q}$ and Z, if we expect to get irreducible unitary representations. However, we may also note that for $k < 0$ we can exchange $\mathfrak{q}$ and $\bar{\mathfrak{q}}$, and get the highest weight discrete series.

The module $\operatorname{ind}(Z^\#) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^\# = U(\mathfrak{u}) \otimes Z^\#$ is obviously equal to $\operatorname{pro}(Z^\#)$ as a $L \cap K = K$-module - the weights are the same. If $k > 0$, it follows that $\operatorname{ind}(Z^\#) = \operatorname{pro}(Z^\#)$ also as $\mathfrak{g}$-modules, simply because there is only one module with this set of weights. If $k \leq 0$, the two modules have the same irreducible subquotients, but they are not isomorphic as $\mathfrak{g}$-modules.

In general, we will show in 6.2.8 below that there is a map

$$\phi_Z : \operatorname{ind}(Z^\#) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^\# \to \operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z^\#) = \operatorname{pro}(Z^\#),$$

given by $\phi_Z(u \otimes z)(v) = \mu(vu)z$, where $\mu : U(\mathfrak{g}) \to U(\mathfrak{l})$ is the Harish-Chandra map. The map ϕ_Z will be important in showing the vanishing of cohomologically induced modules (see Corollary 6.2.10 and the discussion preceding it). The same map is used for inducing Hermitian forms on cohomologically induced modules (see the discussion below Theorem 6.3.2).

Note that one feature that makes the above example too simple is the fact that $L \cap K = K$ means that the produced module is already K-finite and there is no need for Zuckerman functors. This will however happen rarely. In fact, whenever $\bar{\mathfrak{u}} \cap \mathfrak{k} \neq 0$, it acts freely and not finitely on $\operatorname{pro}(Z^\#)$ (and $\mathfrak{u} \cap \mathfrak{k}$ acts freely on $\operatorname{ind}(Z^\#)$). So we immediately see that there are no K-finite vectors, and the 0th Zuckerman functor is 0. The higher derived functors however save the situation.

5.2 Some generalities about adjoint functors

In this section we will review some well known and useful facts which are an essential prerequisite to understand cohomological induction. A good source for the general theory is [ML].

Definition 5.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Consider a pair of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \xleftarrow[G]{} & \end{array}$$

We say F is *left adjoint* to G (or G is *right adjoint* to F) if for any two objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) = \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

More precisely, there should exist mutually inverse isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, Y) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathcal{C}}(X, GY), \\ & \xleftarrow[\beta]{} & \end{array}$$

natural in X and Y. What is meant by naturality in X is this: for any $f : X' \to X$, let f^* be the operation of composing with f, i.e., $f^*(g) = g \circ f$. Then the diagram

$$\begin{array}{ccc} \mathrm{Hom}\,(FX, Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Hom}\,(X, GY) \\ {\scriptstyle (Ff)^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ \mathrm{Hom}\,(FX', Y) & \xrightarrow[\alpha_{X',Y}]{} & \mathrm{Hom}\,(X', GY) \end{array}$$

commutes. Naturality in Y is analogous.

In categories we are interested in, sets of morphisms will always be vector spaces, and it is understood that all functors and natural transformations should respect this linear structure.

Example 5.2.2. Consider an associative algebra A over $\mathbb{C}$ with unit 1. Let B be a subalgebra of A. Then any (left) A-module is also a B-module, by forgetting part of the action. So we have a *forgetful functor* from the category $M(A)$ of A-modules into the category $M(B)$. We want to construct its adjoints, the well known *extension of scalars* functors.

Recall that the complexification of a real vector space V can be thought of as $\mathbb{C} \otimes_{\mathbb{R}} V$, with $\mathbb{C}$ acting by multiplication in the first factor. Similarly, from a complex vector space V we can concoct a (free) A-module

$$A \otimes V = A \otimes_{\mathbb{C}} V,$$

with A acting by left multiplication. We can see a copy of V inside this module, embedded as $1 \otimes V$.

Now if V was a B-module, this action is not at all reflected in the B-action on $A \otimes V$. To glue together the two actions, we want to mod out all the elements of the form $b \otimes v - 1 \otimes bv$. So we divide $A \otimes V$ by the A-submodule generated by these elements:

$$Z = \text{ span } \{ab \otimes v - a \otimes bv\}$$

What we got is of course the tensor product $A \otimes_B V$. In this way we obtain a functor

$$V \mapsto A \otimes_B V$$

from $M(B)$ to $M(A)$; it sends a morphism $f : V \to W$ of B-modules into the morphism

$$\text{id } \otimes f : A \otimes_B V \to A \otimes_B W$$

of A-modules.

Lemma 5.2.3. *The functor $A \otimes_B \bullet$ is left adjoint to the forgetful functor* For $: M(A) \to M(B)$.

Proof. We have to produce natural isomorphisms

$$\text{Hom }_A(A \otimes_B X, Y) \quad \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \quad \text{Hom }_B(X, \text{ For } Y),$$

where X is a B-module and Y is an A-module.

If $f : A \otimes_B X \to Y$ is an A-morphism, we define $\alpha(f) : X \to Y$ by

$$\alpha(f)(x) = f(1 \otimes x).$$

This is a B-morphism, as $\alpha(f)(bx) = f(1 \otimes bx) = f(b \otimes x) = bf(1 \otimes x) = b\alpha(f)(x)$.

If $g : X \to Y$ is a B-morphism, we define $\beta(g) : A \otimes_B X \to Y$ by

$$\beta(g)(a \otimes x) = ag(x).$$

This is an A-morphism, as $\beta(g)(a'a \otimes x) = a'ag(x) = a'\beta(g)(a \otimes x)$.

To see β and α are inverse to each other, we calculate

$$\beta(\alpha(f))(a \otimes x) = a\alpha(f)(x) = af(1 \otimes x) = f(a \otimes x),$$

using the fact that f is an A-morphism. Also,

$$\alpha(\beta(g))(x) = \beta(g)(1 \otimes x) = 1g(x) = g(x).$$

We are still not done, as we have to prove that α and β are natural. Let us prove naturality of α with respect to X: let $h : X' \to X$ be a B-morphism. We have to prove that

$$\alpha(f \circ (\text{ id } \otimes h)) = \alpha(f) \circ h.$$

Let $x \in X'$. Then $\alpha(f \circ (\text{ id } \otimes h))(x) = f((\text{ id } \otimes h)(1 \otimes x)) = f(1 \otimes h(x)) = \alpha(f)(h(x))$ and we are done. Naturality in the other variable is similar.

5.2.4. Adjunction morphisms. In trying to understand why the above example worked, one immediately sees that the main point was the existence of a B-morphism

$$x \mapsto 1 \otimes x$$

from any B-module X to $(\text{ For })A \otimes_B X$, and an A-morphism

$$a \otimes y \mapsto ay$$

from $A \otimes_B (\text{ For })Y$ to any A-module Y.

We used these morphisms to define α and β, and moreover the calculation that α and β are inverse to each other actually came down to seeing that

$$x \mapsto 1 \otimes x \mapsto 1x = x$$

and

$$a \otimes x \mapsto a \otimes 1 \otimes x \mapsto a(1 \otimes x) = a \otimes x$$

are the identity morphisms.

The situation is completely analogous (and also more clear) in general. Assume we have morphisms $\Phi_X : X \to GF(X)$ and $\Psi_Y : FGY \to Y$ for any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then we can get

$$\alpha : \text{ Hom }_{\mathcal{D}}(FX, Y) \to \text{ Hom }_{\mathcal{C}}(X, GY)$$

which sends $f : FX \to Y$ into

$$\alpha(f) : X \xrightarrow{\Phi_X} GF(X) \xrightarrow{G(f)} GY,$$

and β in the opposite direction which sends $g : X \to GY$ into

$$\beta(g) : FX \xrightarrow{F(g)} FGY \xrightarrow{\Psi_Y} Y.$$

If $X \mapsto \Phi_X$ and $Y \mapsto \Psi_Y$ are natural transformations, i.e., if for any map $h : X \to X'$ the diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\Phi_X"'] & X' \arrow[d, "\Phi_{X'}"] \\ FGX \arrow[r, "FG(h)"'] & FGX' \end{tikzcd}$$

commutes and the analogous property holds for Ψ, then one easily proves α and β are natural. Moreover, if for any object X of $\mathcal{C}$, the composition

$$FX \xrightarrow{F(\Phi_X)} FGF(X) \xrightarrow{\Psi_{FX}} FX \tag{5.1}$$

is the identity morphism, and for any object Y of $\mathcal{D}$, the composition

$$GY \xrightarrow{\Phi_{GY}} GFG(Y) \xrightarrow{G(\Psi_Y)} GY \tag{5.2}$$

is the identity morphism, then α and β are inverse to each other. Namely, $\beta \circ \alpha = \mathrm{id}$ follows from naturality of Ψ and (5.1), while $\alpha \circ \beta = \mathrm{id}$ follows from naturality of Φ and (5.2).

The converse holds too: if we do have adjunction, then it comes as above from adjunction morphisms. Namely, assume F is left adjoint to G. Then set $\Phi_X = \alpha_{X,FX}(\mathrm{id}_{FX})$ for $X \in \mathcal{C}$, and $\Psi_Y = \beta_{GY,Y}(\mathrm{id}_{GY})$ for $Y \in \mathcal{D}$. Using naturality of α and β and the fact that they are inverse to each other, one gets

Proposition 5.2.5. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then F is left adjoint to G if and only if there are natural transformations*

$$\Phi : Id_{\mathcal{C}} \longrightarrow GF, \quad \Psi : FG \longrightarrow Id_{\mathcal{D}},$$

such that for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the compositions (5.1) and (5.2) are the identity morphisms.

Example 5.2.6. We now get back to the situation of Example 5.2.2 and construct the right adjoint of the forgetful functor. For a B-module V, consider the space

$$\mathrm{Hom}_B(A, V),$$

where the B-morphisms are taken with respect to the left multiplication in the first variable. Make this space into an A-module by letting A act by right multiplication in the first variable:

$$(af)(a') = f(a'a).$$

This is a left action and it is well defined because left and right multiplication commute. To make $\mathrm{Hom}_B(A, \bullet)$ into a functor, we define it on morphisms: $\phi : V \to W$ gets transformed into the operation of composing with ϕ, $\phi_* = \phi \circ \bullet$. To show that this functor is really right adjoint to For, we exhibit the adjunction morphisms.

For $X \in M(A)$, define an A-morphism $\Phi_X : X \to \mathrm{Hom}_B(A, (\mathrm{For})X)$ by

$$\Phi_X(x)(a) = ax.$$

For $Y \in M(B)$, define a B-morphism $\Psi_Y : (\mathrm{For})\,\mathrm{Hom}_B(A, Y) \to Y$ by

$$\Psi_Y(f) = f(1).$$

Then Φ and Ψ are well defined and natural, and (5.1) and (5.2) are the identity morphisms. Namely, (5.1) comes down to

$$x \longmapsto (a \mapsto ax) \longmapsto 1x = x, \qquad x \in X,$$

while (5.2) comes down to

$$f \longmapsto (a \mapsto af) \longmapsto \big(a \mapsto (af)(1) = f(a)\big), \qquad f \in \mathrm{Hom}_B(A, Y).$$

Remarks 5.2.7. We will often have a situation that A is free as a B-module for the left (right) multiplication. For example, if $A = U(\mathfrak{g})$ and $B = U(\mathfrak{g}_1)$ for a Lie subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$, then A is free over B, for both left and right multiplication, by the Poincaré-Birkhoff-Witt Theorem.

If $A = B \otimes \Lambda$ for a vector space Λ, then $\mathrm{Hom}_B(A, X)$ gets identified with $\mathrm{Hom}_{\mathbb{C}}(\Lambda, X)$. In particular, it is an exact functor. The A-action is however more difficult to see in this description; to recover it we must know some commutation relations. Analogously, if $A = \Lambda \otimes B$, then $A \otimes_B X$ can be identified with $\Lambda \otimes X$.

There is a generalization of the above construction, with B not necessarily a subalgebra of A, but another algebra with an algebra map $\tau : B \to A$. It is clear that in the above discussion we never really needed B to be inside A; we needed it to act on A-modules, which it does in the presence of τ: b simply acts as $\tau(b)$. Now every τ can be decomposed as a surjection followed by an injection, so the new situation is when A is a quotient of B (modulo a two-sided ideal). The extreme such situation is a map $B \to \mathbb{C}$. Again, such a map exists for enveloping algebras; it is the projection to constants along the ideal $\mathfrak{g}U(\mathfrak{g})$. For a $\mathfrak{g}$-module V, the corresponding "extended" modules, $\mathbb{C} \otimes_{U(\mathfrak{g})} V$ and $\mathrm{Hom}_{U(\mathfrak{g})}(\mathbb{C}, V)$ are nothing else but coinvariants and invariants of V with respect to $\mathfrak{g}$. Rather than "extension of scalars", this could be called "killing off the action of (most of) scalars". More generally, if I is an ideal of B, then extension of scalars with respect to the projection $B \to B/I$ is taking coinvariants or invariants with respect to I.

Finally, let us mention that we will often have extra structure on V compatible with the structure of a B-module, which will get carried over to $A \otimes_B V$ and $\mathrm{Hom}_B(A, V)$. This extra structure can for example be an action of a group or another algebra.

5.2.8. Properties of adjoint functors. There are many formal and completely general properties of adjoint functors. It is good to know something about them in order to avoid reproving the same things in the same way in many different situations, without being aware they are actually true automatically, for free.

5.2.9. Uniqueness. Adjoint functors are unique: if F and F' are left (right) adjoint to G then they are isomorphic. Namely, we would have

$$\mathrm{Hom}\,(FX, Y) = \mathrm{Hom}\,(X, GY) = \mathrm{Hom}\,(F'X, Y),$$

for every Y, naturally, and then one gets $FX \cong F'X$ by picking Y to be first FX, and then $F'X$, etc. This is just the usual proof that objects defined by a universal property are unique up to an isomorphism.

5.2.10. Compositions of adjoint functors. If

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C}$$

are functors with F left adjoint to G and F' left adjoint to G', then $F'F$ is left adjoint to GG'. Namely, we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(F'FX, Y) \cong \mathrm{Hom}_{\mathcal{B}}(FX, G'Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, GG'Y).$$

This gives various "induction in stages" theorems, as induction functors are always adjoint to some forgetful functors, and forgetting can obviously be done "in stages".

5.2.11. Preservation of limits and colimits. If F is left adjoint to G, then G preserves all (existing) "limits", while F preserves all "colimits". Examples of limits are: products, kernels, fiber products and inverse limits. Examples of colimits are: sums, cokernels, fibered sums and direct limits.

In particular, all categories we will study are abelian so there are notions of exact sequences and left/right exact functors. Now, since right adjoints preserve kernels, they are left exact. Since left adjoints preserve cokernels, they are right exact. Here for a functor G the notion of "preserving kernels" more precisely means taking the kernel of any morphism f to the kernel of Gf; so $\mathrm{Ker}\, Gf = G(\mathrm{Ker}\, f)$, i.e., G in fact commutes with the functor Ker which attaches to each morphism its kernel.

Viewing things in this way actually takes us half way towards proving the above "preservation statements". Namely the mentioned limits are in fact right adjoint to certain "constant functors", while colimits are left adjoint to constant functors, and hence one can use 5.2.10. See [ML], V.5.

5.2.12. Preservation of projectivity and injectivity. *Projective objects* of an abelian category $\mathcal{A}$ are objects P such that the functor

$$\mathrm{Hom}_{\mathcal{A}}(P, \bullet)$$

is exact. In general, this functor is easily seen to be left exact for any object P. Dually, an object I is *injective* if the (contravariant) functor

$$\mathrm{Hom}_{\mathcal{A}}(\bullet, I)$$

is exact. Again, in general this functor is only left exact.

The point we want to make here is that if $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to G and if G is exact, then F sends projectives to projectives. This is obvious from

$$\mathrm{Hom}_{\mathcal{D}}(F(P), \bullet) = \mathrm{Hom}_{\mathcal{C}}(P, G(\bullet))$$

and the fact that the composition of two exact functors is exact. Analogously, if G is right adjoint to F and if F is exact, then G sends injectives to injectives.

This actually gives the main method of constructing projectives and injectives in various categories. We will use it in 5.3.6 below. In the example of modules over a complex algebra A, consider the (exact) forgetful functor For from $M(A)$ into the category $M(\mathbb{C})$ of complex vector spaces. Since all short exact sequences in $M(\mathbb{C})$ are split, all objects are projective and injective. Thus for any $V \in M(A)$, the adjunction morphism $A \otimes_{\mathbb{C}} V \to V$ exhibits V as a quotient of a projective module, while the adjunction morphism $V \to \operatorname{Hom}_{\mathbb{C}}(A, V)$ exhibits V as a submodule of an injective module.

5.3 Homological algebra of Harish-Chandra modules

5.3.1. Pairs. We will work with *pairs* $(\mathfrak{s}, C)$, where $\mathfrak{s}$ is a complex Lie algebra and C is a compact Lie group such that

- C acts on $\mathfrak{s}$ by automorphisms, i.e., by a Lie group morphism $C \to \operatorname{Aut} \mathfrak{s}$;
- The complexified Lie algebra $\mathfrak{c}$ of C embeds into $\mathfrak{s}$, in such a way that the action of C on $\mathfrak{s}$ extends the adjoint action on $\mathfrak{c}$;
- The differentiated C-action on $\mathfrak{s}$ agrees with the restriction of $\operatorname{ad}_{\mathfrak{s}}$ to $\mathfrak{c}$.

The pairs we are interested in are the ones that already showed up in Section 5.1: $(\mathfrak{g}, K)$ coming from a real reductive group G, $(\mathfrak{l}, L \cap K)$ coming from a Levi subgroup, and also $(\mathfrak{q}, L \cap K)$ which explains why we do not want to assume $\mathfrak{s}$ is reductive.

We denote the action of C on $\mathfrak{s}$ by Ad; in the above examples, it really is the (restricted) adjoint action.

5.3.2. $(\mathfrak{s}, C)$**-modules** These are defined in the same way as $(\mathfrak{g}, K)$-modules in 1.3.7. An $(\mathfrak{s}, C)$-module is a complex vector space with an action of $\mathfrak{s}$ and a (locally) finite smooth action of C, which agree on $\mathfrak{c}$ in the sense that by differentiating the C-action (and then complexifying) we get the same action of $\mathfrak{c}$ as the one obtained by restriction from the $\mathfrak{s}$-action. Furthermore, the following equivariance condition holds:

$$\pi(\operatorname{Ad}(c)X) = \pi(c)\pi(X)\pi(c)^{-1}, \quad c \in C, \; X \in \mathfrak{s};$$

i.e., the action map $\mathfrak{s} \otimes V \to V$ is C-equivariant (where C acts both on $\mathfrak{s}$ and on V.) If C is connected, then the equivariance condition holds automatically.

A morphism between two $(\mathfrak{s}, C)$-modules is a linear map which intertwines both actions.

In many situations we shall impose various finiteness conditions on the modules we want to study. For example, finite generation, admissibility, $Z(\mathfrak{s})$-finiteness, finite length. However, we will also need “big” modules - for example, injective modules are always big.

5.3.3. Change of algebras Suppose $(\mathfrak{s}, C)$ and $(\mathfrak{r}, C)$ are pairs, and assume there is a C-equivariant map $\tau : \mathfrak{r} \to \mathfrak{s}$ whose restriction to $\mathfrak{c}$ is the identity. There is an obvious forgetful functor $\operatorname{For} : M(\mathfrak{s}, C) \to M(\mathfrak{r}, C)$ and we want to construct its adjoints. They are analogous to the change of rings functors from Section 5.2. In fact, let V be an $(\mathfrak{r}, C)$-module. If we for a moment ignore the C-action, we can consider the $U(\mathfrak{s})$-modules

$$U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} V \quad \text{and} \quad \operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)$$

from Section 5.2. Recall that $U(\mathfrak{s})$ acts by left multiplication on $U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} V$ and by right multiplication of the argument on $\operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)$. To turn these into $(\mathfrak{s}, C)$-modules, we define

$$c(u \otimes v) = \operatorname{Ad}(c)u \otimes \pi_V(c)v \quad \text{and} \quad c\alpha = \pi_V(c) \circ \alpha \circ \operatorname{Ad}(c^{-1})$$

for $c \in C$, $u \in U(\mathfrak{s})$, $v \in V$ and $\alpha \in \operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)$. Here π_V is the given C-action on V. One easily checks that these actions are well defined; for example, if $z \in U(\mathfrak{r})$, then

$$\begin{aligned} c(uz \otimes v) = \operatorname{Ad}(c)u \operatorname{Ad}(c)z \otimes \pi_V(c)v &= \operatorname{Ad}(c)u \otimes \pi_V(\operatorname{Ad}(c)z)\pi_V(c)v \\ &= \operatorname{Ad}(c)u \otimes \pi_V(c)\pi_V(z)v = c(u \otimes \pi_V(z)v), \end{aligned}$$

so the C-action is well defined with respect to $\otimes_{U(\mathfrak{r})}$.

We have to show that these C-actions are compatible with the $\mathfrak{s}$-actions. For $X \in \mathfrak{c}$,

$$\operatorname{ad}(X)u \otimes v + u \otimes \pi_V(X)v = (Xu - uX) \otimes v + u \otimes \pi_V(X)v = Xu \otimes v,$$

so the two actions of $\mathfrak{c}$ on $U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} V$ agree. Similarly, the two actions of X also agree on $\operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)$.

Equivariance is also true: for any $X \in \mathfrak{s}$,

$$\begin{aligned} cXc^{-1}(u \otimes v) &= \operatorname{Ad}(c)(X(Ad(c^{-1})u)) \otimes \pi_V(c)\pi_V(c^{-1})v \\ &= ((\operatorname{Ad}(c)X)u) \otimes v = (\operatorname{Ad}(c)X)(u \otimes v). \end{aligned}$$

The equivariance for $\operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)$ is similar.

Finally, the C-action on $U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} V$ is obviously finite, so it is an $(\mathfrak{s}, C)$-module. $\operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)$ is not necessarily C-finite. Therefore, we will take the C-finite part

$$\operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V)_C,$$

consisting of all C-finite vectors. It is an $(\mathfrak{s}, C)$-module. We already know how to transform morphisms under these two constructions: if $f : V \to W$ is an $(\mathfrak{r}, C)$-morphism, we have $\mathfrak{s}$-morphisms

$$\operatorname{id} \otimes f : U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} V \to U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} W$$

and

$$f_* : \operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), V) \to \operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), W)$$

These are trivially checked to be also C-morphisms; in particular, f_* sends the C-finite vectors to C-finite vectors. We thus see that $U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} \bullet$ and $\operatorname{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), \bullet)_C$ are functors from $M(\mathfrak{r}, C)$ to $M(\mathfrak{s}, C)$. To see that they are left respectively right adjoint to the forgetful functor, we only need to show that the adjunction morphisms from Section 5.2 respect also the C-action; everything else is already proved there. This is however obvious if we recall what these were: $v \mapsto 1 \otimes v$ and $\alpha \mapsto \alpha(1)$ are C morphisms since $\mathrm{Ad}\,(c)1 = 1$, and the $\mathfrak{s}$-action map is C-equivariant on any $(\mathfrak{s}, C)$-module by definition.

(Here one again uses the fact that any C-morphism sends C-finite vectors to C-finite ones; so if V is an $(\mathfrak{s}, C)$-module and if W is a space with compatible actions of $\mathfrak{s}$ and C but not necessarily C-finite, then $\mathrm{Hom}_{(\mathfrak{s},C)}(V, W) = \mathrm{Hom}_{(\mathfrak{s},C)}(V, W_C)$.)

So we have proved

Proposition 5.3.4. *Let $(\mathfrak{s}, C)$ and $(\mathfrak{r}, C)$ be pairs, and let $\tau : \mathfrak{r} \to \mathfrak{s}$ be a C-equivariant map whose restriction to $\mathfrak{c}$ is the identity. Then* $\mathrm{For} : M(\mathfrak{s}, C) \to M(\mathfrak{r}, C)$ *has both adjoints. The left adjoint is the functor $U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} \bullet$ and the right adjoint is the functor* $\mathrm{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), \bullet)_C$.

In particular, when τ is one-to-one, we get the (generalized) pro and ind functors from Section 5.1. By Poincaré-Birkhoff-Witt Theorem, we can write $U(\mathfrak{s}) = U(\mathfrak{r}) \otimes \Lambda = \Lambda \otimes U(\mathfrak{r})$, where Λ is spanned by monomials in a basis of a C-invariant direct complement of $\mathfrak{r}$ in $\mathfrak{s}$. It follows that we can write

$$\mathrm{ind}\,(V) = \Lambda \otimes_{\mathbb{C}} V \quad \text{and} \quad \mathrm{pro}\,(V) = \mathrm{Hom}_{\mathbb{C}}(\Lambda, V)_C.$$

It is therefore clear that ind is an exact functor; however, pro is also exact. Namely, the C-module structure of $\mathrm{Hom}_{\mathbb{C}}(\Lambda, V)_C$ depends only on the C-module structure of V, and exactness of a sequence can be checked on the level of C-modules. But the category of (smooth finite) C-modules is semisimple and hence all functors from this category are exact.

Corollary 5.3.5. *If the above map τ is one-to-one, then the adjoints of* $\mathrm{For} : M(\mathfrak{s}, C) \to M(\mathfrak{r}, C)$ *are exact functors* pro *and* ind . *In particular,* For *sends projectives to projectives and injectives to injectives.*

The last claim follows from 5.2.12. A typical example of this situation would be $\mathfrak{r} = \mathfrak{l}$, a Levi subalgebra of $\mathfrak{g}$, $\mathfrak{s} = \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ a parabolic subalgebra, and $C = L \cap K$ - so C preserves $\mathfrak{u}$. In this case, the above Λ can be taken to be $U(\mathfrak{u})$.

The other case we want to single out is the case when τ is a projection along an ideal $\mathfrak{i}$ of $\mathfrak{r}$. In this case, $U(\mathfrak{s}) \otimes_{U(\mathfrak{r})} \bullet$ and $\mathrm{Hom}_{U(\mathfrak{r})}(U(\mathfrak{s}), \bullet)_C$ are the functors of coinvariats respectively invariants with respect to $\mathfrak{i}$, taking $(\mathfrak{r}, C)$-modules into $(\mathfrak{r}/\mathfrak{i}, C)$-modules. These functors are no longer exact, and we will study their derived functors, $\mathfrak{i}$-homology respectively $\mathfrak{i}$-cohomology functors. A typical case like this is $\mathfrak{r} = \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, a parabolic subalgebra, with $\tau : \mathfrak{q} \to \mathfrak{l}$ the projection along $\mathfrak{u}$. Here $\mathfrak{q}/\mathfrak{u}$ can be identified with $\mathfrak{l}$, hence $\mathfrak{u}$-invariants and coinvariants (and $\mathfrak{u}$-cohomology and homology) will be $(\mathfrak{l}, C)$-modules.

5.3.6. Construction of projectives and injectives. To construct enough projectives and injectives for modules over any pair $(\mathfrak{s}, C)$, one can use ind and pro from the category of $(\mathfrak{c}, C)$-modules. Namely, it follows from 5.3.5 and 5.2.12 that pro sends injectives to injectives and ind sends projectives to projectives. On the other hand, the category $M(\mathfrak{c}, C)$ is semisimple, so all its objects are projective and injective. This means that $(\mathfrak{s}, C)$-modules of the form $U(\mathfrak{s}) \otimes_{U(\mathfrak{c})} V$ will be projective and $(\mathfrak{s}, C)$-modules of the form $\mathrm{Hom}_{U(\mathfrak{c})}(U(\mathfrak{s}), V)_C$ will be injective. (These are called standard projectives respectively injectives.)

This is true for any $(\mathfrak{c}, C)$-module V, but if we start with an $(\mathfrak{s}, C)$-module V, then the adjunction morphism $U(\mathfrak{s}) \otimes_{U(\mathfrak{c})} V \to V$ is onto, and the adjunction morphism $V \to \mathrm{Hom}_{U(\mathfrak{c})}(U(\mathfrak{s}), V)_C$ is one to one, and we get that every $(\mathfrak{s}, C)$-module is a quotient of a projective $(\mathfrak{s}, C)$-module and a submodule of an injective $(\mathfrak{s}, C)$-module.

5.3.7. Koszul resolutions. It will be convenient to have at hand some very explicit projective resolutions. They are provided by various *Koszul complexes.* These are resolutions of the trivial module $\mathbb{C}$ in various categories; resolutions of other modules are then obtained simply by tensoring the module with the Koszul complex (or taking Hom). One simple consequence will be that we can always find resolutions which are not longer than the length of the Koszul complex, which is finite (for $(\mathfrak{s}, C)$-modules, the length is $\dim \mathfrak{s}/\mathfrak{c}$). In other words, our categories have finite homological dimension.

We start with the simplest case, the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$. This algebra can be identified with $S(V)$, where V is the dual space of $\mathbb{C}^n$. We have thus already studied the associated Koszul complex, $S(V) \otimes \bigwedge^{\cdot}(V)$, in Section 3.3. Namely, the $S(V)$-modules $S(V) \otimes \bigwedge^i(V)$, with $S(V)$ acting by multiplication in the first factor, are obviously free. Furthermore, it follows from Proposition 3.3.5 that the complex

$$0 \to S(V) \otimes \bigwedge^n(V) \xrightarrow{d} S(V) \otimes \bigwedge^{n-1}(V) \xrightarrow{d} \quad \dots$$

$$\dots \qquad \xrightarrow{d} \quad S(V) \otimes \bigwedge^0(V) \quad \xrightarrow{e} \mathbb{C} \to 0$$

where $n = \dim V$, e is the evaluation at 0, and d is the Koszul differential, is exact. So the Koszul complex is a free (and hence projective) resolution of $\mathbb{C}$.

5.3.8. Koszul complex for a Lie algebra. For a Lie algebra $\mathfrak{s}$, we claim that a resolution of the trivial $\mathfrak{s}$-module $\mathbb{C}$ is given by

$$0 \to U(\mathfrak{s}) \otimes \bigwedge^n(\mathfrak{s}) \xrightarrow{d} \dots \xrightarrow{d} U(\mathfrak{s}) \otimes \bigwedge^0(\mathfrak{s}) \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $n = \dim \mathfrak{s}$, ϵ is the projection along $\mathfrak{s}U(\mathfrak{s}) \otimes 1$, and d is given by

$$d(u \otimes x_1 \wedge \cdots \wedge x_r) = \sum_{i=1}^{r} (-1)^{i-1} u x_i \otimes x_1 \wedge \ldots \hat{x}_i \cdots \wedge x_r$$
$$+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \hat{x}_i \ldots \hat{x}_j \cdots \wedge x_r.$$

It is a routine exercise to show that $d^2 = 0$; note how it is ensured exactly by the introduction of commutators. Furthermore, let us filter $U(\mathfrak{s})$ by degree, grade $\bigwedge^{\cdot}(\mathfrak{s})$ by degree, and consider the total degree on $U(\mathfrak{s}) \otimes \bigwedge^{\cdot}(\mathfrak{s})$. Let F_k be the degree k filtered piece of this filtration. It is clear from the formula for d that F_k is a subcomplex, and that $Gr_k = F_k/F_{k-1}$ is exactly the degree k part of the Koszul complex of $\mathfrak{s}$, $S(\mathfrak{s}) \otimes \bigwedge^{\cdot}(\mathfrak{s})$ considered in 5.3.7. In particular, Gr_k is exact for any k, and we see from the long exact sequence of cohomology corresponding to the short exact sequence

$$0 \to F_{k-1} \to F_k \to Gr_k \to 0$$

of complexes that F_k and F_{k-1} have the same cohomology. But $F_{-1} = 0$ has cohomology 0, hence all F_k are exact. We conclude that their union, $U(\mathfrak{s}) \otimes \bigwedge^{\cdot}(\mathfrak{s})$ is also exact.

This is the original Koszul's proof, taken from [CE].

5.3.9. Koszul complex for a pair $(\mathfrak{s}, C)$**.** Let $\mathfrak{p}$ be a C-invariant direct complement of $\mathfrak{c}$ in $\mathfrak{s}$, and let $\pi : \mathfrak{s} \to \mathfrak{p}$ be the C-equivariant projection. The Koszul resolution of the trivial $(\mathfrak{s}, C)$-module $\mathbb{C}$ is

$$0 \to U(\mathfrak{s}) \otimes_{U(\mathfrak{c})} \bigwedge^{p}(\mathfrak{p}) \xrightarrow{d} \ldots \xrightarrow{d} U(\mathfrak{s}) \otimes_{U(\mathfrak{c})} \bigwedge^{0}(\mathfrak{p}) \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $p =$ dim $\mathfrak{p}$, ϵ is the projection along $\mathfrak{s}U(\mathfrak{s}) \otimes 1$, and d is given by

$$d(u \otimes x_1 \wedge \cdots \wedge x_r) = \sum_{i=1}^{r} (-1)^{i-1} u x_i \otimes x_1 \wedge \ldots \hat{x}_i \cdots \wedge x_r$$
$$+ \sum_{i<j} (-1)^{i+j} u \otimes \pi([x_i, x_j]) \wedge x_1 \wedge \ldots \hat{x}_i \ldots \hat{x}_j \cdots \wedge x_r.$$

Note that if $\mathfrak{c}$ is a symmetric subalgebra of $\mathfrak{s}$, like we have for the pair $(\mathfrak{g}, K)$, then the second (double) sum disappears.

As usual, one checks that $d^2 = 0$ and we want to show this is a resolution (we do know the objects in question are projective - these are the standard projectives of 5.3.6.)

By taking a basis of $\mathfrak{c}$ together with a basis of $\mathfrak{p}$ to form a basis of $\mathfrak{s}$, we can use the Poincaré-Birkhoff-Witt theorem to define a filtration of $U(\mathfrak{s})$ by $\mathfrak{p}$-degree: each PBW monomial gets a degree equal to the total number of elements of the $\mathfrak{p}$-part of the basis in it. (Note that this does not define a grading, but it does define a filtration.) Together with the grading of $\bigwedge^{\cdot}(\mathfrak{p})$ by degree, this gives a filtration F_k of our complex. The graded object is

$$S(\mathfrak{p}) \otimes U(\mathfrak{c}) \otimes_{U(\mathfrak{c})} \bigwedge^{\cdot}(\mathfrak{p}) = S(\mathfrak{p}) \otimes \bigwedge^{\cdot}(\mathfrak{p}),$$

with the differential equal to the Koszul differential $d_{\mathfrak{p}}$ of the polynomials over the vector space $\mathfrak{p}$. In particular, Gr_k is exact for any k, and hence we conclude that our "relative Koszul complex" is also exact, by the same argument as in 5.3.8.

5.3.10. Resolving nontrivial modules. We are now going to use two constructions inside the category $M(\mathfrak{s}, C)$: for any two $(\mathfrak{s}, C)$-modules V and W we can construct $(\mathfrak{s}, C)$ modules $V \otimes W$ and $\mathrm{Hom}_{\mathbb{C}}(V, W)_C$. This construction is very familiar; both $\mathfrak{s}$ and C act on both factors. Moreover, by fixing one variable and varying the other we get four functors from $M(\mathfrak{s}, C)$ into itself. In fact we get three, as $V \otimes W \cong W \otimes V$ in the obvious way. All these functors are exact. The following fact is standard and easy to check:

Lemma 5.3.11. *For any $V \in M(\mathfrak{s}, C)$, the functor $V \otimes \bullet$ is left adjoint to the functor $\mathrm{Hom}_{\mathbb{C}}(V, \bullet)_C$.*

In particular, this means that $V \otimes \bullet$ sends projectives to projectives and that $\mathrm{Hom}_{\mathbb{C}}(V, \bullet)_C$ sends injectives to injectives. Moreover, it follows from the noted symmetry of the tensor product that the contravariant functor $\mathrm{Hom}_{\mathbb{C}}(\bullet, V)_C$ sends projectives to injectives. Namely, if W is projective, then the functor

$$\begin{aligned} \mathrm{Hom}_{(\mathfrak{s},C)}(\bullet, \mathrm{Hom}_{\mathbb{C}}(W, V)_C) &= \mathrm{Hom}_{(\mathfrak{s},C)}(W \otimes \bullet, V) \\ &= \mathrm{Hom}_{(\mathfrak{s},C)}(W, \mathrm{Hom}_{\mathbb{C}}(\bullet, V)_C) \end{aligned}$$

is exact, so $\mathrm{Hom}_{\mathbb{C}}(W, V)_C$ is injective.

This implies the following proposition.

Proposition 5.3.12. *For any $(\mathfrak{s}, C)$ module V,*

$$V \otimes U(\mathfrak{s}) \otimes_{U(\mathfrak{c})} \bigwedge^{\cdot}(\mathfrak{p}) \to V$$

is a projective resolution of V, while

$$V \to \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{s}) \otimes_{U(\mathfrak{c})} \bigwedge^{\cdot}(\mathfrak{p}), V)_C$$

is an injective resolution of V. In particular, the homological dimension of $M(\mathfrak{s}, C)$ is at most $\dim \mathfrak{p}$.

An important special case of the above considerations is when V is finite dimensional. Then $\mathrm{Hom}_{\mathbb{C}}(V, \bullet)_C$ is the same as $V^* \otimes \bullet$, where V^* is contragredient to V. As $V^{**} = V$, this shows that for finite dimensional V, tensoring with V is both left and right adjoint to tensoring with V^*, and in particular

Corollary 5.3.13. *If V is a finite dimensional $(\mathfrak{s}, C)$-module, then the functor $V \otimes \bullet$ sends projectives to projectives and injectives to injectives.*

Another important functor arising in the above context is the *C-finite duality*, which is defined by

$$V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})_C,$$

for an $(\mathfrak{s}, C)$-module V (the actions are contragredient). This functor is exact. The double dual is the identity on admissible modules, i.e., those that contain every irreducible C-module only finitely many times. The main general property is

Corollary 5.3.14. *For any two $(\mathfrak{s}, C)$-modules V and W we have*

$$\operatorname{Hom}_{(\mathfrak{s},C)}(V, W^*) = \operatorname{Hom}_{(\mathfrak{s},C)}(W, V^*).$$

Proof. We can write

$$\begin{aligned} \operatorname{Hom}_{(\mathfrak{s},C)}(V, \operatorname{Hom}_{\mathbb{C}}(W, \mathbb{C})_C) &= \operatorname{Hom}_{(\mathfrak{s},C)}(W \otimes V, \mathbb{C}) \\ &= \operatorname{Hom}_{(\mathfrak{s},C)}(W, \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})_C) \end{aligned}$$

Note that the above property is a kind of "self adjunction" of the duality functor. More precisely, the duality understood as a functor from the opposite category $M(\mathfrak{s}, C)^o$ into $M(\mathfrak{s}, C)$ is right adjoint to its opposite functor, going from $M(\mathfrak{s}, C)$ to $M(\mathfrak{s}, C)^o$. The contravariant functor $\operatorname{Hom}_{\mathbb{C}}(\bullet, V)_C$ has an analogous property for any V.

5.4 Zuckerman functors

The key idea of cohomological induction is to consider the *Zuckerman functor* Γ and its derived functors.

5.4.1. Definition of the functor Γ. Let $(\mathfrak{s}, C)$ be a pair and let T be a closed subgroup of C. Then $(\mathfrak{s}, T)$ is obviously another pair. We are mainly interested in cases $\mathfrak{s} = \mathfrak{g}$ or $\mathfrak{s} = \mathfrak{k}$, $C = K$ and $T = L \cap K$. We will therefore switch to the notation $(\mathfrak{g}, K)$ and $(\mathfrak{g}, T)$, although nothing we do will depend on this particular case.

There is a forgetful functor $\operatorname{For} : M(\mathfrak{g}, K) \to M(\mathfrak{g}, T)$. By definition, Zuckerman functor $\Gamma = \Gamma_{K,T}$ is the right adjoint of this forgetful functor.

To construct Γ, recall that we have a good candidate from Section 5.1:

$$\Gamma V = (R(K) \otimes V)^{(\mathfrak{k},T)},$$

with $(\mathfrak{k}, T)$-invariants taken with respect to the left regular action L tensored with the given action π_V. K-action is given by the right regular action on $R(K)$, which is clearly well defined on invariants, as it commutes with $L \otimes \pi_V$. We have to define a $\mathfrak{g}$-action and show that we get a $(\mathfrak{g}, K)$-module.

Note that we cannot simply take the $\mathfrak{g}$-action π_V we have on V, as this would not be well defined on $(\mathfrak{k}, T)$-invariants, and also it would commute with the K-action $R \otimes 1$ instead of being K-equivariant. What is needed is some twist with respect to the $R(K)$ factor.

To describe and later handle the $\mathfrak{g}$-action it is useful to interpret $R(K) \otimes V$ as the space

$$R(K) \otimes V = R(K, V)$$

of finite range V-valued smooth finite functions on K; here $f \otimes v$ gets identified with the function $k \mapsto f(k)v$. The action of $X \in \mathfrak{g}$ is now given on $F \in R(K, V)$ by

$$(XF)(k) = \pi_V(\operatorname{Ad}(k)X)F(k), \qquad k \in K.$$

We can also write it in tensor notation: if $X_i \in \mathfrak{g}$ and $f_i \in R(K)$ are such that $\operatorname{Ad}(k)X = \sum f_i(k)X_i$, then

$$X(f \otimes v) = \sum_i f f_i \otimes \pi_V(X_i)v.$$

To see that this $\mathfrak{g}$-action is well defined on $(\mathfrak{k}, T)$-invariants, we first note that the action $L \otimes \pi_V$ on $F \in R(K, V)$ is given by

$$(\lambda(t)F)(k) = \pi_V(t)F(t^{-1}k), \qquad t \in T,\, k \in K,$$

and

$$\lambda(X)F = \pi_V(X) \circ F + L_X F, \qquad X \in \mathfrak{k},$$

where as usual $L_X F(k) = \frac{\mathrm{d}}{\mathrm{d}t} F(\exp(-tX)k)\big|_{t=0}$. So we have

$$\begin{aligned}(\lambda(t)XF)(k) &= \pi_V(t)(XF)(t^{-1}k) = \pi_V(t)\pi_V(\operatorname{Ad}(t^{-1}k)X)F(t^{-1}k)\\ &= \pi_V(\operatorname{Ad}(k)X)\pi_V(t)F(t^{-1}k) = (X\lambda(t)F)(k).\end{aligned}$$

Similarly, the $\mathfrak{g}$-action commutes with the λ-action of $\mathfrak{k}$. In particular, the $\mathfrak{g}$-action is well defined on $(\mathfrak{k}, T)$-invariants.

We show next that the $\mathfrak{g}$-action is K-equivariant:

$$\begin{aligned}(kXk^{-1}F)(k') &= (Xk^{-1}F)(k'k) = \pi_V(\operatorname{Ad}(k'k)X)(k^{-1}F)(k'k)\\ &= \pi_V(\operatorname{Ad}(k')\operatorname{Ad}(k)X)F(k') = ((\operatorname{Ad}(k)X)F)(k').\end{aligned}$$

Finally, the action R_X of $X \in \mathfrak{k}$ should agree with the action of X as an element of $\mathfrak{g}$; this will be true on invariants. Namely, if F is $(\mathfrak{k}, T)$-invariant, then

$$L_X F = -\pi_V(X) \circ F.$$

Using this, we see

$$\begin{aligned}(R_X F)(k) &= \frac{\mathrm{d}}{\mathrm{d}t} F(k \exp(tX))\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} F(k \exp(tX)k^{-1}k)\big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t} F(\exp(t \operatorname{Ad}(k)X)k)\big|_{t=0} = -(L_{\operatorname{Ad}(k)X}F)(k)\\ &= \pi_V(\operatorname{Ad}(k)X)F(k) = (XF)(k).\end{aligned}$$

So, we have indeed constructed a $(\mathfrak{g}, K)$-module ΓV. To make Γ into a functor, we define it on morphisms, by

$$\Gamma(\alpha : V \to W) = \text{ id } \otimes \alpha : (R(K) \otimes V)^{(\mathfrak{k},T)} \to (R(K) \otimes V)^{(\mathfrak{k},T)}.$$

One easily checks that this is a well defined $(\mathfrak{g}, K)$-morphism.

5.4.2. Adjunction. We now want to show that Γ is right adjoint to the forgetful functor. For this we need to exhibit adjunction morphisms.

For a $(\mathfrak{g}, K)$-module V, let

$$\Phi_V : V \to R(K, V)^{(\mathfrak{k},T)}, \qquad \Phi_V(v)(k) = \pi_V(k)v$$

be the "matrix coefficient map", given by the action of K on V.

For a $(\mathfrak{g}, T)$-module W, define

$$\Psi_W : R(K, W)^{(\mathfrak{k},T)} \to W, \qquad \Psi_W(F) = F(1).$$

Note the analogy with the change of rings functors.

To show that Φ_V does finish in the $(\mathfrak{k}, T)$-invariants, we calculate

$$\begin{aligned}(\lambda(t)\Phi_V(v))(k) = \pi_V(t)\Phi_V(v)(t^{-1}k) &= \pi_V(t)\pi_V(t^{-1}k)v \\ &= \pi_V(k)v = \Phi_V(v)(k),\end{aligned}$$

i.e., $\Phi_V(v)$ is T-invariant. By a similar calculation it is also $\mathfrak{k}$-invariant. Furthermore, Φ_V is a $(\mathfrak{g}, K)$-morphism: for $X \in \mathfrak{g}$,

$$\Phi_V(\pi_V(X)v)(k) = \pi_V(k)\pi_V(X)v = \pi_V(\text{ Ad }(k)X)\pi_V(k)v = (X\Phi_V(v))(k),$$

so Φ_V is a $\mathfrak{g}$-morphism and similarly it is a K-morphism.

Ψ_W is obviously well defined. It is a $(\mathfrak{g}, T)$-morphism: for $X \in \mathfrak{g}$

$$\Psi_W(XF) = (XF)(1) = \pi_W(\text{ Ad }(1)X)F(1) = \pi_W(X)\Psi_W(F),$$

so Ψ_W is a $\mathfrak{g}$-morphism and similarly it is a T-morphism.

To finish the proof of adjunction, we have to check that the maps (5.1) and (5.2) from 5.2.4 are the identity maps. The map (5.1) is

$$V \xrightarrow{\Phi_V} \Gamma V \xrightarrow{\Psi_V} V, \qquad v \mapsto (k \mapsto \pi_V(k)v) \mapsto \pi_V(1)v = v$$

so it is the identity map. The map (5.2) is the map

$$\Gamma W \xrightarrow{\Phi_{\Gamma W}} \Gamma\Gamma W \xrightarrow{\Gamma(\Psi_W)} \Gamma W,$$

given by $F \mapsto (k \mapsto R_k F) \mapsto (k \mapsto (R_k F)(1) = F(k))$, and this is again the identity map. (Note that we have omitted the forgetful functors in the above calculations.) So we have proved:

Theorem 5.4.3. *The forgetful functor from $M(\mathfrak{g},K)$ to $M(\mathfrak{g},T)$, has a right adjoint, the Zuckerman functor Γ. It can be constructed as $\Gamma V = (R(K) \otimes V)^{(\mathfrak{k},T)}$.*

5.4.4. Derived Zuckerman functors. Since the category $M(\mathfrak{g},T)$ has enough injectives, the right derived functors $\Gamma^i = R^i\Gamma$ are defined. To calculate $\Gamma^i(V)$, one should take an injective resolution $V \to I^\cdot$, and take cohomology of the complex $\Gamma(I^\cdot)$. This is not so great, because injective modules are big and not easy to work with.

However, the formula we used for Γ, $\Gamma(V) = (R(K) \otimes V)^{(\mathfrak{k},T)}$ suggests that we could try to calculate Γ^i as derived invariants functor, i.e., $(\mathfrak{k},T)$-cohomology of the $(\mathfrak{k},T)$-module $R(K) \otimes V$. We are going to show that this is indeed possible.

By definition, if $X \to I^\cdot$ is an injective resolution of X in $M(\mathfrak{g},K)$, then

$$\Gamma^i(X) = H^i(\operatorname{Hom}_{(\mathfrak{k},T)}(\mathbb{C}, R(K) \otimes I^\cdot)).$$

Since the forgetful functor from $M(\mathfrak{g},K)$ to $M(\mathfrak{g},T)$ has an exact left adjoint $\operatorname{ind}{}_{\mathfrak{k}}^{\mathfrak{g}}$, I^j are injective as $(\mathfrak{k},T)$-modules. Tensoring them with finite dimensional $(\mathfrak{k},T)$-modules gives again injectives, by Corollary 5.3.13. On the other hand, $R(K)$ is a direct sum of finite dimensional K-modules. If we could show that $R(K) \otimes I^j$ are injective $(\mathfrak{k},T)$-modules, we could conclude that $\Gamma^i(X)$ is the $(\mathfrak{k},T)$ cohomology of the $(\mathfrak{k},T)$-module $R(K) \otimes V$. However, an infinite direct sum of injectives is not necessarily injective. It is however acyclic for $(\mathfrak{k},T)$-cohomology, i.e., if J_n, $n = 1,2,\dots$ are injective $(\mathfrak{k},T)$-modules, then

$$H^i(\mathfrak{k},T;\oplus_n J_n) = 0, \qquad i > 0.$$

This follows at once from the fact that taking $(\mathfrak{k},T)$-cohomology commutes with direct sums.

What we thus need to know is that we can calculate derived functors not only from injective resolutions, but also from resolutions by objects acyclic for the functor in question. This is a well known and general fact from homological algebra; see for example [T], Theorem 4.4.6. So we can identify $\Gamma^i(V)$ with $H^i(\mathfrak{k},T;R(K) \otimes V)$, which we can in turn get as cohomology of the complex

$$\operatorname{Hom}_{(\mathfrak{k},T)}(U(\mathfrak{k}) \otimes_{U(\mathfrak{t})} \textstyle\bigwedge^\cdot(\mathfrak{o}), R(K) \otimes V),$$

where we denote by $\mathfrak{o}$ a T-invariant direct complement of $\mathfrak{t}$ in $\mathfrak{k}$. In other words, we can resolve the first variable, $\mathbb{C}$, instead of the second in $\operatorname{Hom}_{(\mathfrak{k},T)}(\mathbb{C}, R(K) \otimes V)$. The $(\mathfrak{g},K)$-action is given as before, on the second variable only; this follows from the functoriality of the whole construction. In particular, the cohomology of the above complex is a $(\mathfrak{g},K)$-module.

We can also use the adjunction of $\operatorname{ind}{}_{\mathfrak{t}}^{\mathfrak{k}}$ and For to write $\operatorname{Hom}_{(\mathfrak{k},T)}(U(\mathfrak{k}) \otimes_{U(\mathfrak{t})} Z, W) = \operatorname{Hom}_T(Z,W)$. So we have proved:

Theorem 5.4.5. *The derived Zuckerman functors of a $(\mathfrak{g}, T)$-module V can be expressed as*

$$\Gamma^i(V) = H^i(\operatorname{Hom}_T(\textstyle\bigwedge^{\cdot}(\mathfrak{o}), R(K) \otimes V)),$$

with $(\mathfrak{g}, K)$ action coming from the one described earlier on $R(K) \otimes V$. Here $\mathfrak{o}$ is a T-invariant direct complement of $\mathfrak{t}$ in $\mathfrak{k}$, and T acts by the adjoint action on $\mathfrak{o}$ and by the action $\lambda = L \otimes \pi_V$ on $R(K) \otimes V$.

5.4.6. Some $SL(2)$ examples. As we saw in Example 5.1.5, while cohomologically inducing from $L = K$ in the $SL(2)$ case, we did not need to apply Zuckerman functors to get from $(\mathfrak{g}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules, simply because $L \cap K = K$. We can however see what the above construction gives for $T = \{1\} \subset K$.

In this case, $\mathfrak{t} = 0$, and $\mathfrak{o} = \mathfrak{k}$. Recall the basis

$$W = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad u = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \qquad \bar{u} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$$

of $\mathfrak{g}$ from Example 5.1.5 (and also 1.3.10). Since $\mathfrak{k} = \mathbb{C}W$ is one-dimensional, the complex $C^{\cdot}$ that calculates derived Zuckerman functors is

$$0 \to \operatorname{Hom}_{\mathbb{C}}(\textstyle\bigwedge^0(\mathfrak{k}), R(K) \otimes V) \xrightarrow{\ d\ } \operatorname{Hom}_{\mathbb{C}}(\textstyle\bigwedge^1(\mathfrak{k}), R(K) \otimes V) \to 0.$$

Now both C^0 and C^1 can be identified with $R(K) \otimes V$; for C^0, $f \otimes v$ gets identified with the linear map given by $1 \mapsto f \otimes v$, and for C^1, $f \otimes v$ gets identified with the linear map given by $W \mapsto f \otimes v$. Since the Koszul differential for $\mathfrak{k}$ is given by $d(a \otimes W) = aW$ on $a \otimes W \in U(\mathfrak{k}) \otimes \bigwedge^1(\mathfrak{k})$, in the above identifications we have

$$d(f \otimes v) = (L \otimes \pi_V)(W)(f \otimes v) = L_W f \otimes \pi_V(W)v.$$

In particular, we see that

$$\Gamma^0(V) = (R(K) \otimes V)^{\mathfrak{k}} \qquad \text{and} \qquad \Gamma^1(V) = (R(K) \otimes V)_{\mathfrak{k}};$$

the $\mathfrak{k}$-invariants and coinvariants of $R(K) \otimes V$.

Let us now consider some particular examples of V. The first one is the trivial module $\mathbb{C}$; note that this already is a $(\mathfrak{g}, K)$-module. Now $R(K) \otimes \mathbb{C} = R(K)$. We can decompose

$$R(K) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} f_n,$$

where f_n is the character $k \mapsto k^n$ of $K = S^1$. In particular, $L_W f_n = -n f_n$. So we see that both the invariants and the coinvariants of $R(K)$ with respect to the left regular action consist of the constants $\mathbb{C} = \mathbb{C} f_0$. The $(\mathfrak{g}, K)$-action can only be trivial, so we conclude

$$\Gamma^0(\mathbb{C}) = \mathbb{C}, \qquad \Gamma^1(\mathbb{C}) = \mathbb{C}.$$

Next, let us consider another module which already is a $(\mathfrak{g}, K)$-module: the disrete series representation with lowest weight k, $V = \bigoplus_{i\geq 0} \mathbb{C}v_{k+2i}$. Now

$$R(K) \otimes V = \bigoplus_{n\in\mathbb{Z}, i\in\mathbb{Z}_+} \mathbb{C}f_n \otimes v_{k+2i},$$

and the action of W with respect to which we are taking invariants and coinvariants is

$$W : f_n \otimes v_{k+2i} \mapsto (-n + k + 2i) f_n \otimes v_{k+2i}.$$

We see again that invariants and coinvariants are the same, and equal to

$$\bigoplus_{i\in\mathbb{Z}_+} \mathbb{C}f_{k+2i} \otimes v_{k+2i}.$$

The K-action on this space is the right regular action; so it is isomorphic to V as a K-module. It is thus also isomorphic to V as a $(\mathfrak{g}, K)$-module, as V is the only $(\mathfrak{g}, K)$-module with weights $k, k+2, k+4, \ldots$. So we again see

$$\Gamma^0(V) = V, \qquad \Gamma^1(V) = V.$$

In fact, one can show in a general situation, that whenever V is already a $(\mathfrak{g}, K)$-module, then $\Gamma^i(V) = H^i(\mathfrak{k}, T; R(K)) \otimes V$. Moreover, for connected K, $H^i(\mathfrak{k}, T; R(K)) = H^i(\mathfrak{k}, T; \mathbb{C})$.

We now turn our attention to modules V which are not $(\mathfrak{g}, K)$-modules, but only $\mathfrak{g}$-modules. One such module is $V = U(\mathfrak{g})$, with $\mathfrak{g}$-action being the left multiplication.

The action of W on $R(K) \otimes U(\mathfrak{g})$ is

$$W : f_n \otimes v \mapsto -n f_n \otimes v + f_n \otimes Wv.$$

Since the degree of Wv is greater than the degree of v, it is easy to see that there are no $\mathfrak{k}$-invariants in $R(K) \otimes U(\mathfrak{g})$. The space of coinvariants can be identified with

$$R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}),$$

where the right action of $U(\mathfrak{k})$ on $R(K)$ is given by twisting the action L_W, so that W acts as L_{-W}. Incidentally, this is the same as R_W, but that is only because $\mathfrak{k}$ is abelian. Since $U(\mathfrak{g})$ is free over $U(\mathfrak{k})$, the space is quite large. It can actually be turned into a convolution algebra. This is the Hecke algebra, studied extensively in [KV].

5.5 Bernstein functors

5.5.1. Definition of Π. We keep the same setting as in the previous section: $(\mathfrak{g}, K)$ is a pair (most often the usual $(\mathfrak{g}, K)$ coming from a real reductive group) and T is a closed subgroup of K (for example, $L \cap K$ for L coming from a θ-stable parabolic subalgebra of $\mathfrak{g}$).

We define the *Bernstein functor* $\Pi : M(\mathfrak{g}, T) \to M(\mathfrak{g}, K)$ by

$$\Pi(V) = (R(K) \otimes V)_{(\mathfrak{k},T)}.$$

The $(\mathfrak{k}, T)$-coinvariants are taken with respect to the same $(\mathfrak{k}, T)$-action on $R(K) \otimes V$ as in the last section: $L \otimes \pi_V$, the left regular action on $R(K)$ tensored with the given action on V.

In the last section we defined actions of $\mathfrak{g}$ and K on $R(K) \otimes V$, commuting with the $(\mathfrak{k}, T)$-action $L \otimes \pi_V$. So these actions descend to coinvariants, just like they restricted to actions on invariants. Furthermore, the $\mathfrak{g}$-action, which was given by twisted π_V, was K-equivariant already on the level of $R(K) \otimes V$, so it stays equivariant after passing to $(\mathfrak{k}, T)$-coinvariants. (Recall that the K-action is the right regular representation in the first factor.)

The proof that the two actions of $\mathfrak{k}$, one given by differentiating the K-action and the other given by restricting the $\mathfrak{g}$-action, agree (so that $\Pi(V)$ is a $(\mathfrak{g}, K)$-module), is similar to the proof for the Zuckerman functor, but a little more complicated. Let $F \in R(K, V)$, let $X \in \mathfrak{k}$ and suppose $\mathrm{Ad}\,(k)X = \sum_i f_i(k)X_i$, $k \in K$. Then

$$\begin{aligned}(XF - R_X F)(k) &= \pi_V(\,\mathrm{Ad}\,(k)X)(F(k)) + (L_{\mathrm{Ad}\,(k)X} F)(k) \\ &= \sum_i f_i(k)\pi_V(X_i)(F(k)) + f_i(k)L_{X_i}F(k).\end{aligned}$$

What we want is rather

$$\sum_i \big(\pi_V(X_i) \circ (f_i F) + L_{X_i}(f_i F)\big)(k);$$

this would obviously be zero after passing to coinvariants. The two expressions are actually the same because of the following lemma and the fact that $\mathfrak{k}$ is unimodular.

Lemma 5.5.2. *Let $X \in \mathfrak{k}$, and let $\mathrm{Ad}\,(k)X = \sum_{i=1}^n f_i(k)X_i$, $k \in K$. Then the function*

$$\sum_{i=1}^n L_{X_i} f_i$$

is a constant function on K, with value $\mathrm{tr}\;\mathrm{ad}\,X$. In particular, if $\mathfrak{k}$ is unimodular, then this function is 0.

Proof. Let $F \in R(K, \mathfrak{k})$ be given by $F(k) = \mathrm{Ad}\,(k)X$; the corresponding element of $R(K) \otimes \mathfrak{k}$ is then $\sum_i f_i \otimes X_i$. Since

$$((L \otimes \mathrm{Ad}\,)(k)F)(k') = \mathrm{Ad}\,(k)(F(k^{-1}k')) = \mathrm{Ad}\,(k)\,\mathrm{Ad}\,(k^{-1}k')X$$
$$= \mathrm{Ad}\,(k')X = F(k'),$$

F is invariant for the action $L \otimes \mathrm{Ad}$ of K.

Let $a : R(K) \otimes \mathfrak{k} \to R(K)$ be defined by $a(f \otimes Y) = L_Y f$. This is actually the $\mathfrak{k}$-action map for the $(\mathfrak{k}, K)$-module $R(K)$ with respect to the left regular representation L. Thus, by the equivariance condition, a intertwines the action $L \otimes \mathrm{Ad}$ of K on $R(K) \otimes \mathfrak{k}$ with L. It follows that

$$\sum_i L_{X_i} f_i = a(\sum_i f_i \otimes X_i)$$

is in the K-invariants of $R(K)$ with respect to L, i.e., it is a constant function on K.

To see the value of this function, we calculate it at $e \in K$. In writing

$$\mathrm{Ad}\,(k)X = \sum_{i=1}^{n} f_i(k) X_i, \qquad k \in K,$$

we can clearly assume that $\{X_i\}$ is a basis of $\mathfrak{k}$ (take any X_i's and write them in a basis). Setting $k = \exp tX_j$, $t \in \mathbb{R}$, deriving with respect to t and then setting $t = 0$, we obtain

$$[X_j, X] = \sum_{i=1}^{n} \frac{d}{dt} f_i(\exp tX_j) X_i\big|_{t=0} = \sum_{i=1}^{n} (L_{-X_j} f_i)(e) X_i;$$

hence

$$[X, X_j] = \sum_{i=1}^{n} (L_{X_j} f_i)(e) X_i.$$

In other words, the matrix of $\mathrm{ad}\, X$ is given by

$$(\,\mathrm{ad}\, X)_{ij} = (L_{X_j} f_i)(e)$$

in the basis $\{X_i\}$, and it follows that

$$\sum_{i=1}^{n} (L_{X_i} f_i)(e) = \mathrm{tr}\ \mathrm{ad}\, X.$$

So we see that $\Pi(V)$ is a $(\mathfrak{g}, K)$-module. To make Π into a functor, we define it on morphisms by

$$\Pi(f : V \to W) = \mathrm{id}\, \otimes f : (R(K) \otimes V)_{(\mathfrak{k},T)} \to (R(K) \otimes W)_{(\mathfrak{k},T)}$$

which is obviously well defined on $(\mathfrak{k}, T)$-coinvariants.

Since taking coinvariants is right exact, we conclude that Π is also right exact. So we will consider the left derived functors of Π. We are going to show that they are given by $(\mathfrak{k}, T)$-homology.

5.5.3. Derived Bernstein functors. The proof that

$$\Pi_i(V) = L^i\Pi(V) = H_i(\mathfrak{k}, T; R(K) \otimes V)$$

is analogous and even simpler than the proof in 5.4.4. Namely, by definition, $\Pi_i(V)$ is the $(-i)$-th cohomology of the complex

$$\pi(P^\cdot) = (R(K) \otimes P^\cdot)_{(\mathfrak{k},T)}, \tag{5.3}$$

where $P^\cdot \to V \to 0$ is a projective resolution of the $(\mathfrak{g}, T)$-module V. Since $\mathrm{For} : M(\mathfrak{g}, T) \to M(\mathfrak{k}, T)$ has an exact right adjoint pro, For sends projectives to projectives, i.e., P^i are projective $(\mathfrak{k}, T)$-modules. Furthermore, $R(K) \otimes \bullet$ sends projectives to projectives by Lemma 5.3.11 and the remarks just after it. So $R(K) \otimes P^\cdot$ is a projective resolution of the $(\mathfrak{k}, T)$-module $R(K) \otimes V$, and hence (5.3) calculates the $(\mathfrak{k}, T)$-homology of $R(K) \otimes V$. Since $(\mathfrak{k}, T)$-homology can be calculated using the Koszul complex, we conclude

Theorem 5.5.4. *The derived Bernstein functors of a $(\mathfrak{g}, T)$-module V can be expressed as*

$$\Pi_i(V) = H^i(\textstyle\bigwedge^\cdot(\mathfrak{o}) \otimes_T R(K) \otimes V),$$

with $(\mathfrak{g}, K)$-action coming from the one described earlier on $R(K) \otimes V$. Here $\mathfrak{o}$ is a T-invariant complement of $\mathfrak{t}$ in $\mathfrak{k}$, T acts on $\bigwedge^\cdot(\mathfrak{o})$ by the adjoint action and on $R(K) \otimes V$ by the action $\lambda = L \otimes \pi_V$, and the cohomology is taken with respect to the Koszul differential.

5.5.5. Pseudoforgetful functors. Bernstein functors are not left adjoint to the forgetful functors as one might suppose; in fact, one can show that $\mathrm{For} : M(\mathfrak{g}, K) \to M(\mathfrak{g}, T)$ usually does not have a left adjoint. There is however another functor similar to For which we denote by $\mathrm{For}^\vee$ and call the *pseudoforgetful functor* (following [KV]).

Let V be a $(\mathfrak{g}, K)$-module. We consider the space

$$\mathrm{Hom}_K(R(K), V),$$

the K-morphisms being taken with respect to the left regular action of K in $R(K)$ and the given action on V.

On this vector space, we have a "surviving" action of K, by the contragredient right regular action in the argument:

$$(k\alpha)(f) = \alpha(R_{k^{-1}}f), \qquad \alpha \in \mathrm{Hom}_K(R(K), V).$$

This K-action is not finite; we restrict it to T and take the T-finite part. Thus we define

$$\mathrm{For}^{\vee}(V) = \mathrm{Hom}_K(R(K), V)_T.$$

When we define a $\mathfrak{g}$-action on this space, and make $\mathrm{For}^{\vee}$ into a functor from $M(\mathfrak{g}, K)$ into $M(\mathfrak{g}, T)$, it will be automatic that this functor is exact. The argument is the same as for pro ; to check exactness we can forget about $\mathfrak{g}$ and consider an analogous functor from $M(K)$ to $M(T)$; but every such functor is exact as $M(K)$ is a semisimple category.

The $\mathfrak{g}$-action on $\mathrm{For}^{\vee}(V)$ is the restriction of a $\mathfrak{g}$-action defined on all of $\mathrm{Hom}_K(R(K), V)$. We first interprete this last space in a different way. We are going to use some analytic notation: for $f \in R(K)$, let $\pi_V(f)$ be the operator on V defined by

$$\pi_V(f)v = \int_K f(k)\pi_V(k)v\mathrm{d}k,$$

where $\mathrm{d}k$ denotes the Haar measure on K. In fact, whenever V is K-finite, this can also be written in an algebraic way: if $\pi_V(k)v = \sum_i f_i(k)v_i$, then the above integral can be written as

$$\sum_i \left(\int_K f(k)f_i(k)\mathrm{d}k \right) v_i = \sum_i \epsilon(ff_i)v_i,$$

where $\epsilon : R(K) \to \mathbb{C}$ is the $K \times K$-equivariant projection to the constants. However, it is often easier to follow the integral notation. It is well known (and trivial to check) that

$$\pi_V(L_k f) = \pi_V(k)\pi_V(f) \qquad \text{and} \qquad \pi_V(R_k f) = \pi_V(f)\pi_V(k^{-1}). \tag{5.4}$$

We are going to identify the (non-locally finite) K-module $\mathrm{Hom}_K(R(K), V)$ with

$$\prod_{\delta \in \hat{K}} V(\delta),$$

where for each $(\delta, V_\delta) \in \hat{K}$, $V(\delta) = \mathrm{Hom}_K(V_\delta, V) \otimes V_\delta$ is the δ-isotypic component of V. An abstract way to get this identification is

$$\begin{aligned} \mathrm{Hom}_K(R(K), V) = \mathrm{Hom}_K(\bigoplus_\delta V_\delta \otimes V_\delta^*, V) &= \prod_\delta \mathrm{Hom}_K(V_\delta \otimes V_\delta^*, V) \\ = \prod_\delta \mathrm{Hom}_K(V_\delta, V) \otimes V_\delta &= \prod_\delta V(\delta). \end{aligned}$$

More concretely, $\tilde{v} = (v_\delta) = \sum_\delta v_\delta \in \prod_\delta V(\delta)$ defines a map

$$\tilde{v} : f \mapsto \sum_\delta \pi_V(f)v_\delta$$

from $R(K)$ to V. Although the sum is over an infinite set, for each particular f there are only finitely many nonzero terms, corresponding to those δ for which f has a nonzero component in $R(K)(\delta)$. The map $\tilde{v}$ is a K-morphism because of (5.4). (5.4) also implies that the K-action we defined on $\operatorname{Hom}_K(R(K), V)$ corresponds to the obvious K-action on $\prod_\delta V(\delta)$.

The element $\tilde{v}$ of $\prod_\delta V(\delta)$ is determined by the map from $R(K)$ to V it induces, as v_δ can be recovered as the image of χ_δ, the normalized character of δ. It follows that all K-morphisms are obtained in this way. (This can also be seen from the above abstract description of the identification.)

We now define a $\mathfrak{g}$-action on $\prod_\delta V(\delta)$, by

$$X(\sum_\delta v_\delta) = \sum_\delta \pi_V(X)v_\delta, \quad X \in \mathfrak{g}.$$

The δ-component of this sum is finite for each δ, as $\pi_V(X)v_{\delta'}$ can have nonzero δ-component only for finitely many δ'. Namely, we can consider the lattice of highest weights of all K-types; then the "finitely many δ'" are those with distance between highest weights of δ' and δ smaller than a fixed constant determined by the weights of the K-module $\mathfrak{g}$.

Hence the sum does define an element of $\prod_\delta V(\delta)$. The corresponding element of $\operatorname{Hom}_K(R(K), V)$ is given by

$$\begin{aligned}(X\tilde{v})(f) &= \sum_\delta \int_K f(k)\pi_V(k)\pi_V(X)v_\delta \mathrm{d}k \\ &= \sum_\delta \int_K f(k)\pi_V(\operatorname{Ad}(k)X)\pi_V(k)v_\delta \mathrm{d}k.\end{aligned}$$

If we write $\operatorname{Ad}(k)X = \sum_i f_i(k)X_i$, then this becomes

$$\sum_i \pi_V(X_i) \sum_\delta \int_K f_i(k)f(k)\pi_V(k)v_\delta dk = \sum_i \pi_V(X_i)\tilde{v}(f_i f).$$

This is the description of our action in terms of $\operatorname{Hom}_K(R(K), V)$. One could work with this as the definition, but the calculations are then a bit longer and less obvious.

Our $\mathfrak{g}$-action is obviously K-equivariant, as

$$\begin{aligned}(kXk^{-1})(\sum_\delta v_\delta) = \sum_\delta \pi_V(k)\pi_V(X)\pi_V(k^{-1})v_\delta &= \sum_\delta \pi_V(\operatorname{Ad}(k)X)v_\delta \\ &= (\operatorname{Ad}(k)X)(\sum_\delta v_\delta),\end{aligned}$$

as the $\mathfrak{g}$-action on V is K-equivariant. In the same way, the two actions of $\mathfrak{k}$ agree since they agree on V. It follows that $\operatorname{For}^\vee(V)$ is a $(\mathfrak{g}, T)$-module.

By definition of the actions, it is obvious that

$$j : V = \bigoplus_\delta V(\delta) \to (\prod_\delta V(\delta))_T = \operatorname{For}^\vee(V)$$

given by the inclusion of the direct sum into the direct product is a natural morphism of $(\mathfrak{g}, T)$-modules, i.e., a natural transformation of For into $\operatorname{For}^\vee$. The map j is always one-to-one, but typically not onto.

5.5.6. Adjunction. We are now going to prove that the Bernstein functor Π is left adjoint to $\mathrm{For}^{\vee} : M(\mathfrak{g},K) \to M(\mathfrak{g},T)$. For a $(\mathfrak{g},T)$-module V, we define

$$\Phi_V : V \to \mathrm{For}^{\vee}(\Pi(V)) = \mathrm{Hom}_K(R(K),(R(K)\otimes V)_{(\mathfrak{k},T)})_T$$

by

$$\Phi_V(v)(f) = f^{\vee} \otimes v,$$

where $f^{\vee}$ is the function $k \mapsto f(k^{-1})$. Here we identify $f^{\vee}\otimes v$ with its image in the $(\mathfrak{k},T)$-coinvariants. Clearly, $\Phi_V(v)$ is a well defined K-morphism, as $\Phi_V(v)(L_k f) = (L_k f)^{\vee}\otimes v = R_k f^{\vee}\otimes v = (R_k\otimes 1)\Phi_V(v)(f)$. Furthermore, Φ_V is a T-morphism as

$$(t\Phi_V(v))(f) = \Phi_V(v)(R_{t^{-1}}f) = (R_{t^{-1}}f)^{\vee}\otimes v = L_{t^{-1}}f^{\vee}\otimes v,$$

and in the $(\mathfrak{k},T)$-coinvariants this is the same as $f^{\vee}\otimes\pi_V(t)v = \Phi_V(\pi_V(t)v)(f)$.

To check that Φ_V is a $\mathfrak{g}$-morphism, we use the interpretation of $\Pi(V)$ as $R(K,V)_{(\mathfrak{k},T)}$. Let $X\in\mathfrak{g}$ and let $\mathrm{Ad}(k)X = \sum f_i(k)X_i$. Then

$$(X\Phi_V(v))(f)(k) = \sum_i \big(X_i(\Phi_V(v)(ff_i))\big)(k) = \sum_i \pi_V(\mathrm{Ad}(k)X_i)(ff_i)^{\vee}(k)v$$
$$= \pi_V\big(\mathrm{Ad}(k)(\sum_i f_i(k^{-1})X_i)\big)f^{\vee}(k)v = f^{\vee}(k)\pi_V(X)v = \Phi_V(\pi_V(X)v)(f)(k).$$

The other adjunction morphism is

$$\Psi_W : (R(K)\otimes \mathrm{Hom}_K(R(K),W)_T)_{(\mathfrak{k},T)} \to W$$

given by

$$\Psi_W(f\otimes\alpha) = \alpha(f^{\vee}) = \sum_{\delta}\pi_W(f^{\vee})w_{\delta},$$

where W is a $(\mathfrak{g},K)$-module and α corresponds to $(w_\delta)\in\prod_\delta W(\delta)$. It follows immediately from (5.4) that Ψ_W is well defined on coinvariants and a K-morphism. It is also a $\mathfrak{g}$-morphism, as

$$\begin{aligned}\Psi_W(X(f\otimes\alpha)) &= \sum_i (X_i\alpha)(ff_i)^{\vee} = \sum_{i,\delta}\int_K (ff_i)^{\vee}(k)\pi_W(k)\pi_W(X_i)w_\delta \mathrm{d}k\\ &= \sum_\delta \int_K f^{\vee}(k)\sum_i f_i(k^{-1})\pi_W(\mathrm{Ad}(k)X_i)\pi_W(k)w_\delta \mathrm{d}k\\ &= \pi_W(X)\sum_\delta\int_K f^{\vee}(k)\pi_W(k)w_\delta \mathrm{d}k = \pi_W(X)\Psi_W(f\otimes\alpha).\end{aligned}$$

The compositions (5.1) and (5.2) are now easily checked to be the identity maps by the usual calculation. This finishes the proof of adjunction.

5.5.7. Pseudoforgetful functor and duality. We will denote by $*$ the linear space duality, i.e., $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$, and by $*_K$ and $*_T$ the K-finite respectively T-finite duality operations. So $V^{*_K} = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})_K$ is the K-finite dual of a $(\mathfrak{g}, K)$-module V (with contragredient actions of $\mathfrak{g}$ and K), and analogously for T.

We claim that for any $(\mathfrak{g}, K)$-module V, there is a natural isomorphism

$$\operatorname{For}^{\vee}(V^{*_K}) = (\operatorname{For} V)^{*_T}.$$

This is obvious since if $V = \bigoplus_\delta V(\delta)$ then $V^{*_K} = \bigoplus_\delta V(\delta)^*$, and hence

$$\operatorname{For}^{\vee}(V^{*_K}) = (\prod_\delta V(\delta)^*)_T = (V^*)_T = V^{*_T}.$$

It is clear that all the above equalities are compatible with actions of $\mathfrak{g}$ and T and functorial.

We see now that if V is an admissible $(\mathfrak{g}, K)$-module, then upon writing $V = (V^{*_K})^{*_K}$ we have

$$\operatorname{For}^{\vee}(V) = \operatorname{For}^{\vee}((V^{*_K})^{*_K}) = (\operatorname{For}(V^{*_K}))^{*_T},$$

i.e., we can interpret $\operatorname{For}^{\vee}$ as the T-finite dual of the K-finite dual.

6

Properties of cohomologically induced modules

In this chapter we review the basic properties of the $(\mathfrak{g}, K)$-modules obtained by cohomological induction. These properties are roughly as follows: let Z be an $(\mathfrak{l}, L\cap K)$-module with infinitesimal character λ. Then the cohomologically induced modules have $\mathfrak{g}$-infinitesimal character $\lambda+\rho(\mathfrak{u})$, where $\rho(\mathfrak{u})$ is the half sum of roots corresponding to $\mathfrak{u}$. Under appropriate dominance conditions, they are:

- nonzero only in the middle degree S, and moreover $\mathcal{R}^S(Z) \cong \mathcal{L}_S(Z)$;
- irreducible if Z is irreducible;
- unitary if Z is unitary.

The proofs of these results are rather complicated and they are written in great detail in [KV]. Hence we will omit most of the proofs and only give some general ideas about them.

At the end we describe the results of Salamanca-Riba [SR] about unitary representations with strongly regular infinitesimal character. We are going to use these results in our analysis of the discrete series representations.

6.1 Duality theorems

6.1.1. Easy duality. We have by now constructed essentially two pairs of induction functors. One pair is the change of algebras, by pro or ind, that is, by taking Hom or tensoring. The other pair is the change of groups, by either Zuckerman or Bernstein functors. In both cases the left and right variant are intertwined by the duality operations on the two categories in question. This is a very formal and easy consequence of adjunction and the corresponding property for forgetful functors. Moreover, the extension to derived functors also comes basically for free. We start with Zuckerman and Bernstein functors.

Proposition 6.1.2. *Let $(\mathfrak{g}, K)$ be a pair and let T be a closed subgroup of K. Denote by W^{*_K} the K-finite dual of a $(\mathfrak{g}, K)$-module W, with contragredient*

$(\mathfrak{g}, K)$-action. Analogously, V^{*_T} denotes the T-finite dual of a $(\mathfrak{g}, T)$-module V. Then there is a natural isomorphism of $(\mathfrak{g}, K)$-modules

$$\Pi_i(V)^{*_K} = \Gamma^i(V^{*_T}).$$

Proof. For $i = 0$, this immediately follows from 5.5.7 and the adjunctions by the following series of natural isomorphisms:

$$\begin{aligned} &\operatorname{Hom}_{(\mathfrak{g},K)}(X, \Pi(V)^{*_K}) = \operatorname{Hom}_{(\mathfrak{g},K)}(\Pi(V), X^{*_K}) \\ = &\operatorname{Hom}_{(\mathfrak{g},T)}(V, \operatorname{For}^\vee(X^{*_K})) = \operatorname{Hom}_{(\mathfrak{g},T)}(V, (\operatorname{For} X)^{*_T}) \\ = &\operatorname{Hom}_{(\mathfrak{g},T)}(\operatorname{For} X, V^{*_T}) = \operatorname{Hom}_{(\mathfrak{g},K)}(X, \Gamma(V^{*_T})), \end{aligned}$$

where X is an arbitrary $(\mathfrak{g}, K)$-module. Here we used the "self-adjunction" of the duality functors from Corollary 5.3.14. To get the statement for any i, we calculate the (left) derived functors of the contravariant functor

$$\Pi(\bullet)^{*_K} = \Gamma((\bullet)^{*_T}).$$

Let $P^\cdot \to V \to 0$ be a projective resolution of a $(\mathfrak{g}, T)$-module V. By Lemma 5.3.11 and the remarks after it, $0 \to V^{*_T} \to (P^\cdot)^{*_T}$ is an injective resolution of the $(\mathfrak{g}, T)$-module V^{*_T}. Since both duality operations are exact, we see that

$$\begin{aligned} \Pi_i(V)^{*_K} &= L_i(\Pi(\bullet)^{*_K})(V) = H^i(\Pi(P^\cdot)^{*_K}) \\ &= H^i(\Gamma((P^\cdot)^{*_T})) = R^i(\Gamma((\bullet)^{*_T})(V) = \Gamma^i(V^{*_T}). \end{aligned}$$

Let now $(\mathfrak{s}, C)$ and $(\mathfrak{r}, C)$ be pairs with a C-equivariant map $\tau : \mathfrak{r} \to \mathfrak{s}$ which is the identity on $\mathfrak{c}$. Clearly, the forgetful functor $\operatorname{For} : M(\mathfrak{s}, C) \to M(\mathfrak{r}, C)$ intertwines the C-finite duality operations on the two categories. Thus exactly the same argument as above applies to the adjoints of For. In case τ is an embedding, these adjoints are the exact functors ind and pro, and there are no higher derived functors. In the other case we are interested in, that of projection $\tau : \mathfrak{r} \to \mathfrak{s}$ along an ideal ι, the derived functors are ι-homology and cohomology. The case of main interest here is the projection $\mathfrak{q} \to \mathfrak{l}$ along $\mathfrak{u}$ for a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$. We state the two cases separately because of different notation.

Proposition 6.1.3. *(i) Let $(\mathfrak{s}, C)$ and $(\mathfrak{r}, C)$ be pairs with $\mathfrak{r} \subset \mathfrak{s}$. Then there is a natural isomorphism of $(\mathfrak{s}, C)$ modules*

$$\operatorname{ind}(V)^{*_C} = \operatorname{pro}(V^{*_C}), \qquad V \in M(\mathfrak{r}, C).$$

(ii) Let $(\mathfrak{q}, C)$ and $(\mathfrak{l}, C)$ be pairs, and let $\tau : \mathfrak{q} \to \mathfrak{l}$ be the projection along an ideal $\mathfrak{u}$ of $\mathfrak{q}$. Then for any $i \geq 0$ there is a natural isomorphism of $(\mathfrak{l}, C)$-modules

$$H_i(\mathfrak{u}; V)^{*_C} = H^i(\mathfrak{u}; V^{*_C}), \qquad V \in M(\mathfrak{q}, C).$$

Finally, there is a completely analogous property for the relative homology and cohomology. Let $(\mathfrak{s}, C)$ be a pair, and consider the "forgetful" functor from the category of vector spaces to the category $M(\mathfrak{s}, C)$, which attaches a trivial module V to the vector space V. This functor intertwines the C-finite duality on $M(\mathfrak{s}, C)$ with the full linear duality $V \mapsto V^*$ on the category of vector spaces. (This situation corresponds to the map of pairs $(\mathfrak{s}, C) \to (0, 1)$.)

The adjoints are the functors of $(\mathfrak{s}, C)$-coinvariants and invariants, with derived functors $(\mathfrak{s}, C)$-homology respectively cohomology. We conclude

Proposition 6.1.4. *For any $i \geq 0$, there is a natural isomorphism of vector spaces*

$$H_i(\mathfrak{s}, C; V)^* = H^i(\mathfrak{s}, C; V^{*C}), \qquad V \in M(\mathfrak{s}, C).$$

Note that [KV] contains a common generalization of all these statements, referring to the more general "change of rings" functors P and I corresponding to a general map of pairs $(\mathfrak{r}, C) \to (\mathfrak{s}, T)$. The above cases are however sufficient for our purposes.

6.1.5. Hard duality. Unlike the above statements which involved duality operations, hard duality in fact does not refer to the duality operations. Rather, it says that derived Bernstein and Zuckerman functors are essentially the same (up to a modular twist in the argument), but in complementary degrees. In particular, one could define the Bernstein functor Π as the top derived Zuckerman functor of the twisted module. In fact, the theorem we are about to prove is very closely related to the well known Poincaré duality.

Before we state the theorem, let us explain the "modular twist" mentioned above. Let $(\mathfrak{g}, K)$ be a pair and let T be a closed subgroup of K. Let $\mathfrak{o}$ be a T-invariant direct complement of $\mathfrak{t}$ in $\mathfrak{k}$ and let $n = \dim \mathfrak{o}$. Then T acts on the top exterior power $\bigwedge^{\text{top}} \mathfrak{o}$ by the adjoint action. Let $\mathfrak{g}$ act on $\bigwedge^{\text{top}} \mathfrak{o}$ by zero. This is certainly a T-equivariant action, and in fact it makes $\bigwedge^{\text{top}} \mathfrak{o}$ into a $(\mathfrak{g}, T)$-module. This means that the adjoint action of $\mathfrak{t}$ on $\bigwedge^{\text{top}} \mathfrak{o}$ is zero, and hence the only nontrivial part of the action can come from the component group of T. This follows easily from unimodularity of $\mathfrak{t}$ and $\mathfrak{k}$; in the following lemma we prove a little more.

Lemma 6.1.6. *Let $\mathfrak{k}$, $\mathfrak{t}$ and $\mathfrak{o}$ be as above. Define an action of $\mathfrak{k}$ on $\bigwedge^{\text{top}} \mathfrak{o}$ by*

$$X \cdot \lambda_1 \wedge \cdots \wedge \lambda_n = \sum_i \lambda_1 \wedge \ldots [X, \lambda_i]_{\mathfrak{o}} \cdots \wedge \lambda_n,$$

where $(\cdot)_{\mathfrak{o}}$ denotes the projection of $\mathfrak{k}$ onto $\mathfrak{o}$ along $\mathfrak{t}$. Then this $\mathfrak{k}$-action is zero. In particular, for $X \in \mathfrak{t}$, $\operatorname{ad}(X)$ is zero on $\bigwedge^{\text{top}} \mathfrak{o}$.

Proof. Let $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_n$ be a basis (i.e., nonzero) element of $\bigwedge^{\text{top}} \mathfrak{o}$ and let μ be a basis element of $\bigwedge^{\text{top}} \mathfrak{t}$. Clearly, $\lambda \wedge \mu$ is a basis element of $\bigwedge^{\text{top}} \mathfrak{k}$. Since $\mathfrak{k}$ is unimodular, $\operatorname{ad}(X)(\lambda \wedge \mu) = 0$ for any $X \in \mathfrak{k}$.

Let $X \in \mathfrak{t}$. Since $\mathfrak{o}$ is $\mathfrak{t}$-invariant, $X \cdot \lambda = \operatorname{ad}(X)\lambda$. Since $\mathfrak{t}$ is unimodular, $\operatorname{ad}(X)\mu = 0$. We conclude

$$(X \cdot \lambda) \wedge \mu = \operatorname{ad}(X)(\lambda \wedge \mu) - \lambda \wedge \operatorname{ad}(X)\mu = 0,$$

so $X \cdot \lambda = 0$.

If $X \in \mathfrak{o}$, then each term of $\operatorname{ad}(X)\mu$ has a factor in $\mathfrak{o}$, hence $\lambda \wedge \operatorname{ad}(X)\mu = 0$. Also, each term of $\operatorname{ad}(X)\lambda - X \cdot \lambda = \sum_i \lambda_1 \wedge \dots [X, \lambda_i]_{\mathfrak{t}} \cdots \wedge \lambda_n$ has a factor in $\mathfrak{t}$, hence $(X \cdot \lambda) \wedge \mu = (\operatorname{ad}(X)\lambda) \wedge \mu$. We conclude

$$(X \cdot \lambda) \wedge \mu = \operatorname{ad}(X)(\lambda \wedge \mu) - \lambda \wedge \operatorname{ad}(X)\mu = 0,$$

so $X \cdot \lambda = 0$.

Theorem 6.1.7. *Let $(\mathfrak{g}, K)$ be a pair and let T be a closed subgroup of K. Then for each $p \geq 0$ there is a natural isomorphism of $(\mathfrak{g}, K)$-modules*

$$\Pi_p(V) = \Gamma^{n-p}(\textstyle\bigwedge^{\text{top}} \mathfrak{o} \otimes V), \qquad V \in M(\mathfrak{g}, T).$$

Here $\bigwedge^{\text{top}} \mathfrak{o}$ is a $(\mathfrak{g}, T)$-module with trivial action of $\mathfrak{g}$ and adjoint action of T and $n = \dim \mathfrak{o}$.

Proof. Recall that we can use the Koszul resolutions to identify

$$\Pi_p(V) = H^{-p}(\textstyle\bigwedge^{\cdot} \mathfrak{o} \otimes_T R(K, V))$$

and

$$\Gamma^{n-p}(\textstyle\bigwedge^{\text{top}} \mathfrak{o} \otimes V) = H^{n-p}(\operatorname{Hom}_T(\bigwedge^{\cdot} \mathfrak{o}, \bigwedge^{\text{top}} \mathfrak{o} \otimes R(K, V))).$$

We define a map

$$\phi_p : \textstyle\bigwedge^p \mathfrak{o} \otimes R(K, V) \to \operatorname{Hom}_{\mathbb{C}}(\bigwedge^{n-p} \mathfrak{o}, \bigwedge^{\text{top}} \mathfrak{o} \otimes R(K, V))$$

for each p by

$$\phi_p(\lambda \otimes F)(\mu) = \lambda \wedge \mu \otimes F.$$

It is clear that ϕ_p is a T-morphism and hence restricts to a map between T-invariants, i.e., a map from $\bigwedge^p \mathfrak{o} \otimes_T R(K, V)$ into $\operatorname{Hom}_T(\bigwedge^{n-p} \mathfrak{o}, \bigwedge^{\text{top}} \mathfrak{o} \otimes R(K, V))$. Also, ϕ_p is a linear isomorphism, as the pairing $(\lambda, \mu) \mapsto \lambda \wedge \mu$ is nondegenerate for λ and μ in complementary degrees. Furthermore, it is obvious that ϕ_p is a $(\mathfrak{g}, K)$-morphism, as both $\mathfrak{g}$ and K act only on $F \in R(K, V)$ on each of the sides. To finish the proof of the theorem, we must prove that ϕ_p descends to cohomology.

This is actually exactly what is proved in the standard proof of Poincaré duality for relative $(\mathfrak{k}, T)$ homology and cohomology; see e.g. [Kn2], proofs of Theorem 7.31 and Theorem 6.10. In other words, from now on we are just retelling the standard proof of that well known result.

We will prove that

$$\phi_{p-1} \circ d + (-1)^{p-1} d \circ \phi_p = 0.$$

(We could thus easily turn ϕ into a chain map: put $\psi_p = (-1)^{p(p+1)/2}\phi_p$.) Actually, we will show that the above expression is equal to the expression for $d(\lambda \wedge \mu \otimes F)$. But since $\lambda \wedge \mu$ is in $\bigwedge^{n+1} \mathfrak{o} = 0$, we see that $d(\lambda \wedge \mu \otimes F) = 0$.

Let $\lambda = \lambda_1 \wedge \cdots \wedge \lambda_p$ and $\mu = \mu_1 \wedge \cdots \wedge \mu_{n-p+1}$. By definition of the differentials and the map ϕ, we see

$$\begin{aligned}
\phi_{p-1}(d(\lambda \otimes F))(\mu) &= \phi_{p-1}\Big(\sum_{i=1}^{p} (-1)^{i-1} \lambda_1 \wedge \ldots \hat{\lambda}_i \cdots \wedge \lambda_p \otimes (-\lambda_i) F \\
&\quad + \sum_{i<j} (-1)^{i+j} [\lambda_i, \lambda_j]_{\mathfrak{o}} \wedge \lambda_1 \wedge \ldots \hat{\lambda}_i \ldots \hat{\lambda}_j \cdots \wedge \lambda_p \otimes F \Big)(\mu) \\
&= \sum_{i=1}^{p} (-1)^{i} \lambda_1 \wedge \ldots \hat{\lambda}_i \cdots \wedge \lambda_p \wedge \mu \otimes \lambda_i F \\
&\quad + \sum_{i<j} (-1)^{i+j} [\lambda_i, \lambda_j]_{\mathfrak{o}} \wedge \lambda_1 \wedge \ldots \hat{\lambda}_i \ldots \hat{\lambda}_j \cdots \wedge \lambda_p \wedge \mu \otimes F.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(-1)^{p-1} d(\phi_p(\lambda \otimes F))(\mu) \\
&= \sum_{j=1}^{n-p+1} (-1)^{p+j} \mu_j \phi_p(\lambda \otimes F)(\mu_1 \wedge \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1}) \\
&+ \sum_{i<j} (-1)^{p-1+i+j} \phi_p(\lambda \otimes F)([\mu_i, \mu_j]_{\mathfrak{o}} \wedge \mu_1 \wedge \ldots \hat{\mu}_i \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1}) \\
&= \sum_{j=1}^{n-p+1} (-1)^{p+j} \mu_j (\lambda \wedge \mu_1 \wedge \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1} \otimes F) \\
&+ \sum_{i<j} (-1)^{p-1+i+j} \lambda \wedge [\mu_i, \mu_j]_{\mathfrak{o}} \wedge \mu_1 \wedge \ldots \hat{\mu}_i \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1} \otimes F \\
&= \sum_{j=1}^{n-p+1} (-1)^{p+j} \lambda \wedge \mu_1 \wedge \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1} \otimes \mu_j F \\
&+ \sum_{j=1}^{n-p+1} (-1)^{p+j} \mu_j \cdot (\lambda \wedge \mu_1 \wedge \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1}) \otimes F \\
&+ \sum_{i<j} (-1)^{p+i} \lambda \wedge \mu_1 \wedge \ldots \hat{\mu}_i \ldots [\mu_j, \mu_i]_{\mathfrak{o}} \cdots \wedge \mu_{n-p+1} \otimes F.
\end{aligned}$$

Although we know that $\mu_j \cdot (\lambda \wedge \mu_1 \wedge \ldots \hat{\mu}_j \cdots \wedge \mu_{n-p+1}) = 0$, we will still write it out as it fits nicely into the formulas. It is equal to

$$\sum_{i=1}^{p} \lambda_1 \wedge \dots [\mu_j, \lambda_i]_{\mathfrak{o}} \dots \wedge \lambda_p \wedge \mu + \sum_{i<j} \lambda \wedge \mu_1 \wedge \dots [\mu_j, \mu_i]_{\mathfrak{o}} \dots \hat{\mu}_j \dots \wedge \mu_{n-p+1}$$
$$+ \sum_{i>j} \lambda \wedge \mu_1 \wedge \dots \hat{\mu}_j \dots [\mu_j, \mu_i]_{\mathfrak{o}} \dots \wedge \mu_{n-p+1}.$$

Substituting this into the above expression for $(-1)^{p-1} d(\phi_p(\lambda \otimes F))(\mu)$, we see that the sum with $i > j$ cancels the last sum of that expression (upon exchanging i and j). Hence we see

$$\begin{aligned}
(-1)^{p-1} d(\phi_p(\lambda \otimes F))(\mu) &= \sum_{j=1}^{n-p+1} (-1)^{p+j} \lambda \wedge \mu_1 \wedge \dots \hat{\mu}_j \dots \wedge \mu_{n-p+1} \otimes \mu_j F \\
&\quad + \sum_{j=1}^{n-p+1} (-1)^{p+j} \sum_{i=1}^{p} \lambda_1 \wedge \dots [\mu_j, \lambda_i]_{\mathfrak{o}} \dots \wedge \lambda_p \wedge \mu \\
&\quad + \sum_{i<j} (-1)^{p+j} \lambda \wedge \mu_1 \wedge \dots [\mu_j, \mu_i]_{\mathfrak{o}} \dots \hat{\mu}_j \dots \wedge \mu_{n-p+1} \\
&= \sum_{j=1}^{n-p+1} (-1)^{p+j} \lambda \wedge \mu_1 \wedge \dots \hat{\mu}_j \dots \wedge \mu_{n-p+1} \otimes \mu_j F \\
&\quad + \sum_{i,j} (-1)^{i+(p+j)} [\lambda_i, \mu_j]_{\mathfrak{o}} \wedge \lambda_1 \wedge \dots \hat{\lambda}_1 \dots \lambda_p \wedge \mu_1 \dots \hat{\mu}_j \dots \wedge \mu_{n-p+1} \\
&\quad + \sum_{i<j} (-1)^{(p+i)+(p+j)} [\mu_i, \mu_j]_{\mathfrak{o}} \wedge \lambda \wedge \mu_1 \dots \hat{\mu}_i \dots \hat{\mu}_j \dots \mu_{n-p+1}.
\end{aligned}$$

This now exactly fits with the expression for $\phi_{p-1}(d(\lambda \otimes F))(\mu)$ to give the expression for $d(\lambda \wedge \mu \otimes F)$ and this finishes the proof.

In case T is connected, for example $T = L \cap K$, the twist $\bigwedge^{\mathrm{top}} \mathfrak{o}$ disappears, as it is then also trivial as a T-module. Also, for $T = L \cap K$ we can take $\mathfrak{u} \cap \mathfrak{k} \oplus \bar{\mathfrak{u}} \cap \mathfrak{k}$ for $\mathfrak{o}$. The dimension of this space is $2S$, where $S = \dim \mathfrak{u} \cap \mathfrak{k}$. The degree of particular interest is the middle degree S, and we see that Π_S and Γ^S are the same.

The proof of the theorem also proves the classical Poincaré duality for relative $(\mathfrak{s}, C)$-homology and cohomology, which includes the case of ordinary Lie algebra homology and cohomology (set $C = 1$). Moreover, the version with $\mathfrak{u}$-homology and cohomology with additional $(\mathfrak{l}, C)$-module structure is also included; one only needs to check that the map ϕ from the above proof respects the $(\mathfrak{l}, C)$ action, and this is obvious from the definitions. The action on the top wedge is no longer trivial in general, but it is again determined by the identification $\bigwedge^{\mathrm{top}} \mathfrak{o} \otimes \bigwedge^{\mathrm{top}} \mathfrak{c} = \bigwedge^{\mathrm{top}} \mathfrak{s}$. Note that the above proof did not use triviality of the action on the top wedge.

To conclude:

Corollary 6.1.8. *(i) For any $i \geq 0$, there is a natural isomorphism of vector spaces*

$$H_i(\mathfrak{s}, C; V) = H^{n-i}(\mathfrak{s}, C; \textstyle\bigwedge^{\text{top}} \mathfrak{o} \otimes V), \qquad V \in M(\mathfrak{s}, C),$$

where $\mathfrak{o}$ is a C-invariant complement of $\mathfrak{c}$ in $\mathfrak{s}$ and $n = \dim \mathfrak{o}$.

(ii) Let $(\mathfrak{q}, C)$ and $(\mathfrak{l}, C)$ be pairs, and let $\tau : \mathfrak{q} \to \mathfrak{l}$ be the projection along an ideal $\mathfrak{u}$ of $\mathfrak{q}$. Then for any $i \geq 0$ there is a natural isomorphism of $(\mathfrak{l}, C)$-modules

$$H_i(\mathfrak{u}; V) = H^{n-i}(\mathfrak{u}; \textstyle\bigwedge^{\text{top}} \mathfrak{u} \otimes V), \qquad V \in M(\mathfrak{q}, C),$$

where $n = \dim \mathfrak{u}$.

One can now combine easy and hard duality in the obvious way. Let us state the result only for Zuckerman functors; this is the *Zuckerman duality*, conjectured in [Z] and first proved by Enright and Wallach [EW].

Corollary 6.1.9. *For any $(\mathfrak{g}, T)$-module V and for any $i \geq 0$, there is a natural isomorphism of $(\mathfrak{g}, K)$-modules*

$$\Gamma^i(V^{*_T}) = \Gamma^{n-i}(\textstyle\bigwedge^{\text{top}} \mathfrak{o} \otimes V)^{*_K}.$$

6.2 Infinitesimal character, K-types and vanishing

Let us first show that Zuckerman and Bernstein functors preserve infinitesimal characters. As usual, we will identify infinitesimal characters $\lambda : Z(\mathfrak{g}) \to \mathbb{C}$ with elements $\lambda \in \mathfrak{h}^*$. (More precisely, an infinitesimal character is a W-orbit of λ in $\mathfrak{h}^*$, where W is the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$; see 1.4.7 and 1.4.8.)

Proposition 6.2.1. *Let T be a closed subgroup of K. Assume that the adjoint action of K on $Z(\mathfrak{g})$ is trivial; this is for example true if K is connected.*

Let V be a $(\mathfrak{g}, T)$-module with infinitesimal character λ. Then $\Gamma^i(V)$ and $\Pi_i(V)$ have infinitesimal character λ for any i.

Proof. Recall that the action of $X \in \mathfrak{g}$ on $F \in R(K, V)$ is given by $(XF)(k) = \pi_V(\operatorname{Ad}(k)X)(F(k))$. This formula works also for $X \in U(\mathfrak{g})$. For $z \in Z(\mathfrak{g})$, we have $\operatorname{Ad}(k)z = z$ and so

$$(zF)(k) = \pi_V(\operatorname{Ad}(k)z)(F(k)) = \pi_V(z)(F(k)) = \lambda(\gamma_{\mathfrak{b}}(z))F(k).$$

This then also holds after passing to $(\mathfrak{k}, T)$-cohomology or homology.

Since $\mathfrak{h}$ is a Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{l}$, both $\mathfrak{l}$-infinitesimal characters and $\mathfrak{g}$-infinitesimal characters are described by elements $\lambda \in \mathfrak{h}^*$. We will choose positive roots for $\mathfrak{g}$ and $\mathfrak{l}$ with respect to $\mathfrak{h}$ in a compatible way;

so $\mathfrak{b} = \mathfrak{b} \cap \mathfrak{l} \oplus \mathfrak{u}$. In particular, $\rho(\mathfrak{n}) = \rho(\mathfrak{n} \cap \mathfrak{l}) + \rho(\mathfrak{u})$, where $\rho(\mathfrak{u})$ is the half sum of roots corresponding to $\mathfrak{u}$. We will use the notation from 1.4.7: let $\mu^{\mathfrak{g}}_{\bar{\mathfrak{b}}}$ and $\mu^{\mathfrak{l}}_{\bar{\mathfrak{b}}\cap\mathfrak{l}}$ be the Harish-Chandra projections from $Z(\mathfrak{g})$ respectively $Z(\mathfrak{l})$ to $U(\mathfrak{h}) = P(\mathfrak{h}^*)$ corresponding to the opposite Borel subalgebras $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{b}} \cap \mathfrak{l}$. The corresponding Harish-Chandra isomorphisms are $\gamma^{\mathfrak{g}}_{\bar{\mathfrak{b}}} = s_{-\rho(\mathfrak{n})} \circ \mu^{\mathfrak{g}}_{\bar{\mathfrak{b}}}$ and $\gamma^{\mathfrak{l}}_{\bar{\mathfrak{b}}\cap\mathfrak{l}} = s_{-\rho(\mathfrak{n}\cap\mathfrak{l})} \circ \mu^{\mathfrak{l}}_{\bar{\mathfrak{b}}\cap\mathfrak{l}}$.

We will also use the Harish-Chandra projection

$$\mu_{\bar{\mathfrak{q}}} : U(\mathfrak{g}) \to U(\mathfrak{l}), \quad \text{along} \quad \mathfrak{u}U(\mathfrak{g}) + U(\mathfrak{g})\bar{\mathfrak{u}};$$

it is constructed analogously to $\mu^{\mathfrak{g}}_{\bar{\mathfrak{b}}}$. Clearly, $\mu^{\mathfrak{g}}_{\bar{\mathfrak{b}}} = \mu^{\mathfrak{l}}_{\bar{\mathfrak{b}}\cap\mathfrak{l}} \circ \mu_{\bar{\mathfrak{q}}}$. Using this and $\rho(\mathfrak{n}) = \rho(\mathfrak{n} \cap \mathfrak{l}) + \rho(\mathfrak{u})$, we get

$$\gamma^{\mathfrak{g}}_{\bar{\mathfrak{b}}} = s_{-\rho(\mathfrak{u})} \circ \gamma^{\mathfrak{l}}_{\bar{\mathfrak{b}}\cap\mathfrak{l}} \circ \mu_{\bar{\mathfrak{q}}}. \tag{6.1}$$

(The reason for switching to $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{q}}$ is the fact that $\mathfrak{u}U(\mathfrak{g})$ and $U(\mathfrak{g})\bar{\mathfrak{u}}$ act trivially on $\operatorname{pro}(Z^{\#})$ respectively $\operatorname{ind}(Z^{\#})$, not $\bar{\mathfrak{u}}U(\mathfrak{g})$ and $U(\mathfrak{g})\mathfrak{u}$.)

Assuming that the module Z we start from has $\mathfrak{l}$-infinitesimal character λ, we want to determine the $\mathfrak{g}$-infinitesimal characters of $\mathcal{R}^i(Z)$ and $\mathcal{L}_i(Z)$. By 6.2.1, we only need to determine the infinitesimal characters of $\operatorname{pro}^{\mathfrak{g}}_{\mathfrak{q}}(Z^{\#})$ and $\operatorname{ind}^{\mathfrak{g}}_{\mathfrak{q}}(Z^{\#})$.

Clearly, the infinitesimal character of $Z^{\#}$ is $\lambda + 2\rho(\mathfrak{u})$.

For $z \in Z(\mathfrak{g})$, we can write $z = \mu_{\bar{\mathfrak{q}}}(z) + Xv = \mu_{\bar{\mathfrak{q}}}(z) + wY$, where $X \in \mathfrak{u}$, $Y \in \bar{u}$ and $v, w \in U(\mathfrak{g})$.

Now z acts on $f \in \operatorname{pro}(Z^{\#}) = \operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z^{\#})_{L\cap K}$ by

$$(zf)(u) = f(uz) = f(zu) = f(\mu_{\bar{\mathfrak{q}}}(z)u) + f(Xvu) = \mu_{\bar{\mathfrak{q}}}(z)f(u);$$

namely, since our $Z^{\#}$ has zero action of $\mathfrak{u}$ in the definition of $\operatorname{pro}(Z^{\#})$, $f(Xvu) = Xf(vu) = 0$. Likewise, wY kills $\operatorname{ind}(Z^{\#})$, and z acts on $\operatorname{ind}(Z^{\#})$ again by the action of $\mu_{\bar{\mathfrak{q}}}(z)$ on $Z^{\#}$.

So z acts on both $\operatorname{pro}(Z^{\#})$ and $\operatorname{ind}(Z^{\#})$ by the scalar

$$(\lambda + 2\rho(\mathfrak{u}))(\gamma^{\mathfrak{l}}_{\bar{\mathfrak{b}}\cap\mathfrak{l}}\mu_{\bar{\mathfrak{q}}}(z)) = (\lambda + \rho(\mathfrak{u}))(\gamma^{\mathfrak{g}}_{\bar{\mathfrak{b}}}(z)),$$

with equality following from (6.1). In other words, the infinitesimal character of both $\operatorname{pro}(Z^{\#})$ and $\operatorname{ind}(Z^{\#})$ is $\lambda + \rho(\mathfrak{u})$. This proves the following proposition.

Proposition 6.2.2. *Let Z be an $(\mathfrak{l}, L\cap K)$-module with infinitesimal character λ. Then the $(\mathfrak{g}, K)$-modules $\mathcal{R}^i(Z)$ and $\mathcal{L}_i(Z)$ have infinitesimal character $\lambda + \rho(\mathfrak{u})$ for any i.*

6.2.3. K-types and vanishing above middle degree. There are two results to mention here; the first one gives an upper bound on the multiplicities of K-types of the cohomologically induced modules:

Theorem 6.2.4. *(i) The multiplicity of the K-type V_δ in $\mathcal{R}^i(Z)$ is at most*

$$\sum_{j\geq 0} \dim \operatorname{Hom}_{L\cap K}(H_i(\mathfrak{u}\cap\mathfrak{k}; V_\delta), S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^\#).$$

In particular, $\mathcal{R}^i(Z) = 0$ if $i > S = \dim \mathfrak{u}\cap\mathfrak{k}$.

(ii) The multiplicity of the K-type V_δ in $\mathcal{L}_i(Z)$ is at most

$$\sum_{j\geq 0} \dim \operatorname{Hom}_{L\cap K}(S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^\#, H^i(\bar{\mathfrak{u}}\cap\mathfrak{k}; V_\delta)).$$

In particular, $\mathcal{L}_i(Z) = 0$ if $i > S = \dim \mathfrak{u}\cap\mathfrak{k}$.

Here we wrote an infinite sum over j, but only finitely many j actually produce nonzero terms.

The other result concerns an Euler sum:

Theorem 6.2.5. *Let Z be an admissible $(\mathfrak{l}, L\cap K)$ module in which h acts by a scalar (this is automatic if Z has an infinitesimal character). Then*

(i) the $(\mathfrak{g}, K)$-modules $\mathcal{R}^i(Z)$ are admissible, and

$$\sum_{i=0}^{S}(-1)^i \dim \operatorname{Hom}_K(V_\delta, \mathcal{R}^i(Z))$$
$$= \sum_{i=0}^{S}(-1)^i \sum_{j\geq 0} \dim \operatorname{Hom}_{L\cap K}(H_i(\mathfrak{u}\cap\mathfrak{k}; V_\delta), S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^\#).$$

(ii) the $(\mathfrak{g}, K)$-modules $\mathcal{L}_i(Z)$ are admissible, and

$$\sum_{i=0}^{S}(-1)^i \dim \operatorname{Hom}_K(\mathcal{L}_i(Z), V_\delta)$$
$$= \sum_{i=0}^{S}(-1)^i \sum_{j\geq 0} \dim \operatorname{Hom}_{L\cap K}(S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^\#, H^i(\bar{\mathfrak{u}}\cap\mathfrak{k}; V_\delta)).$$

In both of the above sums, the summands are actually nonzero only for finitely many j.

These Euler sum equalities will be especially concrete in the situations when we know that the only possible degree in which $\mathcal{R}^i(Z)$ and $\mathcal{L}_i(Z)$ can be nonzero is $i = S$. Then the above sums become explicit formulas for the K-types of $\mathcal{R}^S(Z)$ and $\mathcal{L}_S(Z)$. We will return to this point later on and see how to make the right hand side even more explicit in some special cases.

For the proofs of these results, see [KV], Sections V.4 and V.5, [V1], Theorem 6.3.12, or [W], Section 6.5.

To get some idea about these proofs, and also to make the reading of the above references easier, let us make some comments about how certain formulas appearing in the proofs can be obtained.

6.2.6. Deriving adjunction: the easy cases. In certain cases, one can interpret both sides of

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) = \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

as composite functors of either of the variables, X or Y, and try to derive both sides of the equality.

In general, this leads to a spectral sequence, and even this only when the appropriate conditions for deriving the composition are met. There are however the easy cases when one only needs to derive one of the functors; we have met such situations before.

The first case is when both F and G are exact functors. In that case, it is also true that F takes projectives to projectives, and G takes injectives to injectives. We can view the adjunction as the equality of two functors of (say) the second variable. Then if $0 \to Y \to I^{\cdot}$ is an injective resolution, so is $0 \to GY \to GI^{\cdot}$, and by taking cohomology of

$$\mathrm{Hom}_{\mathcal{D}}(FX, I^{\cdot}) = \mathrm{Hom}_{\mathcal{C}}(X, GI^{\cdot}), \tag{6.2}$$

we get

$$\mathrm{Ext}^i_{\mathcal{D}}(FX, Y) = \mathrm{Ext}^i_{\mathcal{C}}(X, GY), \qquad i \geq 0.$$

The situation where we will apply this is when one of the functors is forgetful and the other is pro or ind.

The other good case is when one of the Hom 's is exact, i.e., one of the categories is semisimple. In that case, the functor starting from that category will automatically be exact. For example, if $\mathcal{C}$ is semisimple, then F is exact, and by taking cohomology in (6.2) we obtain

$$\mathrm{Ext}^i_{\mathcal{D}}(FX, Y) = \mathrm{Hom}_{\mathcal{C}}(X, R^iG(Y)), \qquad i \geq 0.$$

Analogously, if $\mathcal{D}$ is semisimple we get

$$\mathrm{Hom}_{\mathcal{D}}(L_iF(X), Y) = \mathrm{Ext}^i_{\mathcal{C}}(X, GY), \qquad i \geq 0.$$

Of course, among the categories we are interested in, the semisimple ones are the categories $M(K) = M(\mathfrak{k}, K)$ and $M(T) = M(\mathfrak{t}, T)$ of K-modules respectively T-modules. (T will usually be $L \cap K$.)

Here is a list of cases which actually get to be applied in the proofs of Theorem 6.2.4 and Theorem 6.2.5.

Corollary 6.2.7. *(i) (Shapiro's Lemma) Let* $(\mathfrak{r}, C)$ *and* $(\mathfrak{s}, C)$ *be pairs with* $\mathfrak{r} \subset \mathfrak{s}$*. Then for any* $i \geq 0$,

$$\operatorname{Ext}^i_{(\mathfrak{s},C)}(\operatorname{ind} X, Y) = \operatorname{Ext}^i_{(\mathfrak{r},C)}(X, Y),$$

and

$$\operatorname{Ext}^i_{(\mathfrak{r},C)}(Y, X) = \operatorname{Ext}^i_{(\mathfrak{s},C)}(Y, \operatorname{pro} X),$$

naturally in X and Y.

(ii) Let $\Gamma_{\mathfrak{k}}$ and $\Pi^{\mathfrak{k}}$ be the Zuckerman and Bernstein functors from $M(\mathfrak{k}, T)$ to $M(\mathfrak{k}, K)$, i.e., $\mathfrak{g}$ is replaced by $\mathfrak{k}$ in the definition. Then for any $i \geq 0$,

$$\operatorname{Hom}_K(X, \Gamma^i_{\mathfrak{k}}(Y)) = \operatorname{Ext}^i_{(\mathfrak{k},T)}(X, Y),$$

and

$$\operatorname{Hom}_K(\Pi^{\mathfrak{k}}_i(Y), X) = \operatorname{Ext}^i_{(\mathfrak{k},T)}(Y, \operatorname{For}{}^{\vee} X),$$

naturally in X and Y.

(iii) Let $\mathfrak{q} \cap \mathfrak{k} \to \mathfrak{l} \cap \mathfrak{k}$ be the projection along $\mathfrak{u} \cap \mathfrak{k}$ and consider the corresponding forgetful functor For *. Then for any $i \geq 0$,*

$$\operatorname{Ext}^i_{(\mathfrak{q}\cap\mathfrak{k}, L\cap K)}(X, \operatorname{For} Y) = \operatorname{Hom}_{L\cap K}(H_i(\mathfrak{u} \cap \mathfrak{k}; X), Y),$$

and

$$\operatorname{Ext}^i_{(\mathfrak{q}\cap\mathfrak{k}, L\cap K)}(\operatorname{For} Y, X) = \operatorname{Hom}_{L\cap K}(Y, H^i(\mathfrak{u} \cap \mathfrak{k}; X)),$$

naturally in X and Y.

We can now say the main idea for the proofs of Theorem 6.2.4 and Theorem 6.2.5. Let us concentrate on the case of $\mathcal{L}_i(Z)$.

The multiplicity of a K-type V_δ in $\mathcal{L}_i(Z)$ is

$$\dim \operatorname{Hom}_K(\Pi_i(\operatorname{ind}{}^{\mathfrak{g}}_{\mathfrak{q}} Z^{\#}), V_\delta).$$

Since we are now interested only in the K-structure, we can replace Π_i in the above formula by $\Pi^{\mathfrak{k}}_i$ as in Corollary 6.2.7 (ii). Then the above multiplicity becomes

$$\dim \operatorname{Hom}_K(\Pi_i(\operatorname{ind}{}^{\mathfrak{g}}_{\mathfrak{q}} Z^{\#}), V_\delta) = \operatorname{Ext}^i_{(\mathfrak{k}, L\cap K)}(\operatorname{ind}{}^{\mathfrak{g}}_{\mathfrak{q}} Z^{\#}, V_\delta); \tag{6.3}$$

the equality follows from Corollary 6.2.7 (ii).

To analyze the last module above, one introduces a filtration of $\operatorname{ind}{}^{\mathfrak{g}}_{\mathfrak{q}} Z^{\#} = U(\mathfrak{u} \otimes Z^{\#}$ by $(\mathfrak{k}, L \cap K)$-submodules, using the "$\mathfrak{p}$-degree" (we already used an analogous filtration in 5.3.9.) This filtration comes from the filtration of $U(\mathfrak{u})$ given by

$$F_j U(\mathfrak{u}) = \operatorname{span}_{\mathbb{C}}\{uX_1 \dots X_k \,\big|\, k \leq j,\ u \in U(\mathfrak{u} \cap \mathfrak{k}),\ X_j \in \mathfrak{u} \cap \mathfrak{p}\}.$$

The associated graded modules are

$$\operatorname{Gr}_j \operatorname{ind}{}^{\mathfrak{g}}_{\mathfrak{q}} Z^{\#} = U(\mathfrak{u} \cap \mathfrak{k}) \otimes S^j(\mathfrak{u} \cap \mathfrak{p}) \otimes Z^{\#} = \operatorname{ind}{}^{\mathfrak{k}}_{\mathfrak{q}\cap\mathfrak{k}} S^j(\mathfrak{u} \cap \mathfrak{p}) \otimes Z^{\#}.$$

Thus the analogue of (6.3) for $\operatorname{Gr}_j \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} Z^{\#}$ is easy to calculate using Corollary 6.2.7 (i) and (iii) (part (iii) is applied to $\bar{\mathfrak{q}}$ instead of $\mathfrak{q}$):

$$\begin{aligned}\operatorname{Ext}^i_{(\mathfrak{k}, L\cap K)}(\operatorname{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}}^{\mathfrak{k}} S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^{\#}, V_\delta) &= \operatorname{Ext}^i_{(\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K)}(S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^{\#}, V_\delta)\\ &= \operatorname{Hom}_{L\cap K}(S^j(\mathfrak{u}\cap\mathfrak{p})\otimes Z^{\#}, H^i(\bar{\mathfrak{u}}\cap\mathfrak{k}; V_\delta)).\end{aligned}$$

One now obtains the statements of Theorem 6.2.4 and Theorem 6.2.5 by passing back to the setting of filtered modules, using an induction on the filtration degree.

6.2.8. Vanishing below middle degree. The main idea that enables one to obtain vanishing below middle degree is using vanishing above middle degree from Theorem 6.2.4, and Hard Duality Theorem 6.1.7 which in our present situation says

$$\Pi_j(V) = \Gamma^{2S-j}(V), \qquad V \in M(\mathfrak{g}, L\cap K) \tag{6.4}$$

We are assuming that K (and hence also $L\cap K$) is connected, so the twist $\bigwedge^{\text{top}} \mathfrak{o}$ disappears.

We are interested in $\mathcal{L}_j(Z) = \Pi_j(\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} Z^{\#})$ and $\mathcal{R}^j(Z) = \Gamma^j(\operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}} Z^{\#})$. If we could show that

$$\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} Z^{\#} \cong \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}} Z^{\#},$$

naturally in Z, then we would immediately get from (6.4) vanishing of both $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ for $j < S$, and also $\mathcal{L}_S(Z) = \mathcal{R}^S(Z)$.

It is not always true that the above ind and pro modules are isomorphic. In fact, we know from Example 5.1.5 that for $SL(2)$ this was true (precisely) when Z satisfied an appropriate positivity condition. We are now going to formulate an analogous result in general.

First, there is a map from ind to pro that was already mentioned in Example 5.1.5. Let $\mu_{\bar{\mathfrak{q}}} : U(\mathfrak{g}) \to U(\mathfrak{l})$ be the projection along $\mathfrak{u}U(\mathfrak{g}) + U(\mathfrak{g})\bar{\mathfrak{u}}$, like in the discussion after Proposition 6.2.1. We define

$$\phi_Z : \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} Z^{\#} = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^{\#} \to \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}} Z^{\#} = \operatorname{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), Z^{\#})_{L\cap K}$$

by

$$\phi_Z(u\otimes z)(v) = \pi(\mu_{\bar{\mathfrak{q}}}(vu))z,$$

where π denotes the representation of $(\mathfrak{l}, L\cap K)$ on $Z^{\#}$. To see that this is well defined with respect to $\otimes_{U(\bar{\mathfrak{q}})}$, let $X \in \bar{\mathfrak{q}}$. Then

$$\phi_Z(uX\otimes z)(v) = \pi(\mu_{\bar{\mathfrak{q}}}(vuX))z = \pi(\mu_{\bar{\mathfrak{q}}}(vu))\pi(X)z = \phi_Z(u\otimes \pi(X)z)(v),$$

since $\mu_{\bar{\mathfrak{q}}}(vuX) = \mu_{\bar{\mathfrak{q}}}(vu)\mu_{\bar{\mathfrak{q}}}(X)$ and $\pi(\mu_{\bar{\mathfrak{q}}}(X)) = \pi(X)$ by triviality of the $\bar{\mathfrak{u}}$-action on $Z^{\#}$. Similarly, $\phi_Z(u\otimes z)$ is a $\mathfrak{q}$-morphism.

So ϕ_Z is a linear map from $\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} Z^{\#}$ to $\operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}} Z^{\#}$. It is a $(\mathfrak{g}, L\cap K)$-morphism since

$$\phi_Z(Xu \otimes z)(v) = \pi(\mu_{\bar{\mathfrak{q}}}(vXu))z = \phi_Z(u \otimes z)(vX), \qquad X \in \mathfrak{g}.$$

Furthermore, it is natural in Z. Finally, it is nonzero, since $\phi_Z(1 \otimes z)(1) = z$.

One can actually obtain this map from general adjunction principles as is done in [KV]; we have chosen a more direct approach here.

Call $\lambda \in \mathfrak{h}^*$ real if it has real values on $i\mathfrak{t}_0 \oplus \mathfrak{a}_0$. Any $\lambda \in \mathfrak{h}^*$ can be written as $\lambda = \operatorname{Re}\lambda + i \operatorname{Im}\lambda$, with $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ real. Note that all roots are real in this sense.

One now proves the following (see [KV], Section V.7).

Proposition 6.2.9. *Suppose Z is admissible for $L \cap K$ and has infinitesimal character $\lambda \in \mathfrak{h}^*$ such that*

$$\langle \operatorname{Re}\lambda + \rho(\mathfrak{u}), \alpha \rangle \geq 0, \qquad \textit{for all} \quad \alpha \in \Delta(\mathfrak{u}). \tag{6.5}$$

Then the above map ϕ_Z is an isomorphism.

By the above remarks, this immediately implies

Corollary 6.2.10. *Under the conditions of Proposition 6.2.9, $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ vanish for $j \neq S$, and $\mathcal{L}_S(Z) \cong \mathcal{R}^S(Z)$.*

To prove Proposition 6.2.9, one first sees that whenever Z is admissible for $L \cap K$, $\operatorname{ind}{}_{\mathfrak{q}}^{\mathfrak{g}} Z^\#$ and $\operatorname{pro}{}_{\mathfrak{q}}^{\mathfrak{g}} Z^\#$ are isomorphic as $L \cap K$-modules (the isomorphism is not necessarily given by ϕ_Z). Namely, $U(\mathfrak{u})$ and $U(\bar{\mathfrak{u}})$ are both easily seen to be admissible for $L \cap K$, and moreover $U(\mathfrak{u})$ is the $L \cap K$-finite dual of $U(\bar{\mathfrak{u}})$. Hence

$$\operatorname{pro}{}_{\mathfrak{q}}^{\mathfrak{g}} Z^\# = \operatorname{Hom}_{\mathbb{C}}(U(\bar{\mathfrak{u}}), Z^\#)_{L\cap K} = U(\bar{\mathfrak{u}})^{*_{L\cap K}} \otimes Z^\# = U(\mathfrak{u}) \otimes Z^\# = \operatorname{ind}{}_{\mathfrak{q}}^{\mathfrak{g}} Z^\#$$

as $L \cap K$-modules.

It follows that if ϕ_Z is one-to-one, then it is actually an isomorphism, as it then induces a one-to-one linear map on each (finite-dimensional!) $L \cap K$-isotypic component. One furthermore proves that any nonzero $(\mathfrak{g}, L \cap K)$-submodule of $\operatorname{ind}{}_{\mathfrak{q}}^{\mathfrak{g}} Z^\#$ has non-zero intersection with $1 \otimes Z^\#$. Since ϕ_Z is one-to-one on $1 \otimes Z^\#$, it follows that it is one-to-one on $\operatorname{ind}{}_{\mathfrak{q}}^{\mathfrak{g}} Z^\#$.

A representation Z, or its infinitesimal character λ satisfying (6.5) are called *weakly good*. They are called good if they satisfy the same condition but with strict inequality. It is also said that λ is in the (weakly) good range. We will meet these and similar conditions again when we study irreducibility and unitarity.

6.3 Irreducibility and unitarity

Under appropriate conditions, the cohomologically induced modules are irreducible if the module we start with is irreducible. One such possible condition is the weakly good range assumption (6.5).

Theorem 6.3.1 (Irreducibility Theorem). *Let* $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ *be a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$*. Let* Z *be an irreducible admissible* $(\mathfrak{l}, L \cap K)$*-module with infinitesimal character* λ *in the weakly good range, i.e., (6.5) holds. Then* $\mathcal{L}_S(Z) \cong \mathcal{R}^S(Z)$ *is irreducible or zero. If* λ *is good, i.e., the inequality in (6.5) is strict for all* α*, then* $\mathcal{L}_S(Z) \cong \mathcal{R}^S(Z)$ *is irreducible and nonzero.*

For a proof of this theorem, see [KV], Section VIII.2 or [W], Section 6.6. We will only give some very general remarks about the main idea.

Since the question of irreducibility is about understanding submodules, one is led to consider

$$\operatorname{Hom}_{(\mathfrak{g},K)}(X, \mathcal{R}^S(Z)),$$

where X is an irreducible $(\mathfrak{g}, K)$-module. The idea is to understand this Hom - space using adjunction. However, to derive for example

$$\operatorname{Hom}_{(\mathfrak{g},K)}(X, \Gamma(V)) = \operatorname{Hom}_{(\mathfrak{g},L\cap K)}(X, V) \tag{6.6}$$

is not as easy as in the cases we considered in 6.2.6. Namely, on the left hand side there is not just one non-exact functor; both $\operatorname{Hom}_{(\mathfrak{g},K)}$ and Γ are only left exact. This situation is handled by spectral sequences; under appropriate conditions, one can (roughly) obtain derived functors of a composition $G \circ F$ from the compositions of derived functors of G and F by a certain inductive procedure. Since the derived functors of the right hand side of (6.6) are $\operatorname{Ext}^{\cdot}_{(\mathfrak{g},L\cap K)}(X, V)$, this means that one gets $\operatorname{Ext}^{p+q}_{(\mathfrak{g},L\cap K)}(X, V)$ as the limit of a spectral sequence which starts from $\operatorname{Ext}^{p}_{(\mathfrak{g},K)}(X, \Gamma^q(V))$. For $V = \operatorname{pro} Z^{\#}$ with Z in the weakly good range, this simplifies greatly because of vanishing for $q \neq S$: the spectral sequence "collapses" and one actually obtains an equality.

One then similarly works on the adjunction

$$\operatorname{Hom}_{(\mathfrak{g},L\cap K)}(X, \operatorname{pro} Z^{\#}) = \operatorname{Hom}_{(\mathfrak{q},L\cap K)}(X, Z^{\#}) = \operatorname{Hom}_{(\mathfrak{l},L\cap K)}(X_{\mathfrak{u}}, Z^{\#}).$$

Deriving this produces another spectral sequence, which does not collapse, but one is eventually able to conclude

$$\operatorname{Hom}_{(\mathfrak{g},K)}(X, \mathcal{R}^S(Z)) = \operatorname{Hom}_{(\mathfrak{l},L\cap K)}(H_S(\mathfrak{u}; X), Z^{\#}). \tag{6.7}$$

After obtaining this, one relates the right hand side to similar expressions involving $\mathfrak{u} \cap \mathfrak{k}$-homology instead of $\mathfrak{u}$-homology; the two kinds of cohomology are related by the Hochschild-Serre spectral sequence, which thus also has to be studied. This relationship is then tied with the analysis of the so called *bottom layer* K-types. The outcome is that if there is a sufficiently dominant K-type in the bottom layer, then it survives through all the spectral sequences and X has to contain it. This then determines X as the submodule of $\mathcal{R}^S(Z)$ generated by this K-type; thus $\mathcal{R}^S(Z)$ has a unique irreducible sub. A dual argument shows that $\mathcal{R}^S(Z)$ also has a unique irreducible quotient, generated by the same K-type. It follows that $\mathcal{R}^S(Z)$ is irreducible.

The situations when there is no such sufficiently dominant K-type as needed for this argument are handled by using translations functors.

Finally, let us remark that the dominance condition for Z with infinitesimal character λ to be in the weakly good range can be weakened to require only

$$\frac{2\langle\lambda+\rho(\mathfrak{u}),\alpha\rangle}{|\alpha|^2}\notin\{-1,-2,-3,\dots\},\qquad \alpha\in\Delta(\mathfrak{u}). \tag{6.8}$$

This is enough to get vanishing of $\mathcal{R}^j(Z)$ and $\mathcal{L}_j(Z)$ for $j\neq S$, and also $\mathcal{R}^S(Z)\cong\mathcal{L}_S(Z)$. Once this is proved, irreducibility follows just as with ordinary dominance, since the needed results about translation functors only require this weaker, integral dominance. For details, see [KV], Section VIII.3.

We now consider unitarity of cohomologically induced modules. Recall that a $(\mathfrak{g},K)$-module V is called unitary (or more precisely unitarizable, or infinitesimally unitary) if there is an invariant Hermitian positive definite form $\langle\,,\rangle$ on V. Invariance means that the operators $X\in\mathfrak{g}_0$ are skew hermitian, while $k\in K$ are unitary with respect to $\langle\,,\rangle$. Operators $X\in\mathfrak{g}$ satisfy

$$\langle Xv,w\rangle=-\langle v,\bar{X}w\rangle,$$

where the bar denotes conjugation with respect to $\mathfrak{g}_0$.

The main result about unitarity roughly says that if Z is unitary for $(\mathfrak{l},L\cap K)$ and weakly good, then $\mathcal{L}_S(Z)$ is unitary for $(\mathfrak{g},K)$. In fact, the condition in the following theorem is weaker than the weakly good condition (6.5), but stronger than (6.8).

Theorem 6.3.2 (Unitarizability Theorem). *Let $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ be a θ-stable parabolic subalgebra, let $\mathfrak{h}\subset\mathfrak{l}$ be a Cartan subalgebra, and choose $\Delta^+(\mathfrak{g},\mathfrak{h})\supset\Delta(\mathfrak{u})$. Let Z be an irreducible unitary $(\mathfrak{l},L\cap K)$-module with infinitesimal character λ satisfying*

$$\frac{2\langle\lambda+\rho(\mathfrak{u}),\alpha\rangle}{|\alpha|^2}\notin(-\infty,-1],\qquad \forall\alpha\in\Delta(\mathfrak{u}).$$

Then the $(\mathfrak{g},K)$-module $\mathcal{L}_S(Z)$ is unitary.

The Hermitian form one can define on $\mathcal{L}_S(Z)$ is the so called *Shapovalov form* $\langle\,,\rangle_G$, which is induced from the given form $\langle\,,\rangle_L$ on Z as follows. Recall first that $\mathcal{L}_S(Z)$ is the S-th cohomology of the complex

$$\dots\to\bigwedge^{S+1}\mathfrak{o}\otimes_{L\cap K}R(K)\otimes\ \mathrm{ind}\ Z^{\#}\to\bigwedge^{S}\mathfrak{o}\otimes_{L\cap K}R(K)\otimes\ \mathrm{ind}\ Z^{\#}\to\dots$$

with the Koszul differential. Here $\otimes_{L\cap K}$ can be identified with the $L\cap K$-invariants in $\otimes_{\mathbb{C}}$. The Shapovalov form on the level of this complex is given as the pairing of the components of degree j and $2S-j$, via

$$\langle\xi\otimes f\otimes u\otimes z,\xi'\otimes f'\otimes u'\otimes z'\rangle_G=\epsilon(\xi\wedge\bar{\xi}')(\int_K f\bar{f}')\langle\mu_{\bar{\mathfrak{q}}}(\bar{u}'^t u)z,z'\rangle_L,$$

where ϵ is a suitably chosen element of $(\bigwedge^{\mathrm{top}} \mathfrak{o})^*$, and $\mu_{\bar{\mathfrak{q}}}$ is the same map we used in 6.2.8. Here ξ and ξ' are elements of $\bigwedge^{\cdot} \mathfrak{o}$ of degrees j and $2S - j$ respectively, $f, f' \in R(K)$ and $u \otimes z, u' \otimes z' \in \operatorname{ind} Z^{\#} = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} Z^{\#}$.

One now proves that the above pairing descends to the level of cohomology and hence in particular defines a form on $\mathcal{L}_S(Z)$. For this, it is convenient to reformulate the pairing in terms of maps (using the notion of Hermitian dual), for then one can use the Hard Duality Theorem 6.1.7. One further shows that if $\langle\,,\rangle_L$ is Hermitian, respectively non-degenerate, then so is $\langle\,,\rangle_G$.

To prove the Unitarizability Theorem, one still has to prove that if $\langle\,,\rangle_L$ is positive definite, then so is $\langle\,,\rangle_G$. This can be done by considering the so called *signature character* of an admissible Hermitian $(\mathfrak{g}, K)$-module V. The signature character is a formal $\mathbb{Z}$-linear combination of the K-isotypic components $V(\delta)$ of V, each taken with the multiplicity equal to the signature of the given form on V restricted to $V(\delta)$. Clearly, the form on V is positive definite if and only if the signature character is equal to the K-character of V, that is, the formal sum of $V(\delta)$ each taken with the multiplicity with which it appears in V.

The point is that the signature character of $\mathcal{L}_S(Z)$ can be calculated explicitly, and seen to indeed be equal to the K-character. This can be found in [KV], Section IX.3–IX.5 or in [W], Section 6.7.

6.4 $A_{\mathfrak{q}}(\lambda)$ modules

By definition, $A_{\mathfrak{q}}(\lambda)$ is the $(\mathfrak{g}, K)$-module $\mathcal{L}_S(\mathbb{C}_\lambda)$, where $\mathbb{C}_\lambda$ is the one-dimensional $(\mathfrak{l}, L \cap K)$-module with weight λ. Here $\lambda \in \mathfrak{h}^*$ is L-integral and orthogonal to the roots of $\mathfrak{l}$, as it has to be for $\mathbb{C}_\lambda$ to be well defined.

Since the $\mathfrak{l}$-infinitesimal character of $\mathbb{C}_\lambda$ is $\lambda + \rho_{\mathfrak{l}}$, Proposition 6.2.2 implies that the $\mathfrak{g}$-infinitesimal character of $A_{\mathfrak{q}}(\lambda)$ is $\lambda + \rho_{\mathfrak{l}} + \rho(\mathfrak{u}) = \lambda + \rho$.

If $\Lambda = \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is dominant for $\mathfrak{k}$, and if in addition $\Lambda + 2\rho_{\mathfrak{k}}$ is dominant for $\Delta(\mathfrak{u} \cap \mathfrak{p})$, then Λ defines the unique *lowest K-type* of $A_{\mathfrak{q}}(\lambda)$. Namely, by definition, a K-type of highest weight μ is a lowest (or minimal) K-type of a $(\mathfrak{g}, K)$-module V if $|\mu + 2\rho_{\mathfrak{k}}|$ is the smallest possible for all K-types of V. By the estimates of multiplicities of K-types (Theorem **??**), if a K-type with highest weight Λ' occurs in $A_{\mathfrak{q}}(\lambda)$, then Λ' can be written as $\Lambda + \sum_j \beta_j$ for some weights β_j of $\mathfrak{t}$ in $\mathfrak{u} \cap \mathfrak{p}$.

It follows that

$$|\Lambda' + 2\rho_{\mathfrak{k}}|^2 = |\Lambda + 2\rho_{\mathfrak{k}} + \sum_j \beta_j|^2 \geq |\Lambda + 2\rho_{\mathfrak{k}}|^2;$$

namely by the assumed dominance, $\langle \Lambda + 2\rho_{\mathfrak{k}}, \beta_j \rangle \geq 0$ for any j. Also, the equality in this inequality is possible only for $\Lambda' = \Lambda$. This means Λ indeed determines the unique lowest K-type.

6.4.1. Improved vanishing results. Since $\mathbb{C}_\lambda$ has $\mathfrak{l}$-infinitesimal character $\lambda+\rho_{\mathfrak{l}}$, the weakly good assumption (6.5) for $\mathbb{C}_\lambda$ reads

$$\langle \operatorname{Re}\lambda+\rho,\alpha\rangle \geq 0, \qquad \text{for all} \quad \alpha\in\Delta(\mathfrak{u}). \tag{6.9}$$

If this holds, then by Corollary 6.2.10 $\mathcal{L}_j(\mathbb{C}_\lambda)$ and $\mathcal{R}^j(\mathbb{C}_\lambda)$ vanish for $j\neq S$, and $\mathcal{L}_S(\mathbb{C}_\lambda)\cong\mathcal{R}^S(\mathbb{C}_\lambda)$.

For $A_{\mathfrak{q}}(\lambda)$ modules, one can however prove a stronger result, namely that the same conclusions hold under the assumption that

$$\langle \operatorname{Re}\lambda+\rho(\mathfrak{u}),\alpha\rangle \geq 0, \qquad \text{for all} \quad \alpha\in\Delta(\mathfrak{u}). \tag{6.10}$$

An L-weight λ orthogonal to the roots of $\mathfrak{l}$ (so that $\mathbb{C}_\lambda$ is defined) is said to be in the weakly fair range (or weakly fair) if (6.10) holds. If the inequality in (6.10) is strict, then λ is said to be in the fair range.

The assumption (6.10) of fair range is easily seen to be weaker than the assumption (6.9) of good range, since λ is orthogonal to the roots of $\mathfrak{l}$.

Theorem 6.4.2. *Let $Z=\mathbb{C}_\lambda$ where $\lambda\in\mathfrak{h}^*$ is integral for L and orthogonal to the roots of $\mathfrak{l}$. Assume that (6.10) holds. Then $\mathcal{L}_j(\mathbb{C}_\lambda)$ and $\mathcal{R}^j(\mathbb{C}_\lambda)$ vanish for $j\neq S$, and $\mathcal{L}_S(\mathbb{C}_\lambda)\cong\mathcal{R}^S(\mathbb{C}_\lambda)$.*

The proof of this result is similar to the proof of the general result in the weakly good range (Corollary 6.2.10). Essentially, one proved that $\operatorname{ind}{}_{\mathfrak{q}}^{\mathfrak{g}}\mathbb{C}_\lambda^{\#}$ is irreducible.

6.4.3. A Blattner formula for K-types of $A_{\mathfrak{q}}(\lambda)$. In case of vanishing given by Theorem 6.4.2, the Euler characteristic formula from Theorem 6.2.5 becomes an explicit formula for multiplicities of the K-types of $A_{\mathfrak{q}}(\lambda)=\mathcal{L}_S(\mathbb{C}_\lambda)\cong\mathcal{R}^S(\mathbb{C}_\lambda)$. By a calculation using Kostant's formula for $\mathfrak{u}\cap\mathfrak{k}$-cohomology, one gets:

Theorem 6.4.4. *With assumptions as in Theorem 6.4.2, the multiplicity of the K-type V_δ in $A_{\mathfrak{q}}(\lambda)$ is*

$$\sum_{w\in W^1}\det w\;\;\mathcal{P}\big(w(\delta+\rho_{\mathfrak{k}})-(\Lambda+\rho_{\mathfrak{k}})\big),$$

where $\Lambda=\lambda+2\rho(\mathfrak{u}\cap\mathfrak{p})$.

Here $W^1\subset W(\mathfrak{k},\mathfrak{t})$ is defined to be the set of those w for which $\alpha\in\Delta^+(\mathfrak{k},\mathfrak{t})$, $w^{-1}\alpha<0$ implies $\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{k})$. $\mathcal{P}(\nu)$ denotes the multiplicity of the weight $\nu\in\mathfrak{h}^*$ in $S(\mathfrak{u}\cap\mathfrak{p})^{\mathfrak{l}\cap\mathfrak{k}\cap\mathfrak{n}}$. The algebra $\mathfrak{n}$ corresponds to a choice of positive roots for $\mathfrak{g}$, compatible with $\mathfrak{u}$ and $\Delta^+(\mathfrak{k},\mathfrak{t})$.

6.4.5. Unitarity of $A_{\mathfrak{q}}(\lambda)$ modules. Let us first note that the proof of Unitarizability Theorem works for any irreducible unitary $(\mathfrak{l},L\cap K)$-module Z such that $\operatorname{ind}(Z\otimes\mathbb{C}_{t\rho(\mathfrak{u})})^{\#}$ is irreducible for all $t\geq 0$. Namely, the dominance assumption of the theorem is used in the proof exactly to deduce irreducibility of $\operatorname{ind} Z_t^{\#}$. For $Z = \mathbb{C}_\lambda$ this is true, and it is proved in the same way as the irreducibility of $\operatorname{ind} \mathbb{C}_\lambda^{\#}$ needed for Theorem 6.4.2. This implies the following improved unitarizability result for $A_{\mathfrak{q}}(\lambda)$:

Proposition 6.4.6. *Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a θ-stable Cartan subalgebra of $\mathfrak{l}_0$, and choose $\Delta^+(\mathfrak{g},\mathfrak{h}) \supset \Delta(\mathfrak{u})$.*

Suppose $\mathbb{C}_\lambda$ is a one-dimensional unitary $(\mathfrak{l}, L \cap K)$-module, i.e., λ is real on $\mathfrak{t}_0$ and imaginary on $\mathfrak{a}_0$ (and vanishes on $[\mathfrak{l},\mathfrak{l}]$). Suppose also that λ is in the weakly fair range, range i.e., that (6.10) holds. Then $A_{\mathfrak{q}}(\lambda)$ is a unitary $(\mathfrak{g}, K)$-module.

6.4.7. Irreducibility of $A_{\mathfrak{q}}(\lambda)$ modules. Suppose λ is in the weakly fair range, i.e., (6.10) holds. From the proof of Irreducibility Theorem, we know that $A_{\mathfrak{q}}(\lambda + 2m\rho(\mathfrak{u}))$ is irreducible for sufficiently large integers $m > 0$. We however cannot conclude that $A_{\mathfrak{q}}(\lambda)$ is irreducible or zero by using the translation functor from $\lambda + \rho + 2m\rho(\mathfrak{u})$ to $\lambda + \rho$. This approach worked for the weakly good range, but not for λ in the weakly fair range, as in the last case $\lambda + \rho$ need not be integrally dominant.

However, what one can do in this case is come up with a different version of the main results about translation principle, that works for $A_{\mathfrak{q}}(\lambda)$ modules. The starting remark is that $A_{\mathfrak{q}}(\lambda)$ modules are obtained from generalized Verma modules by applying Bernstein functors, which commute with translation.

The conclusion is that irreducibility can be proved in the weakly fair range, but only with an additional assumption. Namely, let K' be the maximal compact subgroup of a complex connected Lie group with Lie algebra $\mathfrak{g}$. Let L' be the (connected) subgroup of K' corresponding to the compact form of $\mathfrak{l}$. The $(\bar{\mathfrak{q}}, L')$-map $S^n(\mathfrak{g}) \to S^n(\mathfrak{g}/\bar{\mathfrak{q}})$ induced by the projection $\mathfrak{g} \to \mathfrak{g}/\bar{\mathfrak{q}}$ corresponds under adjunction to

$$\Phi_n : S^n(\mathfrak{g}) \to \Gamma_{L'}^{K'} \operatorname{pro}{}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} S^n(\mathfrak{g}/\bar{\mathfrak{q}}).$$

Theorem 6.4.8. *Assume that Φ_n defined above is onto for every $n \geq 0$. Suppose λ is in the weakly fair range, i.e., (6.10) holds. Then the $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is irreducible or zero.*

The assumption is not easy to check in general; however it can be proved that it holds for example if $\mathfrak{u}$ is abelian (the result is due to Hesselink; see [KV], Proposition 8.75.)

See [KV], Section VIII.5 for a proof of Theorem 6.4.8, and also for some instructive examples.

6.5 Unitary modules with strongly regular infinitesimal character

In this section we state the results of [SR] which will be needed in the next chapter. As before, let $\mathfrak{h}$ be the complexification of a fundamental Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Given any weight $\Lambda \in \mathfrak{h}^*$, fix a choice of positive roots $\Delta^+(\Lambda, \mathfrak{h})$ for Λ so that

$$\Delta^+(\Lambda, \mathfrak{h}) \subset \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \operatorname{Re} \langle \Lambda, \alpha \rangle \geq 0\}.$$

Set

$$\rho(\Lambda) = \frac{1}{2} \sum_{\alpha \in \Delta^+(\Lambda, \mathfrak{h})} \alpha.$$

A weight $\Lambda \in \mathfrak{h}^*$ is said to be real if

$$\Lambda \in i\mathfrak{t}_0^* + \mathfrak{a}_0^*.$$

For a general $\Lambda \in \mathfrak{h}^*$, one defines its real part $\operatorname{Re} \Lambda$ in the obvious way. We say that Λ is strongly regular if its real part satisfies

$$\operatorname{Re} \langle \Lambda - \rho(\Lambda), \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta^+(\Lambda, \mathfrak{h}).$$

Salamanca-Riba [SR] proved:

Theorem 6.5.1. (Salamanca-Riba) *Suppose that X is an irreducible unitary $(\mathfrak{g}, K)$-module with strongly regular infinitesimal character $\Lambda \in \mathfrak{h}^*$. Then there exist a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an admissible character λ of L such that X is isomorphic to $A_{\mathfrak{q}}(\lambda)$.*

In fact, the above result is proved in [SR] only for real Λ. There is however a remark there explaining how to generalize the result to an arbitrary strongly regular Λ using the technique of reduction to real infinitesimal character. We warn the reader that instead of [KV] this remark should actually quote [Kn1].

7

Discrete Series

One of the greatest achievements of mathematics in the 20th century is Harish-Chandra's classification of discrete series representations of semisimple Lie groups. Let G be a noncompact semisimple Lie group with a maximal compact subgroup K. Discrete series representations are those irreducible unitary representations of G which occur as subrepresentations in the Plancherel decomposition of $L^2(G)$. Harish-Chandra proved that a necessary and sufficient condition for G to have a discrete series is to have a compact Cartan subgroup. He constructed the characters of all discrete series representations. Speaking of Harish-Chandra's work on discrete series, we quote Varadarajan in his article "Harish-Chandra, His Work, and its Legacy" [Va]: "In my opinion the character problem and the problem of constructing the discrete series were the ones that defined him, by stretching his formidable powers to their limit. The Harish-Chandra formula for the characters of discrete series is the single most beautiful formula in the theory of infinite dimensional unitary representations." Harish-Chandra "actually wrote down all the proofs in an extraordinary sequence of 8 papers [1946a]-[1966b], totaling 461 journal pages constituting one of the most remarkable series of papers in the annals of scientific research in our times–remarkable because how long it took him to reach his goal, remarkable for how difficult was the journey and how it was punctuated by illness, remarkable for how unaided his achievement was, and finally, remarkable for the beauty and inevitability of his theorems."

Harish-Chandra did not construct the discrete series representations explicitly. The explicit construction was first accomplished by Schmid [S1, S2] using L^2-cohomology. Paratharathy [Par] defined a Dirac operator in the appropriate setting of Lie algebras and showed that most of discrete series representations can be realized as kernels of the Dirac operator acting on spinor bundles over the symmetric space G/K. Atiyah-Schmid [AS] extended this construction to all discrete series representations. Moreover, [AS] also gives an independent proof of the existence and exhaustion of discrete series.

In this chapter we will explain how the use of Dirac cohomology simplifies the proofs of some of the very deep theorems on discrete series representations. We follow essentially the same line as [AS]. With the new tool Dirac cohomology in hand, some of the proofs are much shorter.

7.1 L^2-index Theorem

Throughout this chapter we let G be a noncompact connected semisimple Lie group with finite center. Let K be a maximal compact subgroup of G. The analogue of the Peter-Weyl Theorem for a noncompact semisimple Lie group G is the Plancherel Theorem, which is concerned with decomposing the left and right regular representations, i.e., the action on $L^2(G)$ induced by left and right translations. Let $\widehat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. Then the Plancherel theorem asserts a direct integral decomposition

$$L^2(G) \cong \int_{\widehat{G}} H_j \widehat{\otimes} H_j^* d\mu(j), \tag{7.1}$$

where $\mu(j)$ is a positive measure on $\widehat{G}$, H_j is the irreducible representation indexed by $j \in \widehat{G}$, and $H_j \widehat{\otimes} H_j^*$ is the completed tensor product of the Hilbert space H_j and its dual H_j^*. The isomorphism (7.1) is compatible with both the left and right actions of G. We are concerned with the representations occuring in the discrete spectrum.

Definition 7.1.1. A representation H_{j_0} in the decomposition (7.1) is said to be a *discrete series* representation if $j_0 \in \widehat{G}$ has positive measure $\mu(\{j_0\})$.

If H_{j_0} is a discrete series representation, then it occurs as a direct summand of the left or right regular representation. A representation H_{j_0} is a discrete series representation if and only if the matrix coefficients of H_{j_0} are in $L^2(G)$. Recall that a *matrix coefficient* of a representation (π, H) is the function on G given by $g \mapsto \langle \pi(g)v, w\rangle$ for fixed $v, w \in H$. We state this fact as a proposition.

Proposition 7.1.2. *Let (π, H) be an irreducible unitary representation of G. Then the following three conditions are equivalent:*

i) Some nonzero K-finite matrix coefficient of π is in $L^2(G)$.

ii) All matrix coefficients of π are in $L^2(G)$.

iii) H is equivalent to a direct summand of the right regular representation of G on $L^2(G)$.

For a proof of this proposition we refer to Proposition 9.6 of [Kn1].

If F is a finite-dimensional unitary K-module, then $\mathcal{F} = G \times_K F$ is a homogeneous vector bundle over the symmetric space G/K. Then the L^2-sections $L^2(G/K, F)$ of $\mathcal{F}$ can be identified with the space of right K-invariants in $L^2(G) \otimes F$. It follows from (7.1) that

$$L^2(G/K,F) \cong \int_{\widehat{G}} H_j \otimes W_j \, d\mu(j), \tag{7.2}$$

where $W_j \cong \operatorname{Hom}_K(F^*, H_j^*)$ is the K-invariant part of $H_j^* \otimes F$, which is finite-dimensional by admissibility of irreducible unitary representations. Furthermore, it follows from general theory of von Neumann algebras that any closed G-invariant subspace $U \subset L^2(G/K,F)$ has a compatible decomposition

$$U \cong \int_{\widehat{G}} H_j \otimes U_j d\mu(j), \text{ with } U_j \subset W_j. \tag{7.3}$$

We now assume that $\operatorname{rank} G$ is equal to $\operatorname{rank} K$, i.e., that G has a compact Cartan subgroup T. As a compact Cartan subgroup, T is unique up to conjugacy. Also it is connected, hence is a torus. We fix a maximal compact subgroup $K \supset T$. This is always possible, since any two maximal compact subgroups are conjugate. Let $\mathfrak{t}_0$ and $\mathfrak{k}_0$ denote the Lie algebras of T and K, and let $\mathfrak{t}$ and $\mathfrak{k}$ be their complexifications.

Let Φ be the root system of $(\mathfrak{g}, \mathfrak{t})$. A root $\alpha \in \Phi$ is a *compact root* if $\mathfrak{g}_\alpha \subset \mathfrak{t}$ and is a *noncompact root* if $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{k}$ with respect to the Killing form. Let Φ_c and Φ_n be the sets of compact and noncompact roots, respectively. Also, Φ_c is the root system of $(\mathfrak{k}, \mathfrak{t})$, hence is a root subsystem of Φ. Let $W_\mathfrak{g}$ and $W_\mathfrak{k}$ be the Weyl groups associated with the root systems Φ and Φ_c. Thus $W_\mathfrak{k} \subset W_\mathfrak{g}$.

Let $\widehat{T}$ be the character group of T. Then $\widehat{T}$ is isomorphic to the weight lattice Λ,

$$\widehat{T} \cong \Lambda \subset i\mathfrak{t}_0^*$$

contained in $i\mathfrak{t}_0^*$, the real vector space of all those linear functions on $\mathfrak{t}$, which assume purely imaginary values on $\mathfrak{t}_0$.

The equal rank condition implies that $\dim G/K$ is even, and so the space of spinors S decomposes into a direct sum $S = S^+ \oplus S^-$. We write E_μ for the irreducible unitary representation of $\widetilde{K}$, the two fold cover of K, with highest weight μ. We note that E_μ may or may not descend to K. For the fixed positive root system Φ_c^+ for which μ is dominant, we choose a positive root system Φ^+ so that $\Phi^+ \supset \Phi_c^+$ and $\mu + \rho_c$ is dominant. Here ρ_c is the half sum of compact positive roots. We denote by ρ_n the half sum of noncompact positive roots. Then $\rho = \rho_c + \rho_n$ is the half sum of all positive roots.

We assume that $\lambda = \mu - \rho_n$ is a weight of K. Then the $\widetilde{K}$-modules $S^\pm \otimes E_\mu$ descend to K. The Plancherel decomposition in (7.2) applied to this special case $F = S^\pm \otimes E_\mu$ implies the following decomposition

$$L^2(G/K, S^\pm \otimes E_\mu) \cong \int_{\widehat{G}} H_j \otimes V_j^\pm d\mu(j), \tag{7.4}$$

where $V_j^\pm \cong \operatorname{Hom}_K(E_\mu^*, H_j^* \otimes S^\pm)$ is the K-invariant part of $H_j^* \otimes S^\pm \otimes E_\mu$.

The Dirac operator D acts on the smooth sections of the twisted spin bundle in a similar way as described in the Chapter 4, i.e., the differential part of D acts from the right as left invariant vector fields. Since D switches the two factors S^+ and S^- in the space of spinors S, one has

$$D_\mu^\pm \colon C^\infty(G/K, S^\pm \otimes E_\mu) \to C^\infty(G/K, S^\mp \otimes E_\mu).$$

We extend the Dirac operators $D_\mu^\pm$ to closed operators (still denoted by $D_\mu^\pm$) on the corresponding Hilbert spaces:

$$D_\mu^\pm \colon L^2(G/K, S^\pm \otimes E_\mu) \to L^2(G/K, S^\mp \otimes E_\mu).$$

Let Ker $D_\mu^\pm$ be the L^2 null spaces of the Dirac operators $D_\mu^\pm$. They are G-invariant closed subspaces in $L^2(G/K, S^\pm E_\mu)$. The K-equivariant map defined by the Dirac operator

$$D^\pm \colon H_j^* \otimes S^\pm \to H_j^* \otimes S^\mp$$

induces a map

$$D_j^\pm(\mu) \colon \text{ Hom }_K(E_\mu^*, , H_j^* \otimes S^\pm) \to \text{ Hom }_K(E_\mu^*, , H_j^* \otimes S^\mp).$$

It follows from (7.3) that one has the decomposition

$$\text{Ker } D_\mu^\pm \cong \int_{\widehat{G}} H_j \otimes \text{ Ker } D_j^\pm(\mu) d\mu(j).$$

The L^2-*index theorem* calculates the following

$$\text{Index}\, D_\mu^+ = \int_{\widehat{G}} (\text{ dim Ker } D_j^+(\mu) - \text{ dim Ker } D_j^-(\mu)) d\mu(j). \tag{7.5}$$

Since S^+ and S^- are self-dual if $q = \frac{1}{2} \dim G/K$ is even and dual to each other if q is odd, in above formula

$$\text{dim Ker } D_j^+(\mu) - \text{ dim Ker } D_j^-(\mu)$$

is equal to dim Hom $_K(E_\mu^*, , H_j^* \otimes S^+) -$ dim Hom $_K(E_\mu^*, , H_j^* \otimes S^-)$ and it is in turn equal to

$$(-1)^q (\text{ dim Hom }_K(E_\mu, , H_j \otimes S^+) - \text{ dim Hom }_K(E_\mu, , H_j \otimes S^-)).$$

Theorem 7.1.3. (Atiyah-Schmid [AS]) *Let G be a connected noncompact semisimple Lie group with finite center and a maximal compact subgroup K. Assume that a Cartan subgroup T of K is also a Cartan subgroup of G. Let E_μ be a $\widetilde{K}$-module with highest weight μ so that $E_\mu \otimes S^\pm$ descend to K. Choose a system of positive roots Φ^+ compatible with Φ_c^+ and $\mu + \rho_c$ is Φ^+-dominant. Then one has*

$$\text{Index}\, D_\mu^+ = \frac{\prod_{\alpha \in \Phi^+} \langle \mu + \rho_c, \alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle}. \tag{7.6}$$

We set $\lambda = \mu - \rho_n = (\mu + \rho_c) - \rho$ and denote by $d(\lambda) = d(\mu - \rho_n)$ the right hand side of (7.6). If λ happens to be $\mathfrak{g}$-integral, then $d(\lambda)$ gives the dimension of the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ. In general $d(\lambda)$ is a polynomial in λ.

The calculation of this index was first done by Atiyah-Schmid by assuming there is a torsion-free discrete subgroup Γ of G so that $\Gamma\backslash G$ is compact. In case such Γ exists, the index can be computed by Hirzebruch proportionality principle by reducing the problem to the well-known results for compact groups [AS]. We will describe this calculation in the next chapter when we calculate the dimensions of spaces of automorphic forms. Borel showed that for G linear there exists a torsion free discrete subgroup Γ of G so that $\Gamma\backslash G$ and $X = \Gamma\backslash G/K$ are compact smooth manifolds. However, the existence of such Γ for non-linear G is not guaranteed. Nevertheless, for a non-linear group G one can take Γ to be the pull back of a torsion-free discrete subgroup for the adjoint group Ad (G). Atiyah and Schmid showed that the index theorem is still valid by considering the projective bundles over $\Gamma\backslash G/K$ [AS'].

An immediate consequence of the above L^2-index theorem is the existence of discrete series for equal rank groups. We address this in the next section.

7.2 Existence of discrete series

In this section we retain the notation in the previous section. So G is a connected noncompact semisimple Lie group and G has a compact Cartan subgroup T. The L^2-index theorem implies that $\text{Ker } D_\mu$ is nonzero provided $\mu + \rho_c$ is regular. Since $\text{Ker } D_\mu = \text{Ker } D_\mu^+ \oplus \text{Ker } D_\mu^-$, we can decompose it as

$$\text{Ker } D_\mu \cong \int_{\widehat{G}} H_j \otimes \text{ Ker } D_j^+(\mu) d\mu(j) \oplus \int_{\widehat{G}} H_j \otimes \text{ Ker } D_j^-(\mu) d\mu(j). \tag{7.7}$$

Here $D_j(\mu)\colon \text{ Hom }_K(E_\mu^*, H_j^* \otimes S) \to \text{ Hom }_K(E_\mu^*, H_j^* \otimes S)\}$. Since S is self-dual, $\text{Hom }_K(E_\mu^*, H_j^* \otimes S) \cong \text{ Hom }_K(E_\mu, H_j \otimes S)$. It follows that H_j occurs in the decomposition of $\text{Ker } D_\mu^\pm$ if and only if E_μ is in the Dirac cohomology $\text{Ker } D\colon H_j \otimes S \to H_j \otimes S$. By the proved conjecture of Vogan, the infinitesimal character of H_j is $\mu + \rho_c$. A fundamental theorem of Harish-Chandra asserts that there are only finitely many inequivalent irreducible admissible representations with a fixed infinitesimal character. Thus, if $\text{Ker } D_\mu$ is nonzero, the corresponding H_j in the decomposition must be in the discrete spectrum, i.e., a discrete series representation.

As before, we choose a system of positive roots compatible to the given system of positive compact roots so that $\mu + \rho_c$ is dominant. If $\lambda = \mu - \rho_n = \mu + \rho_c - \rho$ is also dominant, then the infinitesimal character $\lambda + \rho$ of H_j in $\text{Ker } D_\mu^\pm$ is strongly regular in terminology of [SR] and therefore H_j is an $A_{\mathfrak{q}}(\lambda)$-module for some theta-stable parabolic subalgebra $\mathfrak{q}$ [SR] as was discussed in Section 6.4. In case λ is regular with respect to the noncompact roots, $\mathfrak{q}$ must be the

θ-stable Borel subalgebra and this $A_{\mathfrak{q}}(\lambda)$-module is isomorphic to $A_{\mathfrak{b}}(\lambda)$ with the lowest K-type $\lambda + 2\rho_n$. This proves

Proposition 7.2.1. *Let G be a connected noncompact semisimple Lie group having finite center and a compact Cartan subgroup T. Let E_μ be a $\widetilde{K}$-module with highest weight μ so that $E_\mu \otimes S^{\pm}$ decent to K, i.e., $\lambda = \mu - \rho_n$ is in the weight lattice Λ. Choose a system of positive roots Φ^+ compatible to the given system of positive compact roots Φ_c^+ so that $\mu + \rho_c$ is Φ^+-dominant. If in addition*

a) $\lambda = \mu - \rho_n$ is Φ^+-dominant, and

b) $\lambda = \mu - \rho_n$ is regular with respect to noncompact roots

then Ker D_μ^+ *is isomorphic to a discrete series representation $A_{\mathfrak{b}}(\lambda)$ for a θ-stable Borel subalgebra $\mathfrak{b}$ with Plancherel measure $d(\lambda)$ and* Ker $D_\mu^- = 0$. *In particular, this proves the existence of discrete series representations for G with a compact Cartan subgroup T.*

We will show in Section 7.4 that condition (c) in the above proposition can be removed. Moreover, every discrete series representation is realized as Ker D_μ^+ for an appropriate μ. We will also show in the next section that the equal rank condition is also necessary for G to have discrete series.

7.3 Global Characters

Let (π, V) be an irreducible admissible representation of G. Then for each $f \in C_c^\infty(G)$,

$$\pi(f) = \int_G f(g)\pi(g)dg$$

is a well defined bounded linear operator on V. Moreover, $\pi(f)$ is of trace class, and

$$f \mapsto \text{ tr } \pi(f)$$

is a distribution on G. We denote this distribution by $\Theta_G(\pi)$, or simply Θ_π, and name it the *global character* of π, or simply the character of π, when there is no confusion. Harish-Chandra showed that the character Θ_π is an *invariant eigendistribution* on G. That is, it is invariant under group conjugation, and every member of the center $Z(\mathfrak{g})$ of the universal enveloping $U(\mathfrak{g})$ acts on it by a scalar. For an exposition of character theory we refer to Chapter X of [Kn1]. We only include the fundamental results here without going into detailed proofs.

We now recall the definition of G', the set of *regular semisimple elements* of G. Let r denote the rank of G, which is the minimal possible multiplicity of the eigenvalue one for the automorphisms Ad (g) of $\mathfrak{g}$, as g ranges over G. An element $g \in G$ is *regular semisimple* if this minimal multiplicity is attained for g. We can write

$$\det\,(\lambda + 1 - \operatorname{Ad} g) = \sum_{k\geq 0} D_\lambda(g)\lambda^{r+k}.$$

Then one has

$$G' = \{g \in G \,|\, D_0(g) \neq 0\}.$$

According to a fundamental theorem of Harish-Chandra, an invariant distribution is a locally L^1 function on G, which moreover is a real analytic function on G'. (A proof of this theorem for a linear group G can be found in [At]. One can also see [Kn1] Chapter X.) Therefore, it is meaningful to restrict an invariant eigendistribution to a function on the set of regular elements of each Cartan subgroup of G. It follows that each invariant eigendistribution is completely determined by restriction to the Cartan subgroups, and it is enough to choose one Cartan subgroup from each of the finitely many conjugacy classes.

Let $H \subset G$ be a Cartan subgroup. Then relative to a positive system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{h})$ of roots, the Weyl denominator is formally the expression

$$\Delta = \prod_{\alpha\in\Phi^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

We can rewrite it as $\Delta = e^\rho \prod_{\alpha\in\Phi^+}(1-e^{-\alpha})$. Since 2ρ is a weight, $|\Delta|$ is a well defined function on H, independent of the choice of a positive root system. Without loss of generality, we may assume that ρ lies in the weight lattice. If it fails to be so, we may pass to a two fold cover of G. Then Δ is a well defined function on H. Harish-Chandra showed that every eigendistribution Θ has the following properties:

1) The function $\Theta|_{H'}\Delta$ on each component of the regular set H' is a linear combination of exponentials with polynomial coefficients (Here exponentials, respectively polynomial, means that they become such when pulled back to the Lie(H) via exp).

2) If H is maximally compact, then $\Theta|_{H'}\Delta$ extends to a C^∞ function on all of H.

3) The restrictions of $\Theta\Delta$ to two Cartan subgroups that are related by a simple Cayley transform satisfy certain matching conditions due to Hirai and modeled on the corresponding conditions in the Lie algebra case discovered by Harish-Chandra (cf. [Kn1] §XI.7 for an exposition of the matching conditions).

4) If Θ is an irreducible character, then the polynomial coefficients in 1) are all constants.

5) Restricted to K, the Harish-Chandra V module of an admissible representation of G decomposes into a direct sum of K-irreducibles

$$V = \bigoplus_{i\in\widehat{K}} V_i.$$

Then the series

$$\Theta_K(V) = \sum_{i\in\widehat{K}} \Theta_K(V_i)$$

converges to a distribution on K; this *K-character* is real-analytic on $K \cap G'$, and $\Theta_K(V) = \Theta_G(V)$ as functions on $K \cap G'$.

The proof of the above properties can be found in [Kn1]. An invariant eigendistribution Θ is said to decay at ∞ if, for each Cartan subgroup H, the function $\Theta|_H \Delta$ tends to 0 outside of compact subsets of H.

Theorem 7.3.1. (Atiyah-Schmid [AS]) *One has*

(a) The character Θ_π of a discrete series representation decays at infinity.

(b) A nonzero invariant eigendistribution which decays at infinity has a non-trivial restriction on some compact Cartan subgroup.

We refer to Section 7 of [AS] for a complete proof of this theorem. We note that Atiyah and Schmid's proof is different from the Harish-Chandra's original approach. They showed that a discrete series character extends continuously from $C_0^\infty(G)$ to the n-th Sobolev space

$$S_n(G) = \{f \in L^2(G) \,|\, r(Z)f \in L^2(G) \; \forall Z \in U_n(\mathfrak{g})\}$$

for sufficiently large integer n. Here $r(Z)$ refers to infinitesimal right translation by Z and $U_n(\mathfrak{g})$ is the subspace of elements of order $\leq n$ in $U(\mathfrak{g})$. Then they proved that an invariant eigendistribution with such a property decays at infinity. As a consequence of the above theorem, we obtain the following result due to Harish-Chandra.

Theorem 7.3.2. (Harish-Chandra) *The character Θ_π of a discrete series representation decays at infinity and therefore $\Theta_\pi|_T \neq 0$ for some compact Cartan subgroup T. In particular, if G has discrete series representations then G has a compact Cartan subgroup.*

Note that a nontrivial linear combination of discrete series characters with the same infinitesimal character is a nonzero invariant eigendistribution which decays at infinity. Note that all compact Cartan subgroups are conjugate. It follows that it has a nontrivial restriction to compact Cartan subgroups. Hence, we obtain the following lemma which is needed in the next section.

Lemma 7.3.3. *The characters of discrete series representations with the same infinitesimal character are linearly independent on compact Cartan subgroups.*

7.4 Exhaustion of discrete series

The combination of the theorems in the previous two sections shows that a connected semisimple Lie group G with finite center has discrete series if and only if $\operatorname{rank} G$ is equal to $\operatorname{rank} K$. In this section we assume that G satisfies this condition, i.e., we assume that a compact Cartan subgroup T of K is also a Cartan subgroup of G. The goal in this section is to classify all discrete series representations.

Let Θ be an invariant eigendistribution such that $\Theta|_T \neq 0$. First suppose that G is acceptable, i.e., ρ is in the weight lattice of G. Then the property (1) of characters described in the previous section implies that $\Theta|_T\Delta$ is a $\mathbb{C}$-linear combination of expressions e^{μ} with μ in the weight lattice Λ. If e^{μ} occurs with a nonzero coefficient, then Θ has infinitesimal character μ, i.e., any $z \in Z(\mathfrak{g})$ acts on Θ by $z\Theta = \chi_\mu(z)\Theta$. Let Φ^+ be a system of positive roots for $(\mathfrak{g}, \mathfrak{t})$, and let ρ be half the sum of the positive roots. Then we can write

$$\Theta|_T e^{\rho} \prod_{\alpha\in\Phi^+}(1-e^{-\alpha}) = \sum_{w\in W_{\mathfrak{g}}} a_w e^{w\mu}$$

for some constants $a_w \in \mathbb{C}$. If G is not necessarily acceptable, we can make sense of this expression by multiplying through by $e^{-\rho}$, and we see that $w\mu-\rho$ must be in Λ. Putting $\mu = \lambda+\rho$, then for any G acceptable or not, we have $\lambda \in \Lambda$ and $\Theta|_T$ is given by the well defined expression

$$\Theta|_T = \frac{\sum_{w\in W_{\mathfrak{g}}} a_w e^{w(\lambda+\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2}-e^{-\alpha/2})}.$$

The restriction $\Theta|_T$ must be $W_{\mathfrak{k}}$-invariant, and the Weyl denominator is $W_{\mathfrak{k}}$-skew. Thus we can rewrite the above expression as follows: Choose $\lambda_1, \ldots, \lambda_k$ so that $\{w(\lambda+\rho) \mid w \in W_{\mathfrak{g}}\}$ is the disjoint union of the $\{w(\lambda_i + \rho) \mid w \in W_{\mathfrak{k}}\}$. Then there exist constants $a_1, \ldots, a_k$ such that

$$\Theta|_T = \sum_i a_i \frac{\sum_{w\in W_{\mathfrak{k}}} \epsilon(w) e^{w(\lambda_i+\rho)}}{\prod_{\alpha\in\Phi^+}(e^{\alpha/2}-e^{-\alpha/2})}.$$

Moreover, if $\lambda_i+\rho$ happens to be Φ_c-singular, $\sum_{w\in W_{\mathfrak{k}}} \epsilon(w)e^{w(\lambda_i+\rho)}$ vanishes, so a_i can be chosen to be zero in that case.

To summarize, we have that any discrete series representation has infinitesimal character $\chi_{\lambda+\rho}$ for some $\lambda \in \Lambda$. Note that $\chi_{\lambda_1+\rho} = \chi_{\lambda_2+\rho}$ if and only if $w(\lambda_1+\rho) = \lambda_2+\rho$ for some $w \in W_{\mathfrak{g}}$. By Lemma 7.3.3 the global characters of the discrete series representations with the same infinitesimal character must be linearly independent on T. Thus we obtain an upper bound on the number of discrete series representations as following.

Lemma 7.4.1. *The infinitesimal character of a discrete series representation must be $\chi_{\lambda+\rho}$ for some $\lambda \in \Lambda$. There are at most $|W_{\mathfrak{g}}/W_{\mathfrak{k}}|$ discrete series with infinitesimal character $\chi_{\lambda+\rho}$.*

For the moment, we let ρ be the half sum of the positive roots with respect to an arbitrary system Φ^+ of the positive roots. Then

$$\Lambda_\rho = \Lambda + \rho$$

does not depend on the particular ordering, since any two possible choices for ρ differ by a sum of roots and hence an element of Λ.

We now show that there is no discrete series representation with infinitesimal character $\chi_{\lambda+\rho}$ if $\lambda+\rho$ is singular. We enumerate the set of $W_{\mathfrak{g}}$-conjugates of $\lambda+\rho$ in Λ_ρ, which are both ϕ_c-regular and $\Phi_c \cap \Phi^+$-dominant, as

$$\lambda_1 + \rho = \lambda + \rho, \lambda_2 + \rho, \ldots, \lambda_N + \rho.$$

Every $\nu + \rho \in \Lambda_\rho$, if it is Φ_c-nonsingular and $W_{\mathfrak{g}}$-conjugate to $\lambda+\rho$, is then $W_{\mathfrak{k}}$-conjugate to precisely one of $\lambda_i + \rho$. We can and will arrange $\lambda = \lambda_1$. We define

$$\widetilde{\Theta}_{\lambda+\rho} = \sum \mu(j)\Theta_j,$$

the sum is over all discrete series representations with infinitesimal character $\chi_{\lambda+\rho}$. Then there exist constants $a_1, \ldots, a_N$ such that

$$\widetilde{\Theta}_{\lambda+\rho}|_T = \sum_i a_i \frac{\sum_{w \in W_{\mathfrak{k}}} \epsilon(w) e^{w(\lambda_i+\rho)}}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})}.$$

Like every λ_i, λ lies in the interior of the positive Weyl chamber for $\Phi^c \cap \Phi^+$ and is Φ_c-integral. For $\mu = (\lambda+\rho) - \rho_c = \lambda + \rho_n$, one has that $\mu - \rho_n$ is in the weight lattice Λ. Thus, the tensor products $E_\mu \otimes S^\pm$ descend to K. It follows from the remark after (7.5) that the K-character $\tau_j = \Theta_j|_{K\cap G'}$ of a discrete series H_j occured in Ker D_μ satisfies the following condition: $\tau_j(chS^+ - chS^-)$ is a finite integral linear combination of irreducible characters of $\widetilde{K}$, in which the character of E_μ occurs with coefficient

$$(-1)^q(\dim \ \mathrm{Ker}\ D_j^+(\mu) - \dim \ \mathrm{Ker}\ D_j^-(\mu)).$$

Then the L^2-index theorem in Section 7.1 implies that

Lemma 7.4.2. *(a) If $\lambda+\rho$ is singular, then the restriction to T of $\widetilde{\Theta}_{\lambda+\rho}$ is zero and therefore there exist no discrete series representations with infinitesimal character $\chi_{\lambda+\rho}$.*

(b) If $\lambda + \rho$ is regular, then every $\lambda_i + \rho$ is dominant with respect to a unique positive root system Φ_i^+, namely

$$\Phi_i^+ = \{\alpha \in \Phi \,|\, (\lambda_i + \rho, \alpha) > 0\}.$$

Then the restriction of to T of $\widetilde{\Theta}_{\lambda+\rho}$ equals

$$\widetilde{\Theta}_{\lambda+\rho}|T = (-1)^q(\prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}) \sum_{i=1}^{N} \frac{\sum_{w \in W_{\mathfrak{k}}} \epsilon(w) e^{w(\lambda_i+\rho)}}{\prod_{\alpha \in \Phi_i^+}(e^{\alpha/2} - e^{-\alpha/2})}, \tag{7.8}$$

where $q = \frac{1}{2}\dim G/K$.

Now we are ready to prove the celebrated theorem of Harish-Chandra on discrete series.

Theorem 7.4.3. (Harish-Chandra) *Let G be a connected semisimple Lie group having finite center and a compact Cartan subgroup T. For each $\lambda \in \Lambda$ with $\lambda + \rho$ regular, there is a unique invariant eigendistribution $\Theta_{\lambda+\rho}$ so that $\Theta_{\lambda+\rho}$ decays at ∞, and*

$$\Theta_{\lambda+\rho}|_T = (-1)^q \frac{\sum_{w \in W_{\mathfrak{k}}} \epsilon(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})}. \tag{7.9}$$

Moreover, every such $\Theta_{\lambda+\rho}$ is a discrete series character, and conversely every discrete series character is one of $\Theta_{\lambda+\rho}$ with $\lambda + \rho$ regular.

It follows from the theorem that the regular elements $\lambda + \rho \in \Lambda_\rho$ are parameters for all discrete series representations. They are called *Harish-Chandra parameters* for discrete series.

Proof. We choose compatible system of positive roots $\Phi^+ \supset \Phi_c^+$ so that $\mu = \lambda + \rho_n$ is Φ_c^+-dominant and $\lambda + \rho$ is Φ^+-dominant. First, we assume that λ is Φ^+-dominant and Φ_n-regular, then it is known from Section 7.2 that $\operatorname{Ker} D_\mu$ contains a single discrete series $A_{\mathfrak{b}}(\lambda)$ with Dirac cohomology $H_D(A_{\mathfrak{b}}(\lambda)) = E_\mu$ and infinitesimal character $\chi_{\lambda+\rho}$. It follows that

$$A_{\mathfrak{b}}(\lambda)|_K(S^+ - S^-) = (-1)^q E_\mu,$$

and therefore the K-character of $A_{\mathfrak{b}}(\lambda)$ restricted to T is equal to $\Theta_{\lambda+\rho}|_T$. Thus, the character of this discrete series $A_{\mathfrak{b}}(\lambda)$ is $\Theta_{\lambda+\rho}$. Secondly, we remove the conditions that λ is Φ^+-dominant and Φ_n-regular. We need to use the following result of Zuckerman [Z], which is often referred to as the translation principle. Note that G needs not be linear, but we do need to assume that it is a finite cover of a linear group. Since $\lambda + \rho$ is dominant, large positive multiples of $(\lambda + \rho)$ are integral and occur as highest weights of finite-dimensional irreducible representations of G. Let τ be such a representation of highest weight $(m-1)(\lambda+\rho)$ $(m \geq 2)$ and let τ^* be the contragredient representation of τ. We write Θ_τ and Θ_{τ^*} for the characters of τ and τ^* respectively. Let $C(\lambda+\rho)$ denote the set of characters of irreducible admissible representations with infinitesimal character $\chi_{\lambda+\rho}$.

Lemma 7.4.4. (Zuckerman[Z]) *The map*

$$S\colon \Theta \mapsto (\Theta_\tau \Theta)_{\chi_{m(\lambda+\rho)}}$$

is a bijection between $C(\lambda + \rho)$ and $C(m\lambda + m\rho)$, whose inverse is given by

$$T\colon \Theta \mapsto (\Theta_{\tau^*}\Theta)_{\chi_{\lambda+\rho}}.$$

If Θ decays at ∞, then so does $S\Theta$ and vice versa. Moreover, for a compact Cartan subgroup T, if $\Theta \in C(m\lambda + m\rho)$ satisfies

$$\Theta|_T \Delta = \sum_{w \in W_{\mathfrak{g}}} a_w e^{mw(\lambda+\rho)},$$

then

$$(S^{-1}\Theta)|_T \Delta = \sum_{w \in W_{\mathfrak{g}}} a_w e^{w(\lambda+\rho)}.$$

We refer to (8.10) and (8.19) in [AS] for a proof of this lemma.

For a sufficiently large $m \geq 2$, let $\lambda' = m\lambda + (m-1)\rho$. Then λ' is Φ^+-dominant and Φ_n-regular. It follows that there exists a discrete series character $\Theta_{m(\lambda+\rho)}$, such that

$$\Theta_{m(\lambda+\rho)}|_T \Delta = (-1)^q \sum_{w \in W_{\mathfrak{k}}} \epsilon(w) e^{mw(\lambda+\rho)}.$$

Thus, the above lemma implies that $\Theta_{\lambda+\rho}$ is the character of an irreducible representation, and

$$\Theta_{\lambda+\rho}|_T \Delta = (-1)^q \sum_{w \in W_{\mathfrak{k}}} \epsilon(w) e^{w(\lambda+\rho)}.$$

Corollary 7.4.5. *In the setting of the theorem one has*

a) Choose a system of positive roots $\Phi^+ = \Phi_c^+ \cup \Phi_n^+$ so that $\lambda+\rho$ is Φ^+-dominant. Then $A_{\mathfrak{b}}(\lambda)$ is a discrete series representation with lowest K-type $E_{\lambda+2\rho_n}$ and Dirac cohomology $E_{\lambda+\rho_n}$.

b) Every discrete series representation of G is exactly one of $A_{\mathfrak{b}}(\lambda)$ as in (a).

Thus, any discrete series representation is determined completely by its Dirac cohomology $E_{\lambda+\rho_n}$.

Proof. We have already showed in Section 7.2 that the discrete series representation with character $\Theta_{\lambda+\rho}$ is $A_{\mathfrak{b}}(\lambda)$ provided λ is Φ^+-dominant and Φ_n^+ regular. It remains to remove these conditions, i.e. to show that if $\lambda+\rho$ is dominant then the discrete series representation with character $\Theta_{\lambda+\rho}$ is still $A_{\mathfrak{b}}(\lambda)$ for a θ-stable Borel subalgebra. This follows from the fact that the translation functor S^{-1} in Lemma 7.4.4 carries $A_{\mathfrak{b}}(\lambda)$-module to $A_{\mathfrak{b}}(\lambda)$-module.

It follows from the above theorem and (7.8) that $\operatorname{Ker} D_\mu$ contains at most one discrete series. By L^2-index theorem we know that the discrete series representation occurs in $\operatorname{Ker} D_\mu^+$ rather than $\operatorname{Ker} D_\mu^-$. As a consequence we obtain the following theorem on geometric construction of all discrete series representations due to Atiyah and Schmid.

Theorem 7.4.6. (Atiyah-Schmid [AS]) *Let G be a connected noncompact semisimple Lie group with finite center. Let K be a maximal compact subgroup of G. Suppose a Cartan subgroup T of K is also a Cartan subgroup of G. Let E_μ be an irreducible representation of the two fold cover $\widetilde{K}$ of K with highest* *weight* μ. *Suppose that* $\mu - \rho_n$ *is in the weight lattice* Λ *so that* $E_\mu \otimes S^\pm$ *descend to* K *and therefore* $D_\mu^\pm$ *is defined on the spinor bundles over* G/K. *Then one has*

a) Ker $D_\mu^- = 0$.

b) Ker $D_\mu^+ = 0$ *if* $\mu + \rho_c$ *is singular.*

c) For regular $\mu + \rho_c$ *choose a system of positive roots* Φ^+ *compatible with* Φ_c^+ *so that* $\mu + \rho_c$ *is dominant. Then* Ker D_μ^+ *is a discrete series representation and its character* $\Theta_{\mu+\rho_c}$ *restricted to* T *is equal to*

$$\Theta_{\mu+\rho_c}|_T = (-1)^q \frac{\sum_{w \in W_{\mathfrak{k}}} \epsilon(w) e^{w(\mu+\rho_c)}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}.$$

d) Every discrete series representation of G *is isomorphic to one of* Ker D_μ^+ *in (c).*

8

Dimensions of Spaces of Automorphic Forms

In this chapter we prove a formula for dimensions of spaces of automorphic forms which sharpens the result of Langlands and Hotta-Parthasarathy [L], [HoP]. Let G be a connected semisimple noncompact Lie group with finite center. Let $K \subset G$ be a maximal compact subgroup of G, and let $\Gamma \subset G$ be a discrete subgroup. Assume that $\Gamma\backslash G$ is compact and that Γ acts freely on G/K. Then $X = \Gamma\backslash G/K$ is a compact smooth manifolds. Furthermore, the action of G by right translation on the Hilbert space $L^2(\Gamma\backslash G)$ is decomposed discretely with finite multiplicities:

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\pi\in\widehat{G}} m(\Gamma,\pi)\mathcal{H}_\pi.$$

Assume that $\operatorname{rank} G$ is equal to $\operatorname{rank} K$. We calculate the multiplicity $m(\Gamma,\pi)$ for a discrete series representation π.

Denote by X_π the Harish-Chandra module of $\mathcal{H}_\pi$. Let F be a finite-dimensional G-module. Understanding the above multiplicity $m(\Gamma,\pi)$ together with the $(\mathfrak{g},K)$-cohomology $H^*(\mathfrak{g},K,X_\pi\otimes F)$ has topological applications. Borel and Wallach (cf. [BW] Chapter VII, §6) showed the following formula for cohomology:

$$H^*(\Gamma,F) \cong \bigoplus_{\pi\in\widehat{G}} m(\Gamma,\pi)H^*(\mathfrak{g},K,X_\pi\otimes F).$$

We will also explain the relationship between Dirac cohomology and $(\mathfrak{g},K)$-cohomology of a Harish-Chandra module.

8.1 Hirzebruch proportionality principle

To obtain the index of the Dirac operator on twisted spinor bundles on X, we use *Hirzebruch proportionality principle* and the result on the index of the Dirac operator for compact Lie groups.

Recall that in the previous chapter if any E_μ is $\widetilde{K}$-module so that $E_\mu \otimes S$ decent to K then the Dirac operator D acts on the smooth sections of the twisted spin bundle as described in the previous chapter:

$$D^{\pm}_\mu \colon C^\infty(G/K, S^{\pm} \otimes E_\mu) \to C^\infty(G/K, S^{\mp} \otimes E_\mu).$$

Note that the above action of D commutes with the left action of G. So we can consider an elliptic operator

$$D^{+}_\mu(X) \colon C^\infty(\Gamma\backslash G/K, S^{+} \otimes E_\mu) \to C^\infty(\Gamma\backslash G/K, S^{-} \otimes E_\mu).$$

The index of $D^+_\mu(X)$ can be computed by the Atiyah-Singer Index Theorem

$$\operatorname{Index} D^+_\mu(X) = \int_X f(\Theta, \Phi),$$

where Θ is the curvature of X and Φ is the curvature of the twisted spinor bundle over G/K. By the homogeneity, $f(\Theta, \Phi)$ is a multiple of the volume form depending only on μ, i.e., $f(\Theta, \Phi) = c(\mu)dx$. Thus

$$\operatorname{Index} D^+_\mu(X) = c(\mu) vol(\Gamma\backslash G/K).$$

Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of the Lie algebra of G. Then $\mathfrak{u}_0 = \mathfrak{k}_0 + i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Let $G_{\mathbb{C}}$ be the connected and simply connected complex group with Lie algebra $\mathfrak{g}$. Let U be the compact analytic subgroup in the $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{u}_0$. We assume for the moment that G is a real form of $G_{\mathbb{C}}$. Then K is a subgroup of U and $Y = U/K$ is a compact homogeneous space. We consider

$$D^{+}_\mu(Y) \colon C^\infty(U/K, S^{+} \otimes E_\mu) \to C^\infty(U/K, S^{-} \otimes E_\mu).$$

By Hirzebruch proportionality principle, the index of $D^+_\mu(Y)$ can be computed in the same way and we obtain

$$\operatorname{Index} D^+_\mu(Y) = (-1)^q c(\mu) vol(U/K),$$

where $q = \dim G/K = \dim U/K$. It follows that

$$\operatorname{Index} D^+_\mu(X) = (-1)^q \frac{vol(\Gamma\backslash G/K)}{vol(U/K)} \operatorname{Index} D^+_\mu(Y).$$

If we normalize the Haar measure so that $vol(U) = 1$, then

$$\operatorname{Index} D^+_\mu(X) = (-1)^q vol(\Gamma\backslash G) \operatorname{Index} D^+_\mu(Y).$$

On the other hand, $\operatorname{Index} D^+_\mu(Y)$ has been calculated in Chapter 4. If we as before choose a system of positive roots Φ^+ so that $\mu + \rho_c$ is dominant, then

$$\operatorname{Index} D_\mu^+(Y) = \frac{\prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\mu+\rho_c,\alpha)}{\prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\rho,\alpha)}.$$

Thus, one has

$$\operatorname{Index} D_\mu^+(X) = (-1)^q vol(\Gamma\backslash G/K)\frac{\prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\mu+\rho_c,\alpha)}{\prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\rho,\alpha)}.$$

We now drop the requirement that the real of G_1 of the simply connected $G_{\mathbb{C}}$ coincides with G. The maximal compact subgroup K_1 of G_1 is locally isomorphic to K. If $\mu-\rho_n$ is a weight of K_1, then our argument still goes through. If $\mu-\rho_n$ is a weight of K, but not a weight of K_1, we first observe that $c(\mu)$ is a polynomial in μ. This follows from the relations between characters and characteristic classes. Therefore, the above formula continues to hold and we conclude the following

Theorem 8.1.1. *Let G be a connected semisimple Lie group having finite center and a maximal compact subgroup K. Let Γ be a cocompact discrete subgroup of G which acts freely on G/K. If a Cartan subgroup of K is also a Cartan subgroup G, then one has*

$$\operatorname{Index} D_\mu^+(X) = (-1)^q vol(\Gamma\backslash G/K)\frac{\prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\mu+\rho_c,\alpha)}{\prod_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{t})}(\rho,\alpha)}.$$

8.2 Dimensions of spaces of automorphic forms

Recall the decomposition

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\pi\in\widehat{G}} m(\Gamma,\pi)\mathcal{H}_\pi.$$

We now calculate the multiplicity $m(\Gamma,\pi)$ for each π. Let X_π be the Harish-Chandra module of $\mathcal{H}_\pi$. For any $\mu\in\widehat{K}$, It follows that

$$\operatorname{Index} D_\mu^+(X) = \sum_{\pi\in\widehat{G}} m(\Gamma,\pi) \operatorname{Index} D_\mu^+(X_\pi),$$

where $D_\mu^+(X_\pi) : \operatorname{Hom}_{\tilde{K}}(E_\mu^*, X_\pi\otimes S^+) \to \operatorname{Hom}_{\tilde{K}}(E_\mu^*, X_\pi\otimes S^-)$ is the linear map defined by $\phi\mapsto D\circ\phi$ for any $\phi\in \operatorname{Hom}_{\tilde{K}}(E_\mu^*, X_\pi\otimes S^+)$.

If $\operatorname{Index} D_\mu^+(X_\pi)\neq 0$, then the Dirac cohomology $H_D(X_\pi)$ contains E_{μ^*}. It follows from the proved Vogan's conjecture that the infinitesimal character of X_π is given by $\mu^*+\rho_c$. If we assume that $\lambda = w(\mu^*+\rho_c)-\rho$ is dominant for some $w\in W$, then X_π is isomorphic to $A_{\mathfrak{q}}(\lambda)$ for some θ-stable parabolic subalgebra $\mathfrak{q}$. If in addition we assume that λ is regular with respect to the noncompact roots $\Delta^+(\mathfrak{p})$, then X_π is uniquely determined as a discrete series representation $A_{\mathfrak{b}}(\lambda)$. Since $\operatorname{Index} D_\mu^+(A_{\mathfrak{b}}(\lambda)) = \dim D_\mu^+(A_{\mathfrak{b}}(\lambda)) - \operatorname{codim} D_\mu^+(A_{\mathfrak{b}}(\lambda)) = (-1)^q$, we obtain the following theorem.

Theorem 8.2.1. *Let G be a connected semisimple Lie group have finite center and a compact subgroup K satisfying* $\operatorname{rank} K = \operatorname{rank} G$. *Let E_μ be an irreducible module of $\widetilde{K}$ (a two fold cover of K) so that $E_\mu \otimes S^\pm$ decent to K. Choose a system of positive roots compatible to the system of positive compact roots. If $\lambda = \mu - \rho_n$ is dominant and regular with respect to noncompact roots, then the discrete series representation $\pi = A_{\mathfrak{b}}(\lambda)$ occurs in $L^2(\Gamma\backslash G)$ with multiplicity*

$$m(\Gamma, \pi) = \operatorname{vol}(\Gamma\backslash G) d_\pi,$$

where d_π is the formal degree of π:

$$d_\pi = \frac{\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})} (\rho, \alpha)}.$$

This sharpens the result of Langlands [L] and Hotta-Parthasarathy [HoP], who proved the above formula for discrete series representations whose K-finite matrix coefficients are in $L^1(G)$. Trombi-Varadarajan [TV] proved that if the K-finite matrix coefficients of the discrete series $A_{\mathfrak{b}}(\lambda)$ are in $L^1(G)$, then for all $\alpha \in \Delta^+(\mathfrak{p})$ and all $w \in W_{\mathfrak{g}}$

$$\langle \lambda + \rho, \alpha \rangle > |\langle w\rho, \alpha \rangle|.$$

Hecht-Schmid [HS] proved this is also a sufficient condition. Our assumption on the regularity of λ with respect to the noncompact roots amounts to the condition that for all $\alpha \in \Delta^+(\mathfrak{p})$

$$\langle \lambda + \rho, \alpha \rangle > |\langle \rho, \alpha \rangle|.$$

Therefore, our condition is weaker than that assumed by Langlands and Hotta-Parthasarathy.

8.3 Dirac cohomology and $(\mathfrak{g}, K)$-cohomology

In this section we study $(\mathfrak{g}, K)$-cohomology of $(\mathfrak{g}, K)$-modules, where as usual $\mathfrak{g}$ is the complexified Lie algebra of a connected semisimple Lie group G, and K is a maximal compact subgroup of G. It is sometimes important to generalize this setting (recall e.g. from Chapter 5 that one can use relative Lie algebra cohomology with respect to a pair $(\mathfrak{k}, T)$ to describe derived Zuckerman functor).

8.3.1. Definition of $(\mathfrak{g}, K)$-cohomology

We recall the definition of $(\mathfrak{g}, K)$-cohomology. Consider the functor

$$V \mapsto V^{\mathfrak{g},K} = \{v \in V \big| Xv = 0, kv = v, \text{ for all } X \in \mathfrak{g}, k \in K\}$$

of taking $(\mathfrak{g}, K)$-invariants. It is a functor from the category $M(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules into the category of complex vector spaces, which is left exact. The $(\mathfrak{g}, K)$-*cohomology* functors $V \mapsto H^i(\mathfrak{g}, K; V)$ are the right derived functors of $V \mapsto V^{\mathfrak{g},K}$. As we saw in Chapter 5, $(\mathfrak{g}, K)$ cohomology can be calculated by the Duflo-Vergne formula. Namely, we can write

$$V^{\mathfrak{g},K} = \operatorname{Hom}_{(\mathfrak{g},K)}(\mathbb{C}, V),$$

and it follows that

$$H^i(\mathfrak{g}, K; V) = \operatorname{Ext}^i_{(\mathfrak{g},K)}(\mathbb{C}, V).$$

Instead of resolving V by injectives, we use a projective resolution of the trivial module $\mathbb{C}$. This is the relative standard complex

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \textstyle\bigwedge^{\cdot}(\mathfrak{p}) \xrightarrow{\ \epsilon\ } \mathbb{C} \longrightarrow 0;$$

Here $\mathfrak{p}$ is a K-invariant direct complement of $\mathfrak{k}$ in $\mathfrak{g}$. Recall the differential d of the above complex is

$$d(u \otimes X_1 \wedge \cdots \wedge X_k) = \sum_i (-1)^{i-1} u X_i \otimes X_1 \wedge \ldots \widehat{X}_i \cdots \wedge X_k.$$

For more general pairs than $(\mathfrak{g}, K)$, we would have a second double sum, but in our case this second sum vanishes, as $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ projects trivially to $\mathfrak{p}$. The map ϵ is the augmentation map, given by $1 \otimes 1 \mapsto 1$ and $\mathfrak{g}U(\mathfrak{g}) \otimes 1 \mapsto 0$.

Recall that we proved in Chapter 5 that the relative standard complex indeed defines a projective resolution of $\mathbb{C}$.

Using the above resolution, we can now identify $H^i(\mathfrak{g}, K; V)$ with the i^{th} cohomology of the complex

$$\operatorname{Hom}^{\cdot}_{(\mathfrak{g},K)}(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \textstyle\bigwedge^{\cdot}(\mathfrak{p}), V) = \operatorname{Hom}^{\cdot}_K(\textstyle\bigwedge^{\cdot}(\mathfrak{p}), V),$$

with differential

$$df(X_1 \wedge \cdots \wedge X_k) = \sum_i (-1)^{i-1} X_i \cdot f(X_1 \wedge \ldots \widehat{X}_i \cdots \wedge X_k).$$

There is also a less often mentioned theory of $(\mathfrak{g}, K)$-*homology* . It is constructed by deriving the functor of $(\mathfrak{g}, K)$-coinvariants; so

$$H_i(\mathfrak{g}, K; V) = \operatorname{Tor}_i^{(\mathfrak{g},K)}(\mathbb{C}, V),$$

which is calculated using the same resolution of $\mathbb{C}$ as above.

8.3.2. Vogan-Zuckerman classification The central result about $(\mathfrak{g}, K)$-cohomology (for K a maximal compact subgroup of a semisimple Lie group G) is the classification of irreducible unitary $(\mathfrak{g}, K)$-modules with nonzero $(\mathfrak{g}, K)$-cohomology due to Vogan and Zuckerman in [VZ]. Here is their result:

Theorem 8.3.3. *Let G be a semisimple Lie group with finite center and a maximal compact subgroup K. Let V be the Harish-Chandra module of an irreducible unitary representation of G. Let F be an irreducible finite dimensional G-module and F^* be the contragredient. Then $V \otimes F$ has nonzero $(\mathfrak{g}, K)$-cohomology if and only if V has the same infinitesimal character as F^* and V is isomorphic to an $A_{\mathfrak{q}}(\lambda)$-module. In case V has nonzero $(\mathfrak{g}, K)$-cohomology, it is equal to*

$$\operatorname{Hom}_{L \cap K}(\textstyle\bigwedge^{i - \dim(\mathfrak{u} \cap \mathfrak{p})}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}),$$

where L is the Levi subgroup involved in the definition of $A_{\mathfrak{q}}(\lambda)$, $\mathfrak{l}$ is the (complexified) Lie algebra of L and $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$.

8.3.4. Dirac cohomology and $(\mathfrak{g}, K)$-cohomology We now discuss the relationship of Dirac cohomology and $(\mathfrak{g}, K)$-cohomology. It was proved in [HP1] that if X is unitary and has $(\mathfrak{g}, K)$-cohomology, i.e.,

$$H^*(\mathfrak{g}, K; X \otimes F) = H^*(\operatorname{Hom}^{\cdot}_{K}(\textstyle\bigwedge^{\cdot}\mathfrak{p}, X \otimes F)) \neq 0$$

for a finite dimensional F (X necessarily has the same infinitesimal character as F^*), then X also has nonzero Dirac cohomology.

In the following we assume that $\dim \mathfrak{p}$ is even. Then we can write $\mathfrak{p}$ as a direct sum of isotropic vector spaces $\mathfrak{u}$ and $\bar{\mathfrak{u}} \cong \mathfrak{u}^*$. One considers the spinor spaces $S = \bigwedge^{\cdot} \mathfrak{u}$ and $S^* = \bigwedge^{\cdot} \bar{\mathfrak{u}}$; then

$$S \otimes S^* \cong \textstyle\bigwedge^{\cdot}(\mathfrak{u} \oplus \bar{\mathfrak{u}}) = \bigwedge^{\cdot}\mathfrak{p}.$$

It follows that we can identify the $(\mathfrak{g}, K)$-cohomology of $X \otimes F^*$ with

$$H^*(\operatorname{Hom}^{\cdot}_{\tilde{K}}(F \otimes S, X \otimes S)).$$

There are several possible actions of the Dirac operator D on the above complex; similarly as before, they can be related to the coboundary operator d and the boundary operator ∂ for $(\mathfrak{g}, K)$-homology, which also acts on the same complex after appropriate identifications.

Now if X is unitary, Wallach has proved that $d = 0$ (see [W], Proposition 9.4.3, or [BW]). Using similar arguments one can analize the above mentioned Dirac actions and the actions of the corresponding "half-Diracs". In particular, it follows that

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \operatorname{Hom}^{\cdot}_{\tilde{K}}(H_D(F), H_D(X)).$$

This can be concluded from the fact that the eigenvalues of D^2 are of opposite signs on $F \otimes S$ and $X \otimes S$; see [W], 9.4.6.

We note that if $\dim \mathfrak{p}$ is odd, then as K-modules $\bigwedge^{\cdot}\mathfrak{p}$ is isomorphic to two copies of $S \otimes S^*$ where S is a spinor of the Clifford algebra $C(\mathfrak{p})$. It follows that in this case $H^*(\mathfrak{g}, K; X \otimes F^*)$ is isomorphic to two copies of $\operatorname{Hom}^{\cdot}_{\tilde{K}}(H_D(F), H_D(X))$.

8.4 Cohomology of discrete subgroups

For the general definition of cohomology space $H^*(\Gamma, F)$ of discrete subgroup Γ of a Lie group G with finitely many connected components, with coefficients in a finite dimensional complex Γ-module (ρ, F) we refer to Chapter VII of [BW]. This space can be described in terms of $(\mathfrak{g}, K)$-cohomology. It is proved in [BW] that

$$H^*(\Gamma, F) = H^*(\mathfrak{g}, K, I^\infty(F)),$$

where K is a maximal compact subgroup of G and

$$I^\infty(F) = \{f \in C^\infty(G, F) \mid f(\gamma \cdot g) = \rho(\gamma) \cdot f(g),\ \gamma \in \Gamma, g \in G\}.$$

If (ρ, F) is in fact a G-module, then this takes the form

$$H^*(\Gamma, F) = H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G) \otimes F).$$

Theorem 8.4.1. *Let G be a connected semisimple Lie group with finite center and no compact factor. Let F be an irreducible finite-dimensional G-module. One has*

$$H^n(\Gamma, F) = \bigoplus_{\pi \in \widehat{G}} m(\Gamma, \pi) H^n(\mathfrak{g}, K, X_\pi \otimes F)\ (n \in \mathbb{N}).$$

This is Theorem 6.1 in Chapter VII of [BW].

We now assume that $\operatorname{rank} G$ is equal to $\operatorname{rank} K$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{g}$. Fix a system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{t})$ of positive roots and a compatible system of positive compact roots $\Phi_c^+ = \Phi^+(\mathfrak{k}, \mathfrak{t})$. Let Φ_n^+ be the set of noncompact roots so that $\Phi^+ = \Phi_c^+ \cup \Phi_n^+$. Let F be an irreducible finite-dimensional representation of G with lowest weight $-\lambda$. In other words, λ is the the highest weight of the contragredient F^*. Recall that W^1 is the subset of $W_\mathfrak{g}$ consisting of elements which map the dominant $\mathfrak{g}$-chamber inside $\mathfrak{k}$-chamber. The multiplication $(\tau, w) \mapsto \tau w$ induces a bijection

$$W^1 \times W_\mathfrak{k} \to W_\mathfrak{g}.$$

Hence, $|W^1| = |W_\mathfrak{g}|/|W_\mathfrak{k}|$. Set $q = 1/2 \dim G/K$.

Theorem 8.4.2. *Let G be a connected semisimple Lie group with finite center and no compact factor. Assume that G has a compact Cartan subgroup. Let F be an irreducible finite-dimensional G-module with lowest weight $-\lambda$. If λ is regular with respect to the roots in Φ_n^+, then*

$$\dim H^n(\Gamma, F) = \begin{cases} |W_\mathfrak{g}|/|W_\mathfrak{k}| \operatorname{vol}(\Gamma\backslash G) \dim F, & \text{if } n = q, \\ 0, & \text{if } n \neq q. \end{cases}$$

Proof. For any Harish-Chandra module X, $H^*(\mathfrak{g}, K, X \otimes F) = 0$ unless the infinitesimal character of X is the same as F^*, which is $\lambda + \rho$. A unitary irreducible Harish-Chandra module with infinitesimal character $\lambda + \rho$ is an $A_{\mathfrak{q}}(\lambda)$-module. In case that λ is regular with respect to Φ_n^+, this is in fact a discrete series $A_{\mathfrak{b}}(\lambda)$. There are exactly $|W^1|$ discrete series $(\pi, \mathcal{H}_\pi)$ with infinitesimal character $\lambda + \rho$. By Theorem 8.2.1,

$$m(\Gamma, A_{\mathfrak{b}}(\lambda)) = \operatorname{vol}(\Gamma\backslash G) \frac{\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})}(\lambda + \rho, \alpha)}{\prod_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})}(\rho, \alpha)}$$

Hence, one has $m(\Gamma, A_{\mathfrak{b}}(\lambda)) = \operatorname{vol}(\Gamma\backslash G) \dim F^* = \operatorname{vol}(\Gamma\backslash G) \dim F$. Then it follows from Theorem 8.4.1 and the fact

$$\dim H^n(\mathfrak{g}, K, A_{\mathfrak{b}}(\lambda)) = \begin{cases} 1, & \text{if } n = q, \\ 0, & \text{if } n \neq q. \end{cases}$$

that one has

$$\dim H^n(\Gamma, F) = \begin{cases} |W^1| \operatorname{vol}(\Gamma\backslash G) \dim F, & \text{if } n = q, \\ 0, & \text{if } n \neq q. \end{cases}$$

9

Dirac operators and nilpotent Lie algebra cohomology

Let $\mathfrak{g}$ be a complex reductive Lie algebra with an invariant symmetric bilinear form B, equal to the Killing form on the semisimple part of $\mathfrak{g}$. In this chapter we consider a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, with unipotent radical $\mathfrak{u}$ and a Levi subalgebra $\mathfrak{l}$. We will denote by $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$ the opposite parabolic subalgebra. Here the bar notation does not mean complex conjugation in general, but it will be a conjugation in the cases we will study the most, so the notation is convenient.

If we denote $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$, then

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}$$

is a decomposition like in 2.3.3. In particular, the restrictions of B to $\mathfrak{l}$ and $\mathfrak{s}$ are non-degenerate, and the above decomposition is orthogonal. Clearly, $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are isotropic with respect to B, while B restricted to $\mathfrak{u} \times \bar{\mathfrak{u}}$ is nondegenerate. This means we can use B to identify $\bar{\mathfrak{u}}$ with $\mathfrak{u}^*$. This identification is $\mathfrak{l}$-equivariant.

Let $C(\mathfrak{s})$ be the Clifford algebra of $\mathfrak{s}$. Since $\mathfrak{s}$ is even-dimensional, $C(\mathfrak{s})$ has a unique irreducible module, for which we take $S = \bigwedge(\mathfrak{u})$. For a detailed description, see 2.2.2. We denote the Kostant's cubic Dirac operator corresponding to $\mathfrak{l} \subseteq \mathfrak{g}$ by D. It is an element of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$. See 4.1.1. Let us emphasize that we are continuing to use the conventions from Chapter 2 in the definitions of $C(\mathfrak{s})$ and S. Therefore, some of the signs differ from the ones in Kostant's paper, as well as from the ones in [HPR].

In this chapter we will show how in certain cases $\mathfrak{u}$-homology and $\bar{\mathfrak{u}}$-cohomology of a $\mathfrak{g}$-module V can be related to the Dirac cohomology of V with respect to D. In fact, as usual, we will be primarily interested in $(\mathfrak{g}, K)$-modules. Thus we assume right from the start that $\mathfrak{g}$ is the complexified Lie algebra of a connected real reductive group G with maximal compact subgroup K. As usual, θ denotes the corresponding Cartan involution and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ are the corresponding Cartan decompositions. This will however play no role in the first section, and only an inessential role in the second section of this chapter.

The results we are going to present in this chapter are taken from [HPR].

9.1 $\mathfrak{u}$-homology and $\bar{\mathfrak{u}}$-cohomology differentials

For any $\mathfrak{g}$-module V, one can define $\mathfrak{u}$-homology and $\bar{\mathfrak{u}}$-cohomology in the usual way. We already studied analogous notions in Chapter 5, but let us briefly recall the definition in the present case.

9.1.1. $\mathfrak{u}$-homology. By definition, the p-th $\mathfrak{u}$-*homology* of a $\mathfrak{u}$-module V, $H_p(\mathfrak{u};V)$, is the p-th left derived functor of the functor of $\mathfrak{u}$-coinvariants, which sends V into the vector space $V/\mathfrak{u}V$. Since $V/\mathfrak{u}V = \mathbb{C} \otimes_{U(\mathfrak{u})} V$ where $\mathbb{C}$ denotes the trivial right $\mathfrak{u}$-module V, we can calculate the derived functors by using the Koszul resolution of the first variable in the tensor product, $\mathbb{C}$. Since $(U(\mathfrak{u})\otimes\bigwedge^p(\mathfrak{u}))\otimes_{U(\mathfrak{u})} V \cong \bigwedge^p(\mathfrak{u})\otimes V$ in the obvious way, we arrive at the space $V \otimes \bigwedge(\mathfrak{u})$. (We interchange the order of factors in the tensor product, because we want to use the action of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ on $V \otimes \bigwedge(\mathfrak{u})$.)

The differential on this space induced by the Koszul differential of $U(\mathfrak{u}) \otimes \bigwedge(\mathfrak{u})$ is $\partial : V \otimes \bigwedge^p\mathfrak{u} \to V \otimes \bigwedge^{p-1}\mathfrak{u}$ given by the following formula.

$$\partial(v \otimes Y_1 \wedge \ldots \wedge Y_p) = \sum_{i=1}^{p}(-1)^i Y_i \cdot v \otimes Y_1 \wedge \ldots \hat{Y_i} \wedge \ldots \wedge Y_p + \sum_{1\le i<j\le p} (-1)^{i+j} v \otimes [Y_i, Y_j] \wedge Y_1 \wedge \ldots \wedge \hat{Y_i} \wedge \ldots \wedge \hat{Y_j} \wedge \ldots Y_p$$

for $v \in V$ and $Y_1, \ldots, Y_p \in \mathfrak{u}$. The p-th $\mathfrak{u}$-homology $H_p(\mathfrak{u};V)$ is now the p-th cohomology of the complex $V \otimes \bigwedge(\mathfrak{u})$ with respect to ∂.

In case we start from a $\mathfrak{g}$-module V, then it is clear that $\mathfrak{l}$ acts on $V \otimes \bigwedge(\mathfrak{u})$ (the action on $\bigwedge(\mathfrak{u})$ being induced by the adjoint action), and that ∂ is $\mathfrak{l}$-equivariant. Consequently, the $\mathfrak{u}$-homology modules have a natural action of $\mathfrak{l}$.

9.1.2. $\bar{\mathfrak{u}}$-cohomology. One similarly defines the $\bar{\mathfrak{u}}$-*cohomology* modules $H^p(\mathfrak{u};V)$ of V. They are given by the right derived functors of the $\bar{\mathfrak{u}}$-invariants functor, which sends a $\bar{\mathfrak{u}}$-module V to the vector space $V^{\bar{\mathfrak{u}}} = \operatorname{Hom}_{\bar{\mathfrak{u}}}(\mathbb{C}, V)$. By a similar analysis as above, one can calculate $H^p(\mathfrak{u};V)$ as the p-th cohomology of the complex $C(\bar{\mathfrak{u}}, V)$ given by

$$C^p(\bar{\mathfrak{u}}, V) = \operatorname{Hom}(\textstyle\bigwedge^p\bar{\mathfrak{u}}, V).$$

The differential of this complex, which is again induced by the Koszul differential on $U(\bar{\mathfrak{u}}) \otimes \bigwedge(\bar{\mathfrak{u}})$, is given by the usual formula:

$$(d\omega)(X_1 \wedge \ldots \wedge X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i-1}\, X_i \cdot \omega(X_1 \wedge \ldots \wedge \hat{X_i} \wedge \ldots \wedge X_{p+1}) + \sum_{1\le i<j\le p+1} (-1)^{i+j}\, \omega([X_i, X_j] \wedge X_1 \wedge \ldots \wedge \hat{X_i} \wedge \ldots \wedge \hat{X_j} \wedge \ldots \wedge X_{p+1}),$$

for any $\omega \in \operatorname{Hom}(\bigwedge^p \bar{\mathfrak{u}}, V)$ and any $X_1, \dots, X_{p+1} \in \bar{\mathfrak{u}}$.

Since $\bar{\mathfrak{u}}$ can be identified with the dual of $\mathfrak{u}$ via B, we have the following identifications :

$$\operatorname{Hom}(\textstyle\bigwedge^p \bar{\mathfrak{u}}, V) \cong \operatorname{Hom}(\bigwedge^p (\mathfrak{u}^*), V) \cong \operatorname{Hom}((\bigwedge^p \mathfrak{u})^*, V) \cong V \otimes \bigwedge^p \mathfrak{u}.$$

Let us fix a basis u_i of $\mathfrak{u}$ $(i = 1, \dots, n)$, and let u_i^* denote the dual basis of $\bar{\mathfrak{u}}$ with respect to B.

Lemma 9.1.3. *Through the above identifications, the differential $d : V \otimes \bigwedge^p \mathfrak{u} \to V \otimes \bigwedge^{p+1} \mathfrak{u}$ is given by*

$$\begin{aligned} d(v \otimes Y_1 \wedge \ldots \wedge Y_p) = &\sum_{i=1}^n u_i^* \cdot v \otimes u_i \wedge Y_1 \wedge \ldots \wedge Y_p \\ &+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^p v \otimes u_i \wedge Y_1 \wedge \ldots \wedge [u_i^*, Y_j]_{\mathfrak{u}} \wedge \ldots \wedge Y_p \end{aligned}$$

where $[u_i^*, Y_j]_{\mathfrak{u}}$ denotes the projection of $[u_i^*, Y_j]$ on $\mathfrak{u}$.

Proof. This is a straigtforward calculation, starting from the fact that the identification $\bigwedge^p (\mathfrak{u}^*) = (\bigwedge^p \mathfrak{u})^*$ is given via $(f_1 \wedge \cdots \wedge f_p)(X_1 \wedge \cdots \wedge X_p) = \det f_i(X_j)$.

9.1.4. Decomposing the Dirac operator. We now turn to describing how the cubic Dirac operator D from 4.1.1 fits into this picture. We will use the basis b_j, $j = 1, \dots, 2n$ of $\mathfrak{s}$, given by

$$b_1 = u_1, \dots, b_n = u_n,\ b_{n+1} = u_1^*, \dots, b_{2n} = u_n^*.$$

The dual basis is then

$$d_1 = u_1^*, \dots, d_n = u_n^*,\ d_{n+1} = u_1, \dots, d_{2n} = u_n.$$

As in 4.1.1, we write D as

$$D = \sum_{j=1}^{2n} d_j \otimes b_j + 1 \otimes v,$$

where v is Kostant's cubic element

$$v = \frac{1}{2} \sum_{1 \le i < j < k \le 2n} B([d_i, d_j], d_k)\; b_i \wedge b_j \wedge b_k.$$

Here by $b_i \wedge b_j \wedge b_k \in C(\mathfrak{s})$ we mean the image of $b_i \wedge b_j \wedge b_k$ under the Chevalley map $\bigwedge(\mathfrak{s}) \to C(\mathfrak{s})$ from 2.1.8.

The first sum in the expression for D can clearly be rewritten as $\sum_{i=1}^n u_i^* \otimes u_i + \sum_{i=1}^n u_i \otimes u_i^*$. To analyze the element v, first notice that for any i, j, k

$$B([u_i,u_j],u_k)=B([u_i^*,u_j^*],u_k^*)=0.$$

It follows that v breaks up into two sums: one is

$$\begin{aligned}v^+&=\frac{1}{2}\sum_{i<j}\sum_k B([u_i^*,u_j^*],u_k)\,u_i\wedge u_j\wedge u_k^*\\&=\frac{1}{4}\sum_{i,j,k}B([u_i^*,u_j^*],u_k)\,u_i\wedge u_j\wedge u_k^*\end{aligned}$$

while the other is

$$\begin{aligned}v^-&=\frac{1}{2}\sum_k\sum_{i<j}B([u_k^*,u_i],u_j)\,u_k\wedge u_i^*\wedge u_j^*\\&=\frac{1}{4}\sum_{i,j,k}B([u_i,u_j],u_k^*)\,u_i^*\wedge u_j^*\wedge u_k.\end{aligned}$$

An easy calculation shows that the Chevalley map sends $u_i\wedge u_j\wedge u_k^*$ into $u_iu_ju_k^*+\delta_{kj}u_i-\delta_{ki}u_j$. Hence

$$\begin{aligned}v^+=\frac{1}{4}\sum_{i,j,k}B([u_i^*,u_j^*],u_k)u_iu_ju_k^*&+\frac{1}{4}\sum_{i,j}B([u_i^*,u_j^*],u_i)\,u_j\\&-\frac{1}{4}\sum_{i,j}B([u_i^*,u_j^*],u_j)\,u_i.\end{aligned}$$

By Lemma 9.1.5 below, $\sum_{i,j}B([u_i^*,u_j^*],u_i)\,u_j$ is an $\mathfrak{l}$-invariant element of $\mathfrak{u}$. Since $\mathfrak{u}$ clearly does not contain nonzero $\mathfrak{l}$-invariant elements, this sum must be zero. Analogously, $\sum_{i,j}B([u_i^*,u_j^*],u_j)\,u_i=0$, and we conclude that

$$v^+=\frac{1}{4}\sum_{i,j,k}B([u_i^*,u_j^*],u_k)\,u_iu_ju_k^*.$$

Since $\sum_k B([u_i^*,u_j^*],u_k)u_k^*=[u_i^*,u_j^*]$, we finally obtain

$$v^+=\frac{1}{4}\sum_{i,j}u_iu_j[u_i^*,u_j^*].\tag{9.1}$$

Analogously, one calculates

$$v^-=\frac{1}{4}\sum_{i,j}u_i^*u_j^*[u_i,u_j].\tag{9.2}$$

So we see that we decomposed D into a sum of two terms: $D=C+C^-$, where

$$C=\sum_i u_i^*\otimes u_i+\frac{1}{4}\sum_{i,j}u_iu_j[u_i^*,u_j^*],$$

and

$$C^- = \sum_i u_i \otimes u_i^* + \frac{1}{4} \sum_{i,j} u_i^* u_j^* [u_i, u_j].$$

Before going on to compare the actions of C and C^- on $V \otimes S$ with the differentials d and ∂, let us formulate the lemma we needed above. This lemma will also imply that all the parts of D we considered, $\sum_i u_i^* \otimes u_i$, $\sum_i u_i^* \otimes u_i$, v^+ and v^- are $\mathfrak{l}$-invariant and independent of the chosen basis u_i. The lemma is a version of the well known principle of constructing invariants by contracting dual indices.

Lemma 9.1.5. *Let*

$$\psi : \mathfrak{s}^{\otimes 2k} \to U(\mathfrak{g}) \otimes C(\mathfrak{s})$$

be a linear map which is $\mathfrak{l}$-equivariant with respect to the adjoint actions. For example, ψ can be composed of the obvious inclusions $\mathfrak{s} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ and $\mathfrak{s} \hookrightarrow C(\mathfrak{s})$, products, commutators in $\mathfrak{g}$ and the Killing form B. Then

$$\sum_I \psi(u_I \otimes u_I^*) \in U(\mathfrak{g}) \otimes C(\mathfrak{s})$$

is independent of the chosen basis u_i and $\mathfrak{l}$-invariant. Here $I = (i_1, \dots, i_k)$ ranges over all k-tuples of integers in $\{1, \dots, n\}$, $u_I = u_{i_1} \otimes \cdots \otimes u_{i_k}$, and $u_I^* = u_{i_1}^* \otimes \cdots \otimes u_{i_k}^*$.

Proof. The proof is quite easy and essentially reduces to the fact that under the identification Hom $(\mathfrak{u}, \mathfrak{u}) \cong \mathfrak{u}^* \otimes \mathfrak{u}$, the identity map corresponds to the sum $\sum_i u_i^* \otimes u_i$.

Proposition 9.1.6. *Under the action of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ on $V \otimes S$, the operators C and C^- act as d and 2∂ respectively. In particular, the cubic Dirac operator $D = C + C^-$ acts as $d + 2\partial$.*

Proof. We use the description of the action of $C(\mathfrak{s})$ on S from 2.2.2. The first part of C^-, $\sum_i u_i \otimes u_i^*$, acts on an element $x \otimes u_{k_1} \wedge \cdots \wedge u_{k_p}$ of $V \otimes S$ by sending it to

$$\sum_{j=1}^{p} u_{k_j} x \otimes 2(-1)^j u_{k_1} \wedge \dots \widehat{u}_{k_j} \cdots \wedge u_{k_p}.$$

This is exactly twice the first sum in the expression for $\partial(x \otimes u_{k_1} \wedge \cdots \wedge u_{k_p})$. On the other hand, by a similar calculation as the one we used to find expressions for v^+ and v^-,

$$v^- = \frac{1}{2} \sum_{i<j} [u_i, u_j] u_i^* u_j^*.$$

Thus v^- acts on $u_{k_1} \wedge \cdots \wedge u_{k_p}$ by sending it to

$$\frac{1}{2}\sum_{i<j} 2(-1)^i 2(-1)^j [u_{k_i}, u_{k_j}] u_{k_1} \wedge \ldots \widehat{u}_{k_i} \ldots \widehat{u}_{k_j} \cdots \wedge u_{k_p},$$

and this is twice the second sum in the expression for $\partial(x \otimes u_{k_1} \wedge \cdots \wedge u_{k_p})$. Thus we checked that C^- acts as 2∂.

To check that C acts as d, we use the expression for d from Lemma 9.1.3. It is obvious that the action of $\sum_i u_i^* \otimes u_i$ coincides with the first (single) sum in the expression for d. Finally, v^+ acts on $Y_1 \wedge \cdots \wedge Y_p \in S$ by sending it to

$$\begin{aligned} &\frac{1}{4}\sum_{i,j} u_i u_j \; 2\sum_{k=1}^{p} (-1)^{k+1} B([u_i^*, u_j^*], Y_k)\, Y_1 \wedge \ldots \hat{Y}_k \cdots \wedge Y_p \\ = -&\frac{1}{2}\sum_{i,j,k} (-1)^{k+1} B([u_i^*, Y_k], u_j^*)\, u_i \wedge u_j \wedge Y_1 \wedge \ldots \hat{Y}_k \cdots \wedge Y_p. \end{aligned}$$

Now we sum $\sum_j B([u_i^*, Y_k], u_j^*)\, u_j = [u_i^*, Y_k]_{\mathfrak{u}}$, and after commuting $[u_i^*, Y_k]_{\mathfrak{u}}$ into its proper place, we get the second (double) sum in the expression for d.

Remark 9.1.7. To end this section, let us consider the $\mathfrak{l}$-actions under the identifications we have made. The natural action of $\mathfrak{l}$ on $V \otimes S$ is the tensor product of the restriction of the $\mathfrak{g}$-action on V and the spin action on S. On the other hand, the usual $\mathfrak{l}$ action on $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology is given by the adjoint action on $\bigwedge^{\cdot} \bar{\mathfrak{u}}$ and $\bigwedge^{\cdot} \mathfrak{u}$. Thus, our identification of $V \otimes \bigwedge^{\cdot} \mathfrak{u}$ with $V \otimes S$ is not an $\mathfrak{l}$-isomorphism. However, as was proved in [Ko3], Proposition 3.6, the two actions differ only by a twist with the one dimensional $\mathfrak{l}$-module $\mathbb{C}_{\rho(\bar{\mathfrak{u}})}$ of weight $\rho(\bar{\mathfrak{u}})$.

This means that, if we consider C and C^- as operators on $V \otimes S$ via the above identification, then as an $\mathfrak{l}$-module, the cohomology of C gets identified with $H^{\cdot}(\bar{\mathfrak{u}}, V) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{u}})}$, while the homology of C^- gets identified with $H_{\cdot}(\mathfrak{u}, V) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{u}})}$.

9.2 Hodge decomposition in the finite-dimensional case

The results we present in this section are essentially contained in Kostant's famous paper [Ko1]. Namely, it is shown there that the $\bar{\mathfrak{u}}$-cohomology of a finite dimensional $\mathfrak{g}$-module V can be represented by harmonic elements, that is, elements which are killed by the spin Laplacean. There is no mention of the Dirac operator there, but in fact the spin Laplacean is nothing else but $-2D^2$, as follows from Kostant's formula for D^2 (see 4.1.1). This was noted in [Ko3]. Since D is skew self-adjoint (see Lemma 4.2.1), it follows that the kernel of the spin Laplacean also represents the Dirac cohomology. Let us prove all this in detail, as we are going to consider some infinite-dimensional analogues of these results in the subsequent sections.

9.2.1. Adjunction of C and $-C^-$. Let V be a finite-dimensional $\mathfrak{g}$-module. Recall that by Lemma 4.2.1, Kostant's cubic Dirac operator D on $V \otimes S$ is skew self-adjoint with respect to the inner product $\langle\,,\rangle$ defined in Section 4.2. In our present situation, we will strengthen this result by showing that the operators C and C^- are minus adjoints od each other with respect to $\langle\,,\rangle$.

As in Section 4.2, let us choose a basis u_i of $\mathfrak{u}$, such that the dual basis with respect to B is $u_i^* = -\theta\bar{u}_i$. Since the adjoint of any $X \in \mathfrak{s}$ on S is $\theta\bar{X}$, we see that the adjoint of u_i on S is $-u_i^*$. On the other hand, the adjoint of any $X \in \mathfrak{g}$ on V is $-\theta\bar{X}$, so in particular the adjoint of u_i on V is u_i^*. Hence $u_i^* \otimes u_i$ and $u_i \otimes u_i^*$ are minus adjoints of each other on $V \otimes S$. It remains to see that v^+ and v^- are minus adjoints of each other on S. This follows immediately from the formulas (9.1) and (9.2), since the adjoint of $[u_i, u_j]u_i^* u_j^*$ is

$$(-u_j)(-u_i)(\theta\overline{[u_i, u_j]}) = -u_i u_j[\theta\bar{u}_i, \theta\bar{u}_j] = -u_i u_j[u_i^*, u_j^*].$$

We have proved

Proposition 9.2.2. *Let V be a finite-dimensional $\mathfrak{g}$-module, and let $\langle\,,\rangle$ be the inner product on $V \otimes S$ defined in Section 4.2. Then the operators C and C^- on $V \otimes S$ are minus adjoints of each other with respect to $\langle\,,\rangle$.*

The rest of this section repeats the well known arguments leading to a Hodge decomposition in the finite dimensional case. See for example [W], Scholium 9.4.4.

Since D is skew self-adjoint (either by Proposition 9.2.2 or by Lemma 4.2.1), it follows that $\operatorname{Ker} D = \operatorname{Ker} D^2$. Namely, it is clear that $Dx = 0$ implies that $D^2x = 0$. On the other hand, if $D^2x = 0$, then $0 = \langle D^2x, x\rangle = -\langle Dx, Dx\rangle$ shows that $Dx = 0$. Furthermore, since C and C^- are differentials adding up to D, we see that

$$D^2 = (C + C^-)^2 = CC^- + C^-C.$$

Hence $D^2x = 0$ implies

$$0 = \langle D^2x, x\rangle = \langle CC^-x, x\rangle + \langle C^-Cx, x\rangle = -\langle C^-x, C^-x\rangle - \langle Cx, Cx\rangle,$$

and hence $Cx = C^-x = 0$. Conversely, $Cx = C^-x = 0$ implies $D^2x = (CC^- + C^-C)x = 0$. So we get

Lemma 9.2.3. *For any finite-dimensional $\mathfrak{g}$-module V, the operators D, C and C^- on $V \otimes S$ satisfy*

$$\operatorname{Ker} D = \operatorname{Ker} D^2 = \operatorname{Ker} C \cap \operatorname{Ker} C^-.$$

The next easy observation from linear algebra is the fact that for any linear operator A on a finite-dimensional vector space X with an inner product, the kernel of A equals the orthogonal of the image of the adjoint of A. Indeed, if $Ax = 0$, then $\langle x, A^{\mathrm{adj}}\, y\rangle = \langle Ax, y\rangle = 0$, so $\operatorname{Ker} A \perp \operatorname{Im} A^{\mathrm{adj}}$. Since the dimensions of these two spaces add up to $\dim X$, it follows that indeed $X = \operatorname{Ker} A \oplus \operatorname{Im} A^{\mathrm{adj}}$. Applied to our setting, this gives

Lemma 9.2.4. *For any finite-dimensional $\mathfrak{g}$-module V, the operators D, C and C^- on $V \otimes S$ satisfy*

$$\begin{aligned} V \otimes S = \ &\mathrm{Ker}\, D \oplus \mathrm{Im}\, D = \mathrm{Ker}\, D^2 \oplus \mathrm{Im}\, D^2 \\ = \ &\mathrm{Ker}\, C \oplus \mathrm{Im}\, C^- = \mathrm{Ker}\, C^- \oplus \mathrm{Im}\, C. \end{aligned}$$

All the direct sums in this equation are orthogonal with respect to $\langle\, , \rangle$.

We are now ready to prove a Hodge decomposition theorem in our setting. First, we claim that

$$\mathrm{Im}\, D = \mathrm{Im}\, C \oplus \mathrm{Im}\, C^-. \tag{9.3}$$

Since $\mathrm{Im}\, C \subseteq \mathrm{Ker}\, C$, $\mathrm{Im}\, C$ is orthogonal to $\mathrm{Im}\, C^-$ by Lemma 9.2.4. Furthermore, $\mathrm{Im}\, D \subseteq \mathrm{Im}\, C + \mathrm{Im}\, C^-$ since $D = C + C^-$. Finally, $\mathrm{Im}\, C \subseteq \mathrm{Im}\, D$. Namely, since $\mathrm{Ker}\, D \subseteq \mathrm{Ker}\, C^-$ by Lemma 9.2.3, $\mathrm{Im}\, C$ is orthogonal to $\mathrm{Ker}\, D$ because of Lemma 9.2.4. So it follows that $\mathrm{Im}\, C \subseteq \mathrm{Im}\, D$ since $\mathrm{Im}\, D = (\mathrm{Ker}\, D)^\perp$ by Lemma 9.2.4. Analogously, $\mathrm{Im}\, C^- \subseteq \mathrm{Im}\, D$, and this finishes the proof of (9.3).

Using Lemma 9.2.4 again, we now immediately get (a) in the following theorem:

Theorem 9.2.5. *For any finite-dimensional $\mathfrak{g}$-module V, the operators D, C and C^- on $V \otimes S$ satisfy*

(a) $V \otimes S = \mathrm{Ker}\, D \oplus \mathrm{Im}\, C \oplus \mathrm{Im}\, C^-$;

(b) $\mathrm{Ker}\, C = \mathrm{Ker}\, D \oplus \mathrm{Im}\, C$;

(c) $\mathrm{Ker}\, C^- = \mathrm{Ker}\, D \oplus \mathrm{Im}\, C^-$.

Proof. It remains to prove (b) and (c). They are both obtained by combining (a) with Lemma 9.2.4. Namely, (a) says that $(\mathrm{Im}\, C^-)^\perp = \mathrm{Ker}\, D \oplus \mathrm{Im}\, C$, while Lemma 9.2.4 says that $(\mathrm{Im}\, C^-)^\perp = \mathrm{Ker}\, C$. This gives (b) and (c) is obtained analogously.

In view of Remark 9.1.7, the above theorem implies

Corollary 9.2.6. *As $\mathfrak{l}$-modules,*

$$\mathrm{Ker}\, D \cong H^\cdot(\bar{\mathfrak{u}}; V) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{u}})} \cong H_\cdot(\mathfrak{u}; V) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{u}})}.$$

More precisely, (up to modular twists) the Dirac cohomology of V, $\mathrm{Ker}\, D$, is the space of harmonic representatives for both $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology of V.

9.3 Hodge decomposition for $\mathfrak{p}^-$ - cohomology in the unitary case

We now want to obtain analogues of the results of Section 9.2 for unitary $(\mathfrak{g}, K)$-modules V. The idea is to use the Hermitian inner product $\langle\, , \rangle$ on

$V \otimes S$ constructed by tensoring the unitary form on V with the same form $\langle\, ,\rangle$ on S as before.

There are several problems with repeating the proof of the Hodge decomposition from Section 9.2 in this setting. The first one is the fact that on the spin module S we still have the same adjunction as in 9.2.1, i.e., u_i is adjoint to $\theta\bar{u}_i = -u_i^*$, but on the other hand the adjoint of any $X \in \mathfrak{g}$ with respect to the unitary form on V is $-\bar{X}$, not $-\theta\bar{X}$ as before. In particular, the adjoint of u_i on V is $-\bar{u}_i = \theta u_i^*$. Since typically $\mathfrak{u}$ intersects both $\mathfrak{k}$ and $\mathfrak{p}$, $u_i^* \otimes u_i$ will sometimes be the adjoint and sometimes minus the adjoint of $u_i \otimes u_i^*$. So C can be neither the adjoint nor minus the adjoint of C^-. The other problems are related to the fact that we are dealing with an infinite dimensional space now, so the linear algebra can be far more complicated.

A case when the problem with adjunction does not appear is when $\mathfrak{l}$ equals $\mathfrak{k}$ and $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are contained in $\mathfrak{p}$. In that case $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are forced to be abelian, and as usual we denote them by $\mathfrak{p}^+$ respectively $\mathfrak{p}^-$. Of course, the pair $(\mathfrak{g}, \mathfrak{k})$ must be Hermitian symmetric in this case. It is also automatic that the parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is θ-stable. As we will see, the finite-dimensional proof of the Hodge decomposition goes through in this case with almost no changes.

Note that for $\mathfrak{l} = \mathfrak{k}$, the Dirac operator is the one studied in Chapter 3. In particular, there is no cubic term. With notation as before, we have $C = \sum u_i^* \otimes u_i$ and $C^- = \sum u_i \otimes u_i^*$.

Lemma 9.3.1. *Let $(\mathfrak{g}, \mathfrak{k})$ be a Hermitian symmetric pair and set $\mathfrak{l} = \mathfrak{k}$. Let V be a unitary $(\mathfrak{g}, K)$-module and consider the above defined form $\langle\, ,\rangle$ on $V \otimes S$. Then the operators C and C^- are adjoints of each other.*

Proof. Since all u_i are in $\mathfrak{p}$, the adjoint of u_i on V is $-\bar{u}_i = \theta\bar{u}_i = -u_i^*$. Since the adjoint of u_i on S is also $-u_i^*$, we see that the operators $C = \sum u_i^* \otimes u_i$ and $C^- = \sum u_i \otimes u_i^*$ on $V \otimes S$ are adjoints of each other.

It follows that the Dirac operator $D = D(\mathfrak{g}, \mathfrak{l}) = D(\mathfrak{g}, \mathfrak{k})$ is self-adjoint. In particular, the operators D and D^2 have the same kernel on $V \otimes S$.

Since V is now infinite-dimensional, not all the statements from Lemma 9.2.3 and Lemma 9.2.4 are immediately obvious. The key fact we need to proceed is the following Lemma. The assumption on V we need is that the Casimir operator $\Omega_\mathfrak{g}$ acts on V by a scalar. This is certainly true whenever V has infinitesimal character. Since V is also unitary, we will not lose too much generality by assuming that V is in fact irreducible.

Lemma 9.3.2. *Let V be an irreducible unitary $(\mathfrak{g}, K)$-module. Then $V \otimes S =$* Ker $(D^2) \oplus$ Im (D^2).

Proof. By Proposition 3.1.6 we know that

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + C,$$

where $\Omega_{\mathfrak{g}}$ and $\Omega_{\mathfrak{k}_\Delta}$ are the Casimir operators for $\mathfrak{g}$ and diagonally embedded $\mathfrak{k}$, and C is the constant $||\rho_{\mathfrak{k}}||^2 - ||\rho_{\mathfrak{g}}||^2$.

Since $\Omega_{\mathfrak{g}}$ acts on V by a constant, it follows that $\Omega_{\mathfrak{k}_\Delta}$ is up to an additive constant equal to D^2 on $V \otimes S$. Since $\Omega_{\mathfrak{k}_\Delta}$ acts by a scalar on each $\tilde{K}$-type in $V \otimes S$, the same is true for D^2. (Recall that $\tilde{K}$ is the spin double cover of K; see 3.2.1.)

So D^2 is a semisimple operator, i.e., $V \otimes S$ is a direct sum of eigenspaces for D^2. The lemma is now clear: the zero eigenspace is $\operatorname{Ker} D^2$, and the sum of the nonzero eigenspaces is $\operatorname{Im} D^2$.

The following Lemma contains analogues of some of the statements of Lemma 9.2.3 and Lemma 9.2.4. The proof is exactly the same as in the finite-dimensional case.

Lemma 9.3.3. *For any unitary $(\mathfrak{g}, K)$-module V, the operators D, C and C^- on $V \otimes S$ satisfy*

(a) $\operatorname{Ker} D = \operatorname{Ker} C \cap \operatorname{Ker} C^-$*;*

(b) $\operatorname{Im} C^-$ *is orthogonal to* $\operatorname{Ker} C$ *and* $\operatorname{Im} C$ *is orthogonal to* $\operatorname{Ker} C^-$*.*

Combining Lemmas 9.3.2 and 9.3.3 with the fact $\operatorname{Ker} D = \operatorname{Ker} D^2$, we can now prove the following analogue of Theorem 9.2.5. The proof is a minor modification of the proof we presented in the finite-dimensional case.

Theorem 9.3.4. *Let $(\mathfrak{g}, \mathfrak{k})$ be a Hermitian symmetric pair and set $\mathfrak{l} = \mathfrak{k}$ and $\mathfrak{u} = \mathfrak{p}^+$. Let V be an irreducible unitary $(\mathfrak{g}, K)$-module. Then:*

(a) $V \otimes S = \operatorname{Ker} D \oplus \operatorname{Im} C \oplus \operatorname{Im} C^-$*;*

(b) $\operatorname{Ker} C = \operatorname{Ker} D \oplus \operatorname{Im} C$*;*

(c) $\operatorname{Ker} C^- = \operatorname{Ker} D \oplus \operatorname{Im} C^-$*.*

All the above sums are orthogonal with respect to $\langle\, , \rangle$.

Proof. (a) By Lemma 9.3.2 and the fact $\operatorname{Ker} D = \operatorname{Ker} D^2$, we only need to show that $\operatorname{Im} D^2 = \operatorname{Im} C \oplus \operatorname{Im} C^-$. The sum $\operatorname{Im} C \oplus \operatorname{Im} C^-$ is orthogonal by Lemma 9.3.3.(b), since $\operatorname{Im} C \subseteq \operatorname{Ker} C$. It is clear that $\operatorname{Im} D^2 \subseteq \operatorname{Im} C + \operatorname{Im} C^-$, since $D^2 = CC^- + C^-C$. On the other hand, since $\operatorname{Im} C$ is orthogonal to $\operatorname{Ker} C^-$ by Lemma 9.3.3.(b), $\operatorname{Im} C$ is also orthogonal to $\operatorname{Ker} D^2 = \operatorname{Ker} D = \operatorname{Ker} C \cap \operatorname{Ker} C^-$. Hence $\operatorname{Im} C \subseteq (\operatorname{Ker} D^2)^\perp$, and the latter is equal to $\operatorname{Im} D^2$ by Lemma 9.3.2. So $\operatorname{Im} C \subseteq \operatorname{Im} D^2$. Analogously one sees that $\operatorname{Im} C^- \subseteq \operatorname{Im} D^2$ and this finishes the proof of (a).

(b) By Lemma 9.3.3.(b) and (a), $\operatorname{Ker} C \subseteq (\operatorname{Im} C^-)^\perp = \operatorname{Ker} D \oplus \operatorname{Im} C$. Furthermore, $\operatorname{Ker} D \subseteq \operatorname{Ker} C$ by Lemma 9.3.3.(a), and $\operatorname{Im} C \subseteq \operatorname{Ker} C$ since C is a differential.

(c) Analogous to (b).

Corollary 9.3.5. *The Dirac cohomology of V is equal to $\mathfrak{p}^-$-cohomology of V and to $\mathfrak{p}^+$-homology of V, up to modular twists:*

$$\operatorname{Ker} D \cong H^\cdot(\mathfrak{p}^-, V) \otimes \mathbb{C}_{\rho(\mathfrak{p}^-)} \cong H_\cdot(\mathfrak{p}^+, V) \otimes \mathbb{C}_{\rho(\mathfrak{p}^-)}.$$

More precisely, (up to modular twists) the Dirac cohomology Ker D *is the space of harmonic representatives for both* $\mathfrak{p}^-$*-cohomology and* $\mathfrak{p}^+$*-homology.*

Remark 9.3.6. If $\mathfrak{g}$ is not simple, it is also possible that a Levi subalgebra $\mathfrak{l}$ strictly contains $\mathfrak{k}$. For example, if $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, then with the obvious notation $\mathfrak{k} = \mathfrak{k}_1 \times \mathfrak{k}_2$, and $\mathfrak{l}$ can be $\mathfrak{k}_1 \times \mathfrak{g}_2$. Since it is still true in this case that $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are contained in $\mathfrak{p}$, our arguments work without change, and we see that Theorem 9.3.4 and Corollary 9.3.5 generalize to this setting.

9.4 Calculating Dirac cohomology in stages

In this and the next section we will study some other cases where we can obtain a Hodge decomposition for unitary modules similar to that of Theorem 9.3.4. These are the cases when $(\mathfrak{g}, \mathfrak{k})$ is Hermitian symmetric, $\mathfrak{l}$ is contained in $\mathfrak{k}$ and $\mathfrak{u}$ contains $\mathfrak{p}^+$. We start by a more general situation and add the extra assumptions as necessary.

Let $\mathfrak{g}$ be any complex reductive Lie algebra, with a fixed invariant nondegenerate symmetric bilinear form B, and let $\mathfrak{r}$ be a quadratic subalgebra of $\mathfrak{g}$ as in 2.3.3. Then $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is the orthogonal of $\mathfrak{r}$ with respect to B. Let $\mathfrak{r}_1$ be another quadratic subalgebra of $\mathfrak{g}$ with orthogonal $\mathfrak{s}_1$. We assume that $\mathfrak{r}_1 \subseteq \mathfrak{r}$, and hence $\mathfrak{s} \supseteq \mathfrak{s}_1$. In particular,

$$\mathfrak{g} = \mathfrak{r}_1 \oplus \mathfrak{s}_1 = \mathfrak{r}_1 \oplus \mathfrak{s} \oplus (\mathfrak{r} \cap \mathfrak{s}_1).$$

Later on we will specialize to the case when $\mathfrak{r}$ is $\mathfrak{k}$, and then eventually $\mathfrak{r}_1$ will be a Levi subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$.

To write down the Dirac operator $D(\mathfrak{g}, \mathfrak{r}_1)$, we form an orthonormal basis for $\mathfrak{s}_1$ from orthonormal bases Z_i for $\mathfrak{s}$ respectively Z_j' for $\mathfrak{r} \cap \mathfrak{s}_1$. Identifying

$$U(\mathfrak{g}) \otimes C(\mathfrak{s}_1) = U(\mathfrak{g}) \otimes C(\mathfrak{s}) \bar{\otimes} C(\mathfrak{r} \cap \mathfrak{s}_1), \tag{9.4}$$

where $\bar{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product, we can write

$$\begin{aligned} D(\mathfrak{g}, \mathfrak{r}_1) = &\sum_i Z_i \otimes Z_i \otimes 1 + \sum_j Z_j' \otimes 1 \otimes Z_j' \\ &+ \frac{1}{2} \sum_{i<j<k} B([Z_i, Z_j], Z_k) \otimes Z_i Z_j Z_k \otimes 1 \\ &+ \frac{1}{2} \sum_{i<j} \sum_k B([Z_i, Z_j], Z_k') \otimes Z_i Z_j \otimes Z_k' \\ &+ \frac{1}{2} \sum_{i<j<k} B([Z_i', Z_j'], Z_k') \otimes 1 \otimes Z_i' Z_j' Z_k'. \end{aligned} \tag{9.5}$$

Note that while Kostant's original definition (see 5.1.1) uses exterior multiplication to define the cubic term, in the present case we can use Clifford multiplication instead. Namely, there is no difference between exterior and Clifford multiplication for orthogonal vectors.

Note also that the terms with Z_i, Z'_j and Z'_k do not appear in (9.5), because $B([Z_i, Z'_j], Z'_k) = B(Z_i, [Z'_j, Z'_k]) = 0$, as $[Z'_j, Z'_k] \in \mathfrak{r}$ is orthogonal to $\mathfrak{s}$.

We consider $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ as the subalgebra $U(\mathfrak{g}) \otimes C(\mathfrak{s}) \otimes 1$ of $U(\mathfrak{g}) \otimes C(\mathfrak{s}) \bar{\otimes} C(\mathfrak{r} \cap \mathfrak{s}_1)$. In view of this, we see that the first and third sum in (9.5) combine to give $D(\mathfrak{g}, \mathfrak{r})$, the Kostant's cubic Dirac operator corresponding to $\mathfrak{r} \subset \mathfrak{g}$.

The remaining three sums come from the cubic Dirac operator corresponding to $\mathfrak{r}_1 \subset \mathfrak{r}$. However, this is an element of the algebra $U(\mathfrak{r}) \otimes C(\mathfrak{r} \cap \mathfrak{s}_1)$, and this algebra has to be embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{s}) \bar{\otimes} C(\mathfrak{r} \cap \mathfrak{s}_1)$ diagonally, by

$$\Delta : U(\mathfrak{r}) \otimes C(\mathfrak{r} \cap \mathfrak{s}_1) \cong U(\mathfrak{r}_\Delta) \bar{\otimes} C(\mathfrak{r} \cap \mathfrak{s}_1) \subset U(\mathfrak{g}) \otimes C(\mathfrak{s}) \bar{\otimes} C(\mathfrak{r} \cap \mathfrak{s}_1).$$

Here $U(\mathfrak{r}_\Delta)$ is embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ by a diagonal embedding analogous to the embedding of 3.1.4, while the factor $C(\mathfrak{r} \cap \mathfrak{s}_1)$ remains unchanged.

We will denote $\Delta(D(\mathfrak{r}, \mathfrak{r}_1))$ by $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$. In case when there are several subalgebras and confusion might arise, we will use the more precise notation $\Delta_{\mathfrak{g},\mathfrak{r}}$ instead of Δ and give up the notation $D_\Delta(.)$. The above diagonal embedding has already been used by Kostant and Alekseev-Meinrenken; in particular, decompositions like the following one can be found in [AM].

Theorem 9.4.1. *With notation as above, $D(\mathfrak{g}, \mathfrak{r}_1)$ decomposes as $D(\mathfrak{g}, \mathfrak{r}) + D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$. Moreover, the summands $D(\mathfrak{g}, \mathfrak{r})$ and $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$ anticommute.*

Proof. We need to describe the image under Δ of the element

$$D(\mathfrak{r}, \mathfrak{r}_1) = \sum_i Z'_i \otimes Z'_i + \frac{1}{2} \sum_{i<j<k} B([Z'_i, Z'_j], Z'_k) \otimes Z'_i Z'_j Z'_k \tag{9.6}$$

of $U(\mathfrak{r}) \otimes C(\mathfrak{r} \cap \mathfrak{s}_1)$, and see that it matches the second, fourth and fifth sum in (9.5). In fact, it is obvious that the image under Δ of the second sum in (9.6) equals the fifth sum in (9.5), and it remains to show that

$$\sum_i \Delta(Z'_i \otimes Z'_i) = \sum_i Z'_i \otimes 1 \otimes Z'_i + \sum_i 1 \otimes \alpha(Z'_i) \otimes Z'_i. \tag{9.7}$$

matches the second and fourth sum in (9.5). Namely, $\Delta(Z \otimes Z') = Z \otimes 1 \otimes Z' + 1 \otimes \alpha(Z) \otimes Z'$, where $\alpha : \mathfrak{r} \to \mathfrak{so}(\mathfrak{s}) \hookrightarrow C(\mathfrak{s})$ is the map from 2.3.3. Thus we are left with showing that the second sum in (9.7) equals the fourth sum in (9.5), i.e., that

$$\sum_k 1 \otimes \alpha(Z'_k) \otimes Z'_k = \frac{1}{2} \sum_{i<j} \sum_k B([Z_i, Z_j], Z'_k) \otimes Z_i Z_j \otimes Z'_k.$$

This is however true, since by (2.8)

$$\alpha(Z'_k) = \frac{1}{2} \sum_{i<j} B([Z_i, Z_j], Z'_k) Z_i Z_j$$

for any k.

This proves that indeed $D(\mathfrak{g}, \mathfrak{r}_1) = D(\mathfrak{g}, \mathfrak{r}) + D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$ To prove that the summands $D(\mathfrak{g}, \mathfrak{r})$ and $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$ anticommute, we use the fact that $D(\mathfrak{g}, \mathfrak{r})$ commutes with $\mathfrak{r}_\Delta$. It follows that the anticommutator

$$\left[D(\mathfrak{g}, \mathfrak{r}) \otimes 1, (Z'_i \otimes 1 + 1 \otimes \alpha(Z'_i)) \otimes Z'_i\right] = \left[D(\mathfrak{g}, \mathfrak{r}), Z'_i \otimes 1 + 1 \otimes \alpha(Z'_i)\right] \otimes Z'_i$$

is zero for any i. Hence $\left[D(\mathfrak{g}, \mathfrak{r}) \otimes 1, \Delta(\sum_i Z'_i \otimes Z'_i)\right] = 0$. It remains to see that also

$$\left[D(\mathfrak{g}, \mathfrak{r}) \otimes 1, 1 \otimes 1 \otimes \frac{1}{2} \sum_{i<j<k} B([Z'_i, Z'_j], Z'_k) Z'_i Z'_j Z'_k\right] = 0.$$

This follows from the definition of $\bar{\otimes}$, since all the $C(\mathfrak{s})$-parts of the monomial terms of $D(\mathfrak{g}, \mathfrak{r})$, and also all $Z'_i Z'_j Z'_k \in C(\mathfrak{r} \cap \mathfrak{s}_1)$, are odd.

We now want to use Theorem 9.4.1 to relate the Dirac cohomology of the various Dirac operators involved. We can do this only in special cases. Namely, we need to develop some algebra of anticommuting operators. Our approach will require certain assumptions.

We define the cohomology of any linear operator T on a vector space V to be the vector space $H(T) = \operatorname{Ker} T / (\operatorname{Im} T \cap \operatorname{Ker} T)$. We call the operator T semisimple, if V is the (algebraic) direct sum of eigenspaces of T.

Lemma 9.4.2. *Let A and B be anticommuting linear operators on an arbitrary vector space V.*

(i) Assume that A^2 is semisimple, and denote by V_λ the eigenspace of A^2 with the eigenvalue λ. Then the cohomology $H(A+B)$ of $A+B$ on V is the same as the cohomology of the restriction of $A+B$ to $V_0 =$ Ker A^2.

(ii) Assume that A^2 is semisimple, and that Ker $A^2 =$ Ker $A = H(A)$*; so* Ker $A \cap$ Im $A = 0$*. Then $H(A+B)$ is equal to the cohomology of B restricted to the cohomology (i.e., kernel) of A.*

(iii) Assume that A^2 and B are semisimple. Then $H(A+B)$ is the cohomology (i.e., the kernel) of B acting on $H(A)$.

Proof. (i) Since $A+B$ commutes with A^2, its kernel, image and cohomology decompose accordingly to eigenspaces V_λ. We thus have to prove that $A+B$ has no cohomology on V_λ for $\lambda \neq 0$. In other words, we are to prove that Ker $(A+B) \subset$ Im $(A+B)$ on V_λ if $\lambda \neq 0$.

Let $v \in V_\lambda$ be such that $(A+B)v = 0$, i.e., $Av = -Bv$. Then

$$(A+B)Av = A^2 v + BAv = A^2 v - ABv = 2A^2 v = 2\lambda v,$$

and hence $v = \frac{1}{2\lambda}(A+B)Av$ is in the image of $A+B$.

(ii) By (i), $H(A+B)$ is the cohomology of $A+B$ on Ker A. But on Ker A, $A+B=B$.

(iii) Applying (i), we can replace V by Ker A^2, i.e., assume $A^2=0$. On the other hand, by (ii), $H(A+B)$ is the cohomology of A acting on Ker B.

Since B is semisimple, we can decompose

$$V = \text{ Ker } B \oplus \bigoplus_{\lambda} V_\lambda \oplus V_{-\lambda}$$

into the (discrete) sum of eigenspaces for B. Here if both λ and $-\lambda$ are eigenvalues, we choose one of them to represent the pair. Since A anticommutes with B, it preserves Ker B, and maps V_λ to $V_{-\lambda}$ and vice versa. Therefore, $H(A)$ decomposes into a Ker B-part and $V_\lambda \oplus V_{-\lambda}$-parts. The Ker B-part is equal to $H(A+B)$ and we will be done if we show that B has no kernel on $H(A|_{V_\lambda \oplus V_{-\lambda}})$. Let $v=v_1+v_2 \in V_\lambda \oplus V_{-\lambda}$ be in Ker A, and assume that $Bv \in$ Im A. This implies $\lambda v_1 - \lambda v_2$ is in Im A, so $v_1 - v_2 \in$ Im A. This however can only happen if both v_1 and v_2 are in Im A, again because A exchanges V_λ and $V_{-\lambda}$. But then also $v=v_1+v_2$ is in Im A, so v is zero in cohomology and we are done.

To apply the above lemma to Dirac cohomology, denote by $H_D(\mathfrak{g},\mathfrak{r};V)$ the Dirac cohomology of a $(\mathfrak{g},K)$-module V with respect to $D(\mathfrak{g},\mathfrak{r})$; analogous notation will be used for other Dirac operators. The reader should bear in mind that $H_D(\mathfrak{g},\mathfrak{r};V)$ is in fact the cohomology of the operator $D(\mathfrak{g},\mathfrak{r})$ on the space $V\otimes S$.

Proposition 9.4.3. *Let $\mathfrak{r}\subset\mathfrak{k}$ be a quadratic subalgebra of $\mathfrak{g}$. As usual, let $\mathfrak{s}$ be the orthocomplement of $\mathfrak{r}$. Assume that either* $\dim\mathfrak{p}$ *is even, or* $\dim\mathfrak{s}\cap\mathfrak{k}$ *is even. Let V be an irreducible unitary $(\mathfrak{g},K)$-module. Then*

$$H_D(\mathfrak{g},\mathfrak{r};V) = H_D(\mathfrak{k},\mathfrak{r};H_D(\mathfrak{g},\mathfrak{k};V)),$$

i.e., the Dirac cohomology can be calculated "in stages", as the $D(\mathfrak{k},\mathfrak{r})$-cohomology of the $D(\mathfrak{g},\mathfrak{k})$-cohomology.

Proof. Since V is unitary, we can consider the form $\langle\,,\rangle$ on $V\otimes S_\mathfrak{p}$ introduced in Section 9.3, where $S_\mathfrak{p}$ denotes the spin module for $C(\mathfrak{p})$. We can extend $\langle\,,\rangle$ to all of $V\otimes S$, by combining it with the form on the spin module $S_{\mathfrak{s}\cap\mathfrak{k}}$ for $C(\mathfrak{s}\cap\mathfrak{k})$ analogous to the form $\langle\,,\rangle$ on $S_\mathfrak{p}$ (see 2.3.9). Here we identify $S=S_\mathfrak{p}\otimes S_{\mathfrak{s}\cap\mathfrak{k}}$, which can be done by the assumption on dimensions. Let $A=D(\mathfrak{g},\mathfrak{k})$ and $B=D_\Delta(\mathfrak{k},\mathfrak{r})$.

By Lemma 9.3.1, A is self-adjoint, and consequently the conditions of Lemma 9.4.2.(ii) are satisfied. So the cohomology with respect to $D(\mathfrak{g},\mathfrak{r})$ is the cohomology with respect to B of Ker $A=H_D(\mathfrak{g},\mathfrak{k};V)\otimes S_{\mathfrak{s}\cap\mathfrak{k}}$.

Now $H_D(\mathfrak{g},\mathfrak{k};V)\subset V\otimes S_\mathfrak{p}\subset V\otimes S$ is a $\tilde{K}$-module, with Lie algebra $\mathfrak{k}$ acting through $\mathfrak{k}_\Delta$. The Dirac cohomology of this module with respect to $D(\mathfrak{k},\mathfrak{r})$ is thus identified with the cohomology with respect to $B=D_\Delta(\mathfrak{k},\mathfrak{r})$.

Let us now assume that $\mathfrak{r}$ is the complexification of a reductive subalgebra $\mathfrak{r}_0$ of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. In this situation we can generalize Proposition 9.4.3 to nonunitary modules. Like in Proposition 9.4.3, we assume that either dim $\mathfrak{p}$ is even or dim $\mathfrak{s} \cap \mathfrak{k}$ is even, so that we can write the spin module as $S = S_{\mathfrak{p}} \otimes S_{\mathfrak{s}\cap\mathfrak{k}}$. The idea is to reverse the roles of $D(\mathfrak{g},\mathfrak{k})$ and $D_\Delta(\mathfrak{k},\mathfrak{r})$. Namely, for any admissible $(\mathfrak{g},K)$-module V, we can decompose $V \otimes S_{\mathfrak{p}}$ into a direct sum of finite dimensional (unitary) modules for the spin double cover $\tilde{K}$ of K. Hence, using either Proposition 9.2.2 or Lemma 4.2.1, we conclude that there is a positive definite form $\langle\,,\rangle$ on $V \otimes S = V \otimes S_{\mathfrak{p}} \otimes S_{\mathfrak{s}\cap\mathfrak{k}}$, such that $D_\Delta(\mathfrak{k},\mathfrak{r})$ is skew self-adjoint with respect to $\langle\,,\rangle$.

It follows that $B = D_\Delta(\mathfrak{k},\mathfrak{r})$ is a semisimple operator, while for $A = D(\mathfrak{g},\mathfrak{k})$ we still have that A^2 is semisimple. Therefore we can apply Lemma 9.4.2.(iii) and obtain the following result.

Theorem 9.4.4. *Let $\mathfrak{r}_0$ be a reductive subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. Let V be an admissible $(\mathfrak{g},K)$-module. Then the Dirac cohomology with respect to $D(\mathfrak{g},\mathfrak{r})$ can be calculated as the Dirac cohomology with respect to $D(\mathfrak{k},\mathfrak{r})$ of the Dirac cohomology with respect to $D(\mathfrak{g},\mathfrak{k})$ of V. In other words:*

$$H_D(\mathfrak{g},\mathfrak{r};V) = H_D(\mathfrak{k},\mathfrak{r};H_D(\mathfrak{g},\mathfrak{k};V)).$$

Also, we can reverse the order of taking Dirac cohomology, i.e.,

$$H_D(\mathfrak{g},\mathfrak{r};V) = H(D(\mathfrak{g},\mathfrak{k})\big|_{H_D(\mathfrak{k},\mathfrak{r};V)}).$$

Proof. The first formula follows from Lemma 9.4.2.(iii) as explained above. The second formula is a direct application of Lemma 9.4.2.(ii), with $A = D_\Delta(\mathfrak{k},\mathfrak{r})$ and $B = D(\mathfrak{g},\mathfrak{k})$ (opposite from Proposition 9.4.3).

For the rest of this section we consider a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, with the Levi subalgebra $\mathfrak{l}$ contained in $\mathfrak{k}$. In particular, there is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{l} \subset \mathfrak{k}$; so $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank. The opposite parabolic subalgebra is $\bar{\mathfrak{q}} = \mathfrak{l} \oplus \bar{\mathfrak{u}}$. As before, we denote $\mathfrak{s} = \mathfrak{u} \oplus \bar{\mathfrak{u}}$, so $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{s}$.

We apply the above considerations to $\mathfrak{r} = \mathfrak{l}$. Since $H_D(\mathfrak{g},\mathfrak{k};V)$ is a finite dimensional $\tilde{K}$-module, and $\mathfrak{k}$ and $\mathfrak{l}$ have equal rank, $H_D(\mathfrak{k},\mathfrak{l};H_D(\mathfrak{g},\mathfrak{k};V))$ is given by Theorem 4.2.2. This gives $H_D(\mathfrak{g},\mathfrak{l};V)$ very explicitly provided we know $H_D(\mathfrak{g},\mathfrak{k};V)$ explicitly. For example, one can in this way calculate the Dirac cohomology of the discrete series representations with respect to the (compact) Cartan subalgebra $\mathfrak{t}$:

Example 9.4.5. Let $V = A_{\mathfrak{b}}(\lambda)$ be a discrete series representation; here $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$ containing a compact Cartan subalgebra $\mathfrak{t}$. The infinitesimal character of V is $\lambda + \rho$. Then the Dirac cohomology of V with respect to $D(\mathfrak{g},\mathfrak{k})$ consists of a single $\tilde{K}$-type $V(\mu)$, whose highest weight is $\mu = \lambda + \rho_n$, where $\rho_n = \rho(\mathfrak{u} \cap \mathfrak{p})$. This is obtained from the highest weight of the lowest K-type of V, $\lambda + 2\rho_n$, by shifting by $-\rho_n$ (the lowest weight of S).

This result is contained in the work of Parthasarathy and Schmid. One can also prove it as follows: it is shown in [HP1], Proposition 5.4, that this $\tilde{K}$-type is contained in the Dirac cohomology. Since V has a unique lowest K-type, and since $-\rho_n$ is the lowest weight of the spin module, with multiplicity one, it follows that any other $\tilde{K}$-type has strictly larger highest weight, and thus can not contribute to the Dirac cohomology.

We now apply the above mentioned Kostant's formula (Theorem 4.2.2) to calculate the Dirac cohomology with respect to $D(\mathfrak{k},\mathfrak{t})$ (we again stress that $\mathfrak{k}$ and $\mathfrak{t}$ have equal rank):

$$H_D(\mathfrak{k},\mathfrak{t};V(\mu)) = \text{ Ker } D(\mathfrak{k},\mathfrak{t}) = \bigoplus_{w\in W_\mathfrak{k}} \mathbb{C}_{w(\mu+\rho_\mathfrak{k})}.$$

It follows from $\mu+\rho_\mathfrak{k} = \lambda+\rho$ that

$$H_D(\mathfrak{k},\mathfrak{t};V(\mu)) = \bigoplus_{w\in W_\mathfrak{k}} \mathbb{C}_{w(\lambda+\rho)}.$$

Comparing with Schmid's formula in Theorem 4.1 of [S2], we have

$$H^*(\bar{\mathfrak{n}}, A_\mathfrak{b}(\lambda)) = H_D(\mathfrak{g},\mathfrak{t};A_\mathfrak{b}(\lambda)) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{n}})}.$$

(note that Schmid's $\mathfrak{n}$ is our $\bar{\mathfrak{n}}$, and his λ is our $\lambda+\rho$.)

In other words, $\mathfrak{n}$-cohomology of a discrete series representation coincides with the Dirac cohomology up to a ρ-shift. This fact is however not covered by our results in this chapter. This indicates that it should be possible to generalize our results.

9.5 Hodge decomposition for $\bar{\mathfrak{u}}$-cohomology in the unitary case

As before, let $\mathfrak{q} = \mathfrak{l}\oplus\mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ and assume that $\mathfrak{l}\subseteq\mathfrak{k}$. In the last section we decomposed the Dirac operator $D(\mathfrak{g},\mathfrak{l})$ as $D(\mathfrak{g},\mathfrak{k}) + D_\Delta(\mathfrak{k},\mathfrak{l})$, and saw that we can use this decomposition to calculate the Dirac cohomology in stages. We would now like to obtain similar results for the "half-Dirac" operators C and C^-.

Let $u_1,\dots,u_k$ be a basis for $\mathfrak{u}\cap\mathfrak{k}$ and let $v_1,\dots,v_p$ be a basis for $\mathfrak{u}\cap\mathfrak{p}$. These can be taken to be the root vectors corresponding to compact, respectively noncompact positive roots, with respect to some $\Delta^+(\mathfrak{g},\mathfrak{t})$ compatible with $\mathfrak{u}$. We normalize these bases so that the dual bases for $\bar{\mathfrak{u}}\cap\mathfrak{k}$ respectively $\bar{\mathfrak{u}}\cap\mathfrak{p}$ with respect to the Killing form are $u_i^* = -\bar{u}_i$ respectively $v_i^* = \bar{v}_i$.

As before, $D = D(\mathfrak{g},\mathfrak{l})$, $C = C(\mathfrak{g},\mathfrak{l})$ and $C^- = C^-(\mathfrak{g},\mathfrak{l})$ denote the Dirac operator for the pair $(\mathfrak{g},\mathfrak{l})$ and its parts. Recall that $C = \sum u_i^*\otimes u_i + \sum v_i^*\otimes$

$v_i + 1 \otimes v^+$, where v^+ denotes a part of the cubic term v. Analogously, $C^- = \sum u_i \otimes u_i^* + \sum v_i \otimes v_i^* + 1 \otimes v^-$. We can further decompose $v^\pm$ as

$$v^\pm = v^\pm_{\mathfrak{k}} + v^\pm_{\mathfrak{k}\mathfrak{p}} + v^\pm_{\mathfrak{p}},$$

where

$$v^\pm_{\mathfrak{k}} = \frac{1}{4}\sum_{i,j}[u_i^*, u_j^*]u_i u_j; \quad v^\pm_{\mathfrak{k}\mathfrak{p}} = \frac{1}{2}\sum_{i,j}[u_i^*, v_j^*]u_i v_j; \quad a_{\mathfrak{p}} = \frac{1}{4}\sum_{i,j}[v_i^*, v_j^*]v_i v_j.$$

In the following, we will consider the Clifford algebra $C(\mathfrak{s})$ as a subalgebra of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, embedded as $1 \otimes C(\mathfrak{s})$. In particular, $1 \otimes v^\pm_{\mathfrak{k}}$ gets identified with $v^\pm_{\mathfrak{k}}$, $1 \otimes v^\pm_{\mathfrak{k}\mathfrak{p}}$ with $v^\pm_{\mathfrak{k}\mathfrak{p}}$ and so on.

Recall that by Theorem 9.4.1, $D(\mathfrak{g},\mathfrak{l}) = D(\mathfrak{g},\mathfrak{k}) + D_\Delta(\mathfrak{k},\mathfrak{l})$, where $D_\Delta(\mathfrak{k},\mathfrak{l})$ is the image of $D(\mathfrak{k},\mathfrak{l})$ under the diagonal embedding $\Delta : U(\mathfrak{k}) \otimes C(\mathfrak{s}\cap\mathfrak{k}) \to U(\mathfrak{g})\otimes C(\mathfrak{p})\bar{\otimes} C(\mathfrak{s}\cap\mathfrak{k})$. Here Δ sends $1\otimes C(\mathfrak{s}\cap\mathfrak{k})$ identically onto $1\otimes 1\otimes C(\mathfrak{s}\cap\mathfrak{k})$, and for $X \in \mathfrak{k}$,

$$\Delta(X \otimes 1) = X \otimes 1 \otimes 1 + 1 \otimes \alpha(X) \otimes 1,$$

with $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ defined in 2.3.3. See 3.1.4.

Clearly, $D(\mathfrak{g},\mathfrak{k}) = \sum v_i^* \otimes v_i + \sum v_i \otimes v_i^*$, while $D_\Delta(\mathfrak{k},\mathfrak{l})$ is the sum of all the other parts of $D(\mathfrak{g},\mathfrak{l})$. We want to make this more precise; namely, in the obvious notation, $D(\mathfrak{k},\mathfrak{l}) = C(\mathfrak{k},\mathfrak{l}) + C^-(\mathfrak{k},\mathfrak{l})$, and we want to identify the images of the summands $C(\mathfrak{k},\mathfrak{l})$ and $C^-(\mathfrak{k},\mathfrak{l})$ under Δ. Denote these images by $C_\Delta(\mathfrak{k},\mathfrak{l})$ and $C_\Delta^-(\mathfrak{k},\mathfrak{l})$.

To do this, we recall an expression for $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ in terms of a basis and a dual basis given in (2.11): if b_i is a basis of $\mathfrak{p}$ with dual basis d_i, then

$$\alpha(X) = \frac{1}{4}\sum_{i,j} B([d_i, d_j], X) b_i b_j.$$

If the basis b_i is $v_1, \dots, v_p, v_1^*, \dots, v_p^*$, then the dual basis d_i is $v_1^*, \dots, v_p^*, v_1, \dots, v_p$, and we get

$$\begin{aligned}\alpha(X) = \frac{1}{4}\sum_{j,k} B([v_j^*, v_k^*], X) v_j v_k + \frac{1}{2}\sum_{j,k} B([v_j, v_k^*], X) v_j^* v_k \\ -\frac{1}{2}\sum_{j} B([v_j, v_j^*], X) + \frac{1}{4}\sum_{j,k} B([v_j, v_k], X) v_j^* v_k^*\end{aligned}$$

(we used $v_j^* v_k = -v_k v_j^* - 2\delta_{jk}$).

Since $C(\mathfrak{k},\mathfrak{l}) = \sum_i u_i^* \otimes u_i + \frac{1}{4} \otimes \sum_{i,j}[u_i^*, u_j^*]u_i u_j$, we see that $\Delta(C(\mathfrak{k},\mathfrak{l})) = \sum_i u_i^* \otimes u_i + \sum_i 1 \otimes \alpha(u_i^*) \otimes u_i + v^+_{\mathfrak{k}}$. We need to calculate the middle term, $\sum_i 1 \otimes \alpha(u_i^*) \otimes u_i$. Applying the above expression for α, we get four sums over i, j and k.

We first notice that the first of these four sums is 0, since B is 0 on $\bar{\mathfrak{u}}$. To calculate the second sum, write $B([v_j, v_k^*], u_i^*) = B(v_j, [v_k^*, u_i^*])$, and observe that since $[v_k^*, u_i^*] \in \bar{\mathfrak{u}} \cap \mathfrak{p}$, $\sum_j B(v_j, [v_k^*, u_i^*])v_j^* = [v_k^*, u_i^*]$. Therefore the second of the four sums is

$$\frac{1}{2} \otimes \sum_{i,k} [v_k^*, u_i^*] v_k u_i = v_{\mathfrak{kp}}^+.$$

The third sum is 0, since we can assume $[v_j, v_j^*]$ is in $\mathfrak{l}$ and hence orthogonal to u_i^*. Namely, we can choose v_j and v_j^* (and also u_i and u_i^*) to be root vectors with respect to $\mathfrak{t}$.

Finally, the fourth sum is calculated by noting that since $[v_j, v_k] \in \mathfrak{u} \cap \mathfrak{k}$, $\sum_i B([v_j, v_k], u_i^*)u_i = [v_j, v_k]$. It follows that the fourth sum is

$$\frac{1}{4} \otimes \sum_{j,k} v_j^* v_k^* [v_j, v_k] = v_{\mathfrak{p}}^- .$$

A completely analogous calculation applies to $C^-(\mathfrak{k}, \mathfrak{l})$, so we proved:

Proposition 9.5.1. *Under the diagonal map $\Delta : U(\mathfrak{k}) \otimes C(\mathfrak{s} \cap \mathfrak{k}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p}) \bar{\otimes} C(\mathfrak{s} \cap \mathfrak{k})$, $C(\mathfrak{k}, \mathfrak{l})$ and $C^-(\mathfrak{k}, \mathfrak{l})$ correspond to*

$$C_\Delta(\mathfrak{k}, \mathfrak{l}) = \sum_i u_i^* \otimes u_i + v_{\mathfrak{k}}^+ + v_{\mathfrak{kp}}^+ + v_{\mathfrak{p}}^-$$

and $$C_\Delta^-(\mathfrak{k}, \mathfrak{l}) = \sum_i u_i \otimes u_i^* + v_{\mathfrak{k}}^- + v_{\mathfrak{kp}}^- + v_{\mathfrak{p}}^+ .$$

Note the unexpected feature of this result, the mixing of the positive and negative parts under the diagonal embedding. Namely, $v_{\mathfrak{p}}^+$ and $v_{\mathfrak{p}}^-$ have opposite positions from the ones one would expect. So we do not have an analogue of Theorem 9.4.1 for C and C^-, unless $v_{\mathfrak{p}}^+ = v_{\mathfrak{p}}^- = 0$. This last thing happens precisely when the pair $(\mathfrak{g}, \mathfrak{k})$ is Hermitian symmetric. This is the reason why we are able to obtain results about $\mathfrak{u}$-cohomology only in the Hermitian case. Maybe this peculiar behavior has something to do with the fact that some of the most concrete results about $\mathfrak{n}$-cohomology, like [E], [Co] or [A], are also obtained in Hermitian situation only.

Let us also point out that although we can write $D(\mathfrak{g}, \mathfrak{k}) = \sum v_i^* \otimes v_i + \sum v_i \otimes v_i^*$, in general the two summands here are not differentials and they are not K-invariant.

In the following we are assuming that the pair $(\mathfrak{g}, \mathfrak{k})$ is Hermitian symmetric. Let V be a unitary $(\mathfrak{g}, K)$-module, and consider the form $\langle\ ,\ \rangle$ on $V \otimes S$ introduced at the beginning of Section 9.3. To apply the results of Section 9.3, we decompose

$$V \otimes S = V \otimes S_{\mathfrak{p}} \otimes S_{\mathfrak{s} \cap \mathfrak{k}} = V \otimes {\textstyle\bigwedge}^{\cdot} \mathfrak{p}^+ \otimes {\textstyle\bigwedge}^{\cdot} \mathfrak{u} \cap \mathfrak{k},$$

and embed $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$ as $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+ \otimes 1$. The form $\langle\ ,\ \rangle$ restricts to the analogous definite form on $V \otimes \bigwedge^{\cdot} \mathfrak{p}^+$.

Denote as before by $D = D(\mathfrak{g}, \mathfrak{l})$ the Dirac operator for the pair $(\mathfrak{g}, \mathfrak{l})$ and by $C = C(\mathfrak{g}, \mathfrak{l})$ and $C^- = C^-(\mathfrak{g}, \mathfrak{l})$ its parts coresponding to $\mathfrak{u}$ and $\bar{\mathfrak{u}}$. By Theorem 9.4.1 and Proposition 9.5.1, $C = C(\mathfrak{g}, \mathfrak{k}) + C_\Delta(\mathfrak{k}, \mathfrak{l})$ and $C^- = C^-(\mathfrak{g}, \mathfrak{k}) + C_\Delta^-(\mathfrak{k}, \mathfrak{l})$. Moreover, by the same arguments that we used in the proof of Theorem 9.4.1 to see that $D(\mathfrak{g}, \mathfrak{r})$ supercommutes with $D_\Delta(\mathfrak{r}, \mathfrak{r}_1)$, one sees that $C^\pm(\mathfrak{g}, \mathfrak{k})$ supercommute with $C_\Delta^\pm(\mathfrak{k}, \mathfrak{l})$.

By Proposition 9.2.2 and Lemma 9.3.1, the adjoints of $C(\mathfrak{g}, \mathfrak{k})$ and $C_\Delta(\mathfrak{k}, \mathfrak{l})$ are respectively $C^-(\mathfrak{g}, \mathfrak{k})$ and $-C_\Delta^-(\mathfrak{k}, \mathfrak{l})$. So the adjoint of C is $C^{\,\mathrm{adj}} = C^-(\mathfrak{g}, \mathfrak{k}) - C_\Delta^-(\mathfrak{k}, \mathfrak{l})$.

The operator D^2 is not as good in the present situation as it was in Section 9.3. Its role is to a large extent taken by the operator $\varDelta = CC^{\,\mathrm{adj}} + C^{\,\mathrm{adj}} C = [C, C^{\,\mathrm{adj}}]$, where we denoted by $[,]$ the supercommutator of the superalgebra $U(\mathfrak{g}) \otimes C(\mathfrak{s})$.

Note first that $\varDelta$ is positive semidefinite. Furthermore, by the above remarks we have

$$\begin{aligned} \varDelta &= [C(\mathfrak{g}, \mathfrak{k}) + C_\Delta(\mathfrak{k}, \mathfrak{l}), C^-(\mathfrak{g}, \mathfrak{k}) - C_\Delta^-(\mathfrak{k}, \mathfrak{l})] \\ &= [C(\mathfrak{g}, \mathfrak{k}), C^-(\mathfrak{g}, \mathfrak{k})] - [C_\Delta(\mathfrak{k}, \mathfrak{l}), C_\Delta^-(\mathfrak{k}, \mathfrak{l})] = D(\mathfrak{g}, \mathfrak{k})^2 - D_\Delta(\mathfrak{k}, \mathfrak{l})^2. \end{aligned}$$

Since we know that $D(\mathfrak{g}, \mathfrak{k})^2$ and $-D_\Delta(\mathfrak{k}, \mathfrak{l})^2$ are both positive semidefinite, it follows that

$$\operatorname{Ker} \varDelta = \operatorname{Ker} D_\Delta(\mathfrak{k}, \mathfrak{l})^2 \cap \operatorname{Ker} D(\mathfrak{g}, \mathfrak{k})^2. \tag{9.8}$$

Moreover, since $D_\Delta(\mathfrak{k}, \mathfrak{l})$ is self-adjoint, $\operatorname{Ker} D_\Delta(\mathfrak{k}, \mathfrak{l})^2 = \operatorname{Ker} D_\Delta(\mathfrak{k}, \mathfrak{l})$. Also, since $D(\mathfrak{g}, \mathfrak{k})$ is anti-self-adjoint, $\operatorname{Ker} D(\mathfrak{g}, \mathfrak{k})^2 = \operatorname{Ker} D(\mathfrak{g}, \mathfrak{k})$. Thus (9.8) implies

$$\operatorname{Ker} \varDelta = \operatorname{Ker} D_\Delta(\mathfrak{k}, \mathfrak{l}) \cap \operatorname{Ker} D(\mathfrak{g}, \mathfrak{k}). \tag{9.9}$$

We know by Lemma 9.3.3 that $\operatorname{Ker} D(\mathfrak{g}, \mathfrak{k}) = \operatorname{Ker} C(\mathfrak{g}, \mathfrak{k}) \cap \operatorname{Ker} C^-(\mathfrak{g}, \mathfrak{k})$. Analogously, by Lemma 9.2.3, $\operatorname{Ker} D_\Delta(\mathfrak{k}, \mathfrak{l}) = \operatorname{Ker} C_\Delta(\mathfrak{k}, \mathfrak{l}) \cap \operatorname{Ker} C_\Delta^-(\mathfrak{k}, \mathfrak{l})$. So (9.9) implies

$$\operatorname{Ker} \varDelta = \operatorname{Ker} C_\Delta(\mathfrak{k}, \mathfrak{l}) \cap \operatorname{Ker} C_\Delta^-(\mathfrak{k}, \mathfrak{l}) \cap \operatorname{Ker} C(\mathfrak{g}, \mathfrak{k}) \cap \operatorname{Ker} C^-(\mathfrak{g}, \mathfrak{k}). \tag{9.10}$$

On the other hand, we have the following analogue of Lemma 9.3.2 for $\varDelta$ in place of D^2.

Lemma 9.5.2. *$V \otimes S$ is a direct sum of eigenspaces for $\varDelta$. In particular, $V \otimes S = \operatorname{Ker} \varDelta \oplus \operatorname{Im} \varDelta$.*

Proof. We know from Lemma 9.3.2 that $V \otimes \bigwedge^\cdot \mathfrak{p}^+$ decomposes into eigenspaces of $D(\mathfrak{g}, \mathfrak{k})^2$ for eigenvalues $\lambda \geq 0$. Each eigenspace is $\tilde{K}$-invariant, and each $\tilde{K}$-isotypic component of $V \otimes \bigwedge^\cdot \mathfrak{p}^+$ is contained in an eigenspace. We assume V is admissible, so the eigenspaces are finite-dimensional.

Passing from $V \otimes \bigwedge^\cdot \mathfrak{p}^+$ to $V \otimes S$ is tensoring with the finite-dimensional $\mathfrak{l}$-module $\bigwedge^\cdot \mathfrak{u} \cap \mathfrak{k}$. On this last space, there is no action of $U(\mathfrak{g})$ or $U(\mathfrak{k}_\Delta)$. So every eigenspace of $D(\mathfrak{g}, \mathfrak{k})^2$ on $V \otimes \bigwedge^\cdot \mathfrak{p}^+$ just gets tensored with $\bigwedge^\cdot \mathfrak{u} \cap \mathfrak{k}$, and this gives the eigenspace of $D(\mathfrak{g}, \mathfrak{k})^2$ on $V \otimes S$ for the same eigenvalue.

Since $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ commutes with $D(\mathfrak{g},\mathfrak{k})^2$, it preserves these eigenspaces. Moreover, the Levi subgroup $L \subset K$ corresponding to $\mathfrak{l}$ is compact. So is then the spin double cover $\tilde{L}$ of L, which acts on $V \otimes S$. Since $\tilde{L}$ commutes with $D(\mathfrak{g},\mathfrak{k})^2$, it also preserves its eigenspaces and hence these eigenspaces decompose into $\tilde{L}$-irreducibles. Since $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ is up to an additive constant equal to the Casimir element of $\mathfrak{l}_\Delta$, it follows that $D_\Delta(\mathfrak{k},\mathfrak{l})^2$ diagonalizes on each eigenspace of $D(\mathfrak{g},\mathfrak{k})^2$.

Now the arguments proving Lemma 9.3.3 and Theorem 9.3.4 work without change, and we obtain

$$\begin{aligned} \operatorname{Ker}\Delta &= \operatorname{Ker} C \cap \operatorname{Ker} C^{\,\mathrm{adj}}\,; \\ V\otimes S &= \operatorname{Ker}\Delta \oplus \operatorname{Im} C \oplus \operatorname{Im} C^{\,\mathrm{adj}}\,; \\ \operatorname{Ker} C &= \operatorname{Ker}\Delta \oplus \operatorname{Im} C; \\ \operatorname{Ker} C^{\,\mathrm{adj}} &= \operatorname{Ker}\Delta \oplus \operatorname{Im} C^{\,\mathrm{adj}}\,. \end{aligned} \tag{9.11}$$

In other words, we have obtained a Hodge decomposition for $\bar{\mathfrak{u}}$-cohomology.

To obtain a Hodge decomposition also for $\mathfrak{u}$-homology, we note that $(C^-)^{\,\mathrm{adj}} = C^-(\mathfrak{g},\mathfrak{k}))^{\,\mathrm{adj}} + (C^-_\Delta(\mathfrak{k},\mathfrak{l}) = C(\mathfrak{g},\mathfrak{k}) - C_\Delta(\mathfrak{k},\mathfrak{l})$, and so

$$\begin{aligned} [C^-,(C^-)^{\,\mathrm{adj}}\,] &= [C^-(\mathfrak{g},\mathfrak{k}) + C^-_\Delta(\mathfrak{k},\mathfrak{l}), C(\mathfrak{g},\mathfrak{k}) - C_\Delta(\mathfrak{k},\mathfrak{l})] \\ &= [C^-(\mathfrak{g},\mathfrak{k}), C(\mathfrak{g},\mathfrak{k})] - [C^-_\Delta(\mathfrak{k},\mathfrak{l}), C_\Delta(\mathfrak{k},\mathfrak{l})] = \Delta. \end{aligned}$$

So the situation for C^- is exactly the same as for C and we conclude

$$\begin{aligned} \operatorname{Ker}\Delta &= \operatorname{Ker} C^- \cap \operatorname{Ker} (C^-)^{\,\mathrm{adj}}\,; \\ V\otimes S &= \operatorname{Ker}\Delta \oplus \operatorname{Im} C^- \oplus \operatorname{Im} (C^-)^{\,\mathrm{adj}}\,; \\ \operatorname{Ker} C^- &= \operatorname{Ker}\Delta \oplus \operatorname{Im} C^-; \\ \operatorname{Ker} (C^-)^{\,\mathrm{adj}} &= \operatorname{Ker}\Delta \oplus \operatorname{Im} (C^-)^{\,\mathrm{adj}}\,. \end{aligned} \tag{9.12}$$

In other words, Hodge decomposition also holds for $\mathfrak{u}$-homology. Moreover, we see that $\bar{\mathfrak{u}}$-cohomology and $\mathfrak{u}$-homology have the same set of harmonic representatives, $\operatorname{Ker}\Delta$. In particular they are isomorphic.

To bring Dirac cohomology into the picture, we first combine (9.9) with Lemma 9.3.3 and Lemma 9.2.3 to obtain

$$\operatorname{Ker}\Delta = \operatorname{Ker} C(\mathfrak{g},\mathfrak{k}) \cap \operatorname{Ker} C^-(\mathfrak{g},\mathfrak{k}) \cap \operatorname{Ker} C_\Delta(\mathfrak{k},\mathfrak{l}) \cap \operatorname{Ker} C^-_\Delta(\mathfrak{k},\mathfrak{l}). \tag{9.13}$$

Since $\operatorname{Ker} C_\Delta(\mathfrak{k},\mathfrak{l}) \cap \operatorname{Ker} C(\mathfrak{g},\mathfrak{k})$ can be thought of as the kernel of $C_\Delta(\mathfrak{k},\mathfrak{l})$ acting on the kernel of $C(\mathfrak{g},\mathfrak{k})$, and similarly for the C^--operators, in view of Theorem 9.3.4 and Theorem 9.2.5 we can reinterprete (9.13) as follows:

Corollary 9.5.3. *To calculate the $\bar{\mathfrak{u}}$-cohomology of V, one can first calculate the $\mathfrak{p}^-$-cohomology of V to obtain a $\tilde{K}$-module, and then calculate the $\bar{\mathfrak{u}}\cap\mathfrak{k}$-cohomology of this module. Analogously, to calculate the $\mathfrak{u}$-homology of V, one can first calculate the $\mathfrak{p}^+$-homology of V, and then the $\mathfrak{u}\cap\mathfrak{k}$-homology of the resulting $\tilde{K}$-module.*

This is in fact the Hochschild-Serre spectral sequence for the ideal $\mathfrak{p}^-$ of $\bar{\mathfrak{u}}$ respectively the ideal $\mathfrak{p}^+$ of $\mathfrak{u}$. What we have obtained is that these Hochschild-Serre spectral sequences are always degenerate for a unitary $(\mathfrak{g}, K)$-module V.

We now turn our attention to the Dirac cohomology of $D = D(\mathfrak{g}, \mathfrak{l})$. In addition to the above considerations, we bring in Theorem 9.4.4, and note that for both $D_\Delta(\mathfrak{k}, \mathfrak{l})$ and $D(\mathfrak{g}, \mathfrak{k})$ the cohomology is the same as the kernel or the kernel of the square. Thus we obtain the main result of [HPR]:

Theorem 9.5.4. *The Dirac cohomology $H_D(\mathfrak{g}, \mathfrak{l}; V)$ of a unitary $(\mathfrak{g}, K)$-module V is isomorphic to the $\bar{\mathfrak{u}}$-cohomology of V and the $\mathfrak{u}$-homology of V up to appropriate modular twists. Moreover, all three cohomologies have the same set of harmonic representatives,* Ker Δ.

9.6 Homological properties of Dirac cohomology

Let us start by showing that although we proved that in some cases Dirac cohomology of a unitary $(\mathfrak{g}, K)$-module with respect to $D(\mathfrak{g}, \mathfrak{l})$ can be identified with $\bar{\mathfrak{u}}$-cohomology or $\mathfrak{u}$-homology, one should by no means expect that these notions agree for general $(\mathfrak{g}, K)$-modules. The reason for this is different behavior with respect to extensions.

9.6.1. Long exact sequences of homology and cohomology. Let

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of $(\mathfrak{g}, K)$-modules. As is well known, there are long exact sequences for $\mathfrak{u}$-homology and $\bar{\mathfrak{u}}$-cohomology attached to this short exact sequence. These long exact sequences are

$$\begin{aligned} \cdots \to H_2(\mathfrak{u}; W) \to H_1(\mathfrak{u}; U) \to H_1(\mathfrak{u}; V) \to H_1(\mathfrak{u}; W) \to \\ \to H_0(\mathfrak{u}; U) \to H_0(\mathfrak{u}; V) \to H_0(\mathfrak{u}; W) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \to H^0(\bar{\mathfrak{u}}; U) \to H^0(\bar{\mathfrak{u}}; V) \to H^0(\bar{\mathfrak{u}}; W) \to \\ \to H^1(\bar{\mathfrak{u}}; U) \to H^1(\bar{\mathfrak{u}}; V) \to H^1(\bar{\mathfrak{u}}; W) \to H^2(\bar{\mathfrak{u}}; U) \to \dots \end{aligned}$$

This suggests that even if the $\bar{\mathfrak{u}}$-cohomology equals $\mathfrak{u}$-homology for U and W, it will not necessarily be so for their extension V. We are going to show that indeed this is what happens even in relatively simple examples.

9.6.2. Six-term exact sequence of Dirac cohomology. Since Dirac cohomology is not $\mathbb{Z}$-graded in a natural way, one can not expect existence of long exact sequences as above for Dirac cohomology. If the spin module S used in the definition of Dirac cohomology is $\mathbb{Z}_2$-graded, there is however a chance that there is a six-term exact sequence, reminiscent of K-theory.

Let $\mathfrak{r}$ be any quadratic subalgebra of $\mathfrak{g}$ such that the orthocomplement $\mathfrak{s}$ of $\mathfrak{r}$ is even dimensional. Then the spin module S for $C(\mathfrak{s})$ is indeed $\mathbb{Z}_2$-graded. In particular, this is true when $\mathfrak{r} = \mathfrak{l}$ is a Levi subalgebra. Let

$$0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$$

be a short exact sequence of $(\mathfrak{g}, K)$-modules. Let us tensor this sequence by S, and denote the arrows still by i and p (they get tensored by the identity on S). Assuming that D^2 is a semisimple operator for each of the three modules, we can construct a six-term exact sequence

$$\begin{array}{ccccc}
H_D^0(U) & \longrightarrow & H_D^0(V) & \longrightarrow & H_D^0(W) \\
\uparrow & & & & \downarrow \\
H_D^1(W) & \longleftarrow & H_D^1(V) & \longleftarrow & H_D^1(U)
\end{array}$$

The horizontal arrows are induced by i and p. The vertical arrows are the connecting homomorphisms, defined as follows. Let $w \in W \otimes S$ represent a Dirac cohomology class, so $Dw = 0$. Choose $v \in V \otimes S$ such that $pv = w$. Since D^2 is semisimple, we can assume $D^2 v = 0$. Since $pDv = Dpv = Dw = 0$, we see that $Dv = iu$ for some $u \in U$. Since $D^2 v = 0$, we see that $Du = 0$, so u defines a cohomology class. This class is by definition the image of the class of v under the connecting homomorphism. Clearly, we changed parity when we applied D, and this defines both vertical arrows at once.

It is easy to see that this map is well defined, and that the resulting six-term sequence is exact. To conclude:

Theorem 9.6.3. *Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be an orthogonal decomposition, with $\mathfrak{r}$ a reductive subalgebra and $\mathfrak{s}$ even-dimensional. Let $0 \to U \to V \to W \to 0$ be a short exact sequence of $(\mathfrak{g}, K)$-modules and assume that the square of the Dirac operator $D(\mathfrak{g}, \mathfrak{r})$ is a semisimple operator for U, V and W. Then there is a six-term exact sequence of Dirac cohomology corresponding to this short exact sequence, as described above.*

9.6.4. Odd-dimensional case. In case $\mathfrak{s}$ is odd dimensional, we can instead of one of the ordinary spin modules S_1, S_2 from 2.2.7 consider the unique irreducible graded module $\tilde{S}$ of $C(\mathfrak{s})$ of 2.2.8. Recall that as a non-graded module, $\tilde{S}$ decomposes as $S_1 \oplus S_2$. If we define Dirac cohomology using $\tilde{S}$ in place of S_1 or S_2, it becomes larger, but we do get a $\mathbb{Z}_2$-grading. Then the above construction works also in the odd case. Thus, this is probably a more natural definition of Dirac cohomology in the odd case.

At present we do not know what to do when D^2 is not a semisimple operator.

9.6.5. Some $\mathfrak{sl}(2)$ examples. To illustrate the above facts, we study some examples of $(\mathfrak{sl}(2, \mathbb{C}), SO(2))$-modules.

Consider the module V which is a nontrivial extension of the discrete series representation W of highest weight $-k-2$ by the finite-dimensional module U of highest weight $k \geq 0$. Thus

$$0 \to U \to V \to W \to 0.$$

(In other words, V is a dual Verma module.)

Let us recall some facts and notation from 1.3.10. The $\mathfrak{k}$-weights of any $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-module are determined by the eigenvalues of the basis element $\left[\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right]$ of $\mathfrak{k}$. These eigenvalues are $k, k-2, k-4, \dots, -k$ for U, $-k-2, -k-4, -k-6, \dots$ for W and the union of these two sets for V. We are considering the case $\mathfrak{l} = \mathfrak{k}$, $\mathfrak{u}$ is spanned by $u = \frac{1}{2}\left[\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right]$ and $\bar{\mathfrak{u}}$ is spanned by $u^* = \frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right]$. Note that u respectively u^* were denoted by X respectively Y in 1.3.10.

In the following we ignore the $\mathfrak{l}$-actions which are of course easy to read off. Thus all the equalities are equalities of vector spaces.

For any $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-module X, we have

$$X \otimes S = X \otimes 1 \quad \oplus \quad X \otimes u,$$

with $d : X \otimes 1 \to X \otimes u$ given by $d(x \otimes 1) = u^* x \otimes u$, $\partial : X \otimes u \to X \otimes 1$ given by $\partial(x \otimes u) = -ux \otimes 1$, and $D = d + 2\partial$.

It is thus clear that the $\bar{\mathfrak{u}}$-cohomology of X, i.e., the cohomology of the differential d, equals

$$\operatorname{Ker} u^* \otimes 1 \quad \oplus \quad \operatorname{Coker} u^* \otimes u,$$

with the first summand being the 0-th $\bar{\mathfrak{u}}$-cohomology of X, and the second summand being the 1-st $\bar{\mathfrak{u}}$-cohomology of X. Let us fix a nonzero weight vector v_i of V for each weight $i = k, k-2, k-4, \dots$. The same symbol v_i will denote the image of v_i in any subquotient of V. We see

$$\begin{gathered} H^0(\bar{\mathfrak{u}}; V) = \mathbb{C}v_{-k} \otimes 1; \quad H^0(\bar{\mathfrak{u}}; U) = \mathbb{C}v_{-k} \otimes 1; \quad H^0(\bar{\mathfrak{u}}; W) = 0; \\ H^1(\bar{\mathfrak{u}}; V) = \mathbb{C}v_k \otimes u \oplus \mathbb{C}v_{-k-2} \otimes u; \quad H^1(\bar{\mathfrak{u}}; U) = \mathbb{C}v_k \otimes u; \\ H^1(\bar{\mathfrak{u}}; W) = \mathbb{C}v_{-k-2} \otimes u. \end{gathered}$$

Similarly, the $\mathfrak{u}$-homology of an arbitrary $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-module X, i.e., the cohomology of the differential ∂, equals

$$\operatorname{Coker} u \otimes 1 \quad \oplus \quad \operatorname{Ker} u \otimes u;$$

again the first summand is the 0-th $\mathfrak{u}$-homology of X, while the second summand is the 1-st $\mathfrak{u}$-homology of X. For our modules U, V and W we get

$$\begin{gathered} H_0(\mathfrak{u}; V) = 0; \quad H_0(\mathfrak{u}; U) = \mathbb{C}v_{-k} \otimes 1; \quad H_0(\mathfrak{u}; W) = 0; \\ H_1(\mathfrak{u}; V) = \mathbb{C}v_k \otimes u; \quad H_1(\mathfrak{u}; U) = \mathbb{C}v_k \otimes u; \quad H_1(\mathfrak{u}; W) = \mathbb{C}v_{-k-2} \otimes u. \end{gathered}$$

Note how the $\mathfrak{u}$-homology and $\bar{\mathfrak{u}}$-cohomology agree for U and W, as predicted by the results of Sections 9.2 and 9.3. They however do not agree for V. The long exact sequence of $\mathfrak{u}$-homology corresponding to $0 \to U \to V \to W \to 0$ is

$$0 \to \mathbb{C}1 \otimes u \to \mathbb{C}v_0 \otimes u \xrightarrow{0} \mathbb{C}v_{-2} \otimes u \to \mathbb{C}1 \otimes 1 \to 0 \to 0 \to 0.$$

The long exact sequence of $\bar{\mathfrak{u}}$-cohomology corresponding to $0 \to U \to V \to W \to 0$ is

$$0 \to \mathbb{C}1 \otimes 1 \to \mathbb{C}v_0 \otimes 1 \to 0 \to \mathbb{C}1 \otimes u \to \mathbb{C}v_{-2} \otimes u \oplus \mathbb{C}v_0 \otimes u \to \mathbb{C}v_{-2} \otimes u \to 0.$$

In both sequences all arrows are the obvious ones except for the one labelled by 0.

Let us now calculate the Dirac cohomology of an $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-module X. Since the Dirac operator D agrees with d on $X \otimes 1$ and with 2∂ on $X \otimes u$, it follows that the Dirac cohomology of X equals

$$\text{Ker}\, u^*/(\,\text{Im}\, u \cap \text{Ker}\, u^*) \quad \oplus \quad \text{Ker}\, u/(\,\text{Im}\, u^* \cap \text{Ker}\, u).$$

As before, the first summand is the 0-th Dirac cohomology of X, while the second summand is the 1-st Dirac cohomology of X. For our modules U, V and W we get

$$\begin{aligned} H_D^0(V) = 0; \quad H_D^0(U) = \mathbb{C}v_{-k} \otimes 1; \quad H_D^0(W) = 0; \\ H_D^1(V) = \mathbb{C}v_k \otimes u; \quad H_D^1(U) = \mathbb{C}v_k \otimes u; \quad H_D^1(W) = \mathbb{C}v_{-k-2} \otimes u. \end{aligned}$$

The six-term exact sequence of Dirac cohomology corresponding to $0 \to U \to V \to W \to 0$ is

$$\begin{tikzcd} \mathbb{C}1 \otimes 1 \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\ \mathbb{C}v_{-2} \otimes u \arrow[u, "\cong"] & \mathbb{C}v_0 \otimes u \arrow[l, "0"'] & \mathbb{C}1 \otimes u \arrow[l, "\cong"'] \end{tikzcd}$$

Here all the maps are the obvious ones, except for the map labelled by 0.

Incidentally, in our example Dirac cohomology coincides with $\mathfrak{u}$-homology, not only for U and W, but also for V. The above six-term sequence thus agrees with the long exact sequence of $\mathfrak{u}$-homology. This is of course related to the presence of zeros. To see that nothing like this should be expected in general, the reader may consider the full principal series module Z, fitting into a short exact sequence

$$0 \to V \to Z \to W' \to 0.$$

Here V is as above, and W' is the lowest weight discrete series representation, with weights $k+2, k+4, k+6, \dots$. Using the above formulas, one can easily see that

$$
\begin{aligned}
H_0(\mathfrak{u}; Z) &= \mathbb{C}v_{k+2} \otimes 1; & H_1(\mathfrak{u}; Z) &= \mathbb{C}v_k \otimes u;\\
H^0(\bar{\mathfrak{u}}; Z) &= \mathbb{C}v_{-k} \otimes 1; & H^1(\bar{\mathfrak{u}}; Z) &= \mathbb{C}v_{-k-2} \otimes u;\\
& & H_D^0(Z) &= H_D^1(Z) = 0.
\end{aligned}
$$

Thus, all three types of cohomology are entirely different for Z.

10

Dirac cohomology for Lie superalgebras

The Dirac operators which we have discussed so far were all associated to nondegenerate symmetric bilinear forms on subspaces of reductive Lie algebras and the Clifford algebras corresponding to these symmetric forms. The Dirac operator to be defined in this chapter is associated to a symplectic form on the odd part of a Lie superalgebra and the corresponding Weyl algebra. In [HP3] we obtain an analog of Vogan's conjecture for this Dirac operator. Our results build upon the results of [Ko6].

In this chapter we discuss the Dirac operator and Dirac cohomology for the Lie superalgebras of Riemannian type and present the above mentioned results. We hope that Dirac cohomology will prove to be a useful tool in representation theory of Lie superalgebras.

Many results in this section referring to a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ are direct analogs of our earlier results for an ordinary Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We will not be recalling these earlier results as we go along, as we feel this would interrupt the reading of the present chapter. Also, we wish to emphasize that for the results in this chapter we do not need to know their analogs in the $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ setting. The interested reader will identify the parallels easily.

10.1 Lie Superalgebras of Riemannian type

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra. In other words, it is an algebra A with a vector space decomposition $A = A_0 \oplus A_1$, such that if $a \in A_\alpha, b \in A_\beta$, $\alpha, \beta \in \mathbb{Z}_2$, then $ab \in A_{\alpha+\beta}$.

A *Lie superalgebra* is a superalgebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with a bracket $[\cdot\,,\cdot]$ satisfying the following axioms:

$$[X,Y] + (-1)^{\deg X \deg Y}[Y,X] = 0;$$
$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{\deg X \deg Y}[Y,[X,Z]]\,.$$

for all (homogoeneous) $X, Y, Z \in \mathfrak{g}$. In other words, the operator ad X sending $Y \in \mathfrak{g}$ to $[X, Y]$ is a superderivation of $\mathfrak{g}$ for any $X \in \mathfrak{g}$. All Lie superalgebras we are going to consider will be over $\mathbb{C}$ and finite-dimensional.

10.1.1. The form B. A Lie superalgebra is said to be of *Riemannian type* if there is a nondegenerate supersymmetric invariant bilinear form B on $\mathfrak{g}$. A bilinear form B on $\mathfrak{g}$ is called supersymmetric if B is symmetric on $\mathfrak{g}_0$ and skew-symmetric on $\mathfrak{g}_1$, and $B(\mathfrak{g}_0, \mathfrak{g}_1) = 0$. In particular, this implies that

$$B(X, Y) = (-1)^{\deg X \deg Y} B(Y, X)$$

for any homogeneous $X, Y \in \mathfrak{g}$. This explains the term supersymmetric.

The form B is invariant if

$$B([X, Y], Z) = B(X, [Y, Z]),$$

for all $X, Y, Z \in \mathfrak{g}$.

In the following we assume that $\mathfrak{g}$ is of Riemannian type, and fix a form B as above. This form should be viewed as a sort of analog of the Killing form in the super setting. In fact, there is also a notion of the Killing form for Lie superalgebras, but it is often degenerate, even though a form B like above may exist. Of course, the same is true for ordinary Lie algebras.

10.1.2. Universal enveloping algebra. Universal enveloping algebras of Lie superalgebras are defined in an analagous manner as for ordinary Lie algebras. Namely, any associative superalgebra A may be viewed as a Lie superalgebra, with the bracket being the supercommutator

$$[a, b] = ab - (-1)^{\deg a \deg b} ba,$$

for homogeneous $a, b \in A$. This defines a forgetful functor from the category of associative superalgebras into the category of Lie superalgebras. This functor has a left adjoint, which attaches to any Lie superalgebra $\mathfrak{g}$ its universal enveloping algebra $U(\mathfrak{g})$. In other words, $U(\mathfrak{g})$ is an associative superalgebra, with a canonical morphism $i : \mathfrak{g} \to U(\mathfrak{g})$ of Lie superalgebras, satisfying the following universal property. For any morphism of Lie superalgebras from $\mathfrak{g}$ into an associative superalgebra A, there is a unique morphism of associative superalgebras $\tilde{\phi} : U(\mathfrak{g}) \to A$ such that $\tilde{\phi} \circ i = \phi$.

To construct $U(\mathfrak{g})$, one makes the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by all elements of the form

$$X \otimes Y - (-1)^{\deg X \deg Y} Y \otimes X - [X, Y]$$

for homogeneous $X, Y \in \mathfrak{g}$. Let us also mention that there is an analog of the Poincaré-Birkhoff-Witt theorem for $U(\mathfrak{g})$. In particular, the morphism $i : \mathfrak{g} \to U(\mathfrak{g})$ is an embedding. Moreover, there is a filtration on $U(\mathfrak{g})$ by degree, such that the associated graded algebra is $S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1)$. For more details, see for example [Sch].

10.1.3. The Casimir element. Having the form B on our Lie superalgebra $\mathfrak{g}$ at hand, we would like to define the associated Casimir of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

For ordinary Lie algebras, the first step in defining the Casimir element would be to take an orthonormal basis of $\mathfrak{g}$ with respect to B. This can not quite be done here, as a symplectic form does not have an orthonormal basis. Namely, skew symmetry of B on $\mathfrak{g}_1$ forces $B(X, X) = 0$ for every $X \in \mathfrak{g}_1$. On the other hand, instead of an orthonormal basis it is possible to choose a basis of $\mathfrak{g}_1$ and consider the dual basis with respect to B. Namely, B is nondegenerate on $\mathfrak{g}_1$. We choose a basis of a special kind: recall that $\mathfrak{g}_1$ has a maximal isotropic subspace, a so called Lagrangean subspace, with dimension equal to half the dimension of $\mathfrak{g}_1$. Moreover, we can choose a pair of complementary Lagrangean subspaces, which are then necessarily nondegenerately paired under B. We choose bases ∂_i, x_i in these Lagrangean subspaces, such that

$$B(\partial_i, x_j) = \frac{1}{2}\delta_{ij}. \tag{10.1}$$

We will see a little bit later why we choose this particular notation for the basis elements, and why we wanted the factor 1/2 in (10.1). Note that if we take

$$\partial_1, \ldots, \partial_n, x_1, \ldots, x_n$$

for a basis of $\mathfrak{g}_1$, then the dual basis (with respect to B) is

$$2x_1, \ldots, 2x_n, -2\partial_1, \ldots, -2\partial_n.$$

A little care is needed when talking about dual bases in the symplectic setting. We say that a basis f_i is dual to a basis e_i if $B(e_i, f_j) = \delta_{ij}$. Note that this does not mean that the basis e_i is dual to the basis f_i; in fact, the basis dual to f_i is $-e_i$. The reason for our choice in this definition is the fact that for super spaces it is the identification $V \otimes V^* = \operatorname{Hom}(V, V)$ that involves no signs (and not $V^* \otimes V = \operatorname{Hom}(V, V)$).

Since B is nondegenerate and symmetric on $\mathfrak{g}_0$, we can choose an orthonormal basis W_k for $\mathfrak{g}_0$ with respect to B. The *Casimir element* of $\mathfrak{g}$ is now defined as

$$\Omega_{\mathfrak{g}} = \sum_k W_k^2 + 2\sum_i (x_i\partial_i - \partial_i x_i). \tag{10.2}$$

It is easy to check that $\Omega_{\mathfrak{g}}$ is an element of the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Using the relation $\partial_i x_i + x_i \partial_i = [\partial_i, x_i]$ in $U(\mathfrak{g})$, one can also write $\Omega_{\mathfrak{g}}$ as

$$\Omega_{\mathfrak{g}} = \sum_k W_k^2 + 4\sum_i x_i\partial_i - 2\sum_i [\partial_i, x_i]. \tag{10.3}$$

(Note that ∂_i and x_i are both odd, so it is their anticommutator in $U(\mathfrak{g})$ that equals their bracket in $\mathfrak{g}$.)

It is also easy to check that $\Omega_{\mathfrak{g}}$ is independent of the choice of basis: if e_j is any basis of $\mathfrak{g}$, with dual basis f_j with respect to B, then $\Omega_{\mathfrak{g}} = \sum f_j e_j$.

10.1.4. The Weyl algebra. One way to define the *Weyl algebra* $W(\mathbb{C}^n)$ of $\mathbb{C}^n$ is as the algebra of differential operators in n variables, with polynomial coefficients. By definition, these diferential operators act on polynomials $\mathbb{C}[x_1, \dots, x_n]$. It is clear that $W(\mathbb{C}^n)$ is generated by the partials $\partial_i = \partial/\partial x_i$ and the coordinate functions x_i, understood as multiplication operators. Moreover, the relations satisfied by these generators are the following commutation relations:

$$x_i x_j - x_j x_i = 0; \qquad \partial_i \partial_j - \partial_j \partial_i = 0; \qquad \partial_i x_j - x_j \partial_i = \delta_{ij}. \tag{10.4}$$

This definition can be made slightly more abstract: let V be a vector space with a symplectic form B. Then the Weyl algebra $W(V)$ of V is the algebra generated by vectors $v \in V$, with relations

$$vw - wv = 2B(v, w), \qquad v, w \in V. \tag{10.5}$$

In other words, $W(V)$ can be constructed as the quotient of the tensor algebra $T(V)$ by the ideal generated by all elements of the form $v \otimes w - w \otimes v - 2B(v, w)$ for $v, w \in V$. Note that this definition is formally very similar to the definition of the Clifford algebra.

To get back to the above more concrete description, one can choose a pair of complementary Lagrangean subspaces of V and bases ∂_i, x_i in them satisfying (10.1). The relations (10.5) then become the relations (10.4). This explains our choice of notation for basis elements in 10.1.3. Namely, the Dirac operator we are going to study in this chapter is an element of $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$, which is the analog of the algebra $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ for an ordinary Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

We will denote the commutators in $W(\mathfrak{g}_1)$ with $[,]_W$ to distinguish them from the (completely different) brackets in $\mathfrak{g}$.

10.1.5. Embedding $\mathfrak{sp}(V)$ into $W(V)$. One can embed the symplectic Lie algebra $\mathfrak{sp}(V)$ into the Weyl algebra $W(V)$ as a Lie subalgebra consisting of quadratic elements.

To construct this embedding, we first note that the symmetrization map $\sigma : S(V) \to W(V)$ is a linear isomorphism. (The map σ is obtained by first sending an element of $S(V)$ into the corresponding symmetric tensor in $T(V)$, and then projecting to $W(V)$.) Next, we can consider the action of $\sigma(S^2(V))$ on $V \subset W(V)$ by commutators in $W(V)$. To describe this action, we first choose a basis $\partial_1, \dots, \partial_n, x_1, \dots, x_n$ of V as before. Then $x_i x_j$ for $i \leq j$, $\partial_i \partial_j$ for $i \leq j$ and $\partial_i x_j$ for all i, j form a basis for $S^2(V)$. Applying σ to this basis we get the basis

$$\begin{aligned} \sigma(x_i x_j) &= x_i x_j, && i \leq j; \\ \sigma(\partial_i \partial_j) &= \partial_i \partial_j, && i \leq j; \\ \sigma(\partial_i x_j) &= \partial_i x_j - \frac{1}{2}\delta_{ij}, && \text{all } i, j \end{aligned} \tag{10.6}$$

of $\sigma(S^2(V))$.

Now a short calculation using the relations (10.4) shows that commuting with $\sigma(x_i x_j)$, $\sigma(\partial_i\partial_j)$ and $\sigma(\partial_i x_j)$ really defines operators on $V \subset W(V)$, and that the corresponding matrices in the basis $\partial_1, \dots, \partial_n, x_1, \dots, x_n$ are

$$\begin{aligned} \sigma(x_i x_j) &\longleftrightarrow -E_{n+i\, j} - E_{n+j\, i}; \\ \sigma(\partial_i\partial_j) &\longleftrightarrow \quad E_{i\, n+j} + E_{j\, n+i}; \\ \sigma(\partial_i x_j) &\longleftrightarrow -E_{ij} + E_{n+j\, n+i}. \end{aligned} \tag{10.7}$$

(Here as usual E_{kl} is the matrix unit, having the kl entry equal to 1 and other entries equal to 0.)

Recall now that the matrices of the operators in $\mathfrak{sp}(V)$ in the basis $\partial_1, \dots, \partial_n, x_1, \dots, x_n$ are block matrices of the form

$$\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$$

where A is an arbitrary $n \times n$ matrix and B and C are symmetric $n \times n$ matrices. Hence the operators on V defined by commuting with $\sigma(x_i x_j)$, $\sigma(\partial_i\partial_j)$ and $\sigma(\partial_i x_j)$ form a basis for $\mathfrak{sp}(V)$. It follows that $\sigma(S^2(V))$ is a Lie subalgebra of $W(V)$ isomorphic to $\mathfrak{sp}(V)$ via the isomorphism described above.

10.1.6. Diagonal embedding of $\mathfrak{g}_0$ into $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$. The action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ via the bracket defines a map

$$\nu : \mathfrak{g}_0 \longrightarrow \mathfrak{sp}(\mathfrak{g}_1).$$

On the other hand, as we saw above, $\mathfrak{sp}(\mathfrak{g}_1)$ embeds into $W(\mathfrak{g}_1)$. Composing this embedding with the map ν, we get a Lie algebra morphism

$$\alpha : \mathfrak{g}_0 \longrightarrow W(\mathfrak{g}_1).$$

Compared to [Ko6], our α is his ν_* followed by the symmetrization map.

We need an explicit formula for $\alpha(X)$, $X \in \mathfrak{g}_0$. To obtain this formula, note first that the matrix coefficients of $\operatorname{ad} X$ in our basis $\partial_1, \dots, \partial_n, x_1, \dots, x_n$ are:

$$\begin{aligned} (\operatorname{ad} X)_{ij} &= 2B(X, [\partial_i, x_j]); & (\operatorname{ad} X)_{i\, n+j} &= 2B(X, [x_i, x_j]) \\ (\operatorname{ad} X)_{n+i\, j} &= -2B(X, [\partial_i, \partial_j]); & (\operatorname{ad} X)_{n+i\, n+j} &= -2B(X, [\partial_i, \partial_j]) \end{aligned}$$

for $i, j = 1, \dots, n$. To see this, first write

$$\operatorname{ad} X(\partial_i) = [X, \partial_i] = \sum \alpha_k \partial_k + \sum \beta_k x_k.$$

Then applying $B(\cdot, x_j)$ to both sides we get $(\operatorname{ad} X)_{ij} = 2B([X, \partial_i], x_j) = 2B(X, [\partial_i, x_j])$, and applying $B(\cdot, \partial_j)$ to both sides we get $(\operatorname{ad} X)_{n+i\, j} = -2B(X, [\partial_i, \partial_j])$. The rest is analogous.

In view of (10.7), we now conclude

$$\begin{aligned}\alpha(X) = \sum_{i<j} 2B(X, [\partial_i, \partial_j])\sigma(x_i x_j) + \sum_i B(X, [\partial_i, \partial_i])\sigma(x_i^2) \\ + \sum_{i<j} 2B(X, [x_i, x_j])\sigma(\partial_i \partial_j) + \sum_i B(X, [x_i, x_i])\sigma(\partial_i^2) \\ - \sum_{i,j} 2B(X, [\partial_i, \partial_j])\sigma(\partial_i x_j).\end{aligned}$$

Since both sums over $i < j$ have summands that are symmetric under exchanging i and j, each of the first two rows of the above formula can be combined into one sum over all i and j. Using (10.6), we finally obtain

$$\begin{aligned}\alpha(X) = \sum_{i,j} (B(X, [\partial_i, \partial_j])x_i x_j + B(X, [x_i, x_j])\partial_i \partial_j - 2B(X, [\partial_i, \partial_j])\partial_i x_j) \\ + \sum_i B(X, [\partial_i, x_i]). \qquad (10.8)\end{aligned}$$

Using the map α, we define a diagonal embedding

$$\mathfrak{g}_0 \to U(\mathfrak{g}) \otimes W(\mathfrak{g}_1),$$

given by

$$X \mapsto X \otimes 1 + 1 \otimes \alpha(X).$$

We denote the image of this map by $\mathfrak{g}_{0\Delta}$; this is a diagonal copy of $\mathfrak{g}_0$. We denote by $U(\mathfrak{g}_{0\Delta})$ and $Z(\mathfrak{g}_{0\Delta})$ the corresponding images of $U(\mathfrak{g}_0)$ respectively its center in $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$. We will be particularly interested in the image of the Casimir element of $\mathfrak{g}_0$, $\Omega_{\mathfrak{g}_0} = \sum_k W_k^2$. This image is equal to

$$\Omega_{\mathfrak{g}_{0\Delta}} = \sum_k (W_k^2 \otimes 1 + 2W_k \otimes \alpha(W_k) + 1 \otimes \alpha(W_k)^2). \qquad (10.9)$$

Kostant [Ko6] has shown that $\alpha(\Omega_{\mathfrak{g}_0}) = \sum_k \alpha(W_k)^2$ is a constant which we denote by C. This constant is equal to $1/8$ of the trace of $\Omega_{\mathfrak{g}_0}$ on $\mathfrak{g}_1$.

Kostant actually proved the following result, analogous to a result about ordinary Lie algebras from [Ko2].

Theorem 10.1.7. *Let $\mathfrak{g}_0$ be a Lie algebra, with a nonsingular invariant symmetric bilinear form B. Let $\mathfrak{g}_1$ be a vector space with nonsingular alternating bilinear form B. Suppose $\mathfrak{g}_1$ is a symplectic representation of $\mathfrak{g}_0$, i.e., there is a Lie algebra map $\mathfrak{g}_0 \to \mathfrak{sp}(\mathfrak{g}_1)$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has a structure of a Lie superalgebra of Riemannian type compatible with the given data if and only if $\alpha(\Omega_{\mathfrak{g}_0})$ is a constant. Here α is the map from $\mathfrak{g}_0$ into $W(\mathfrak{g}_1)$ defined above.*

We can now write out the middle term $2\sum_k W_k \otimes \alpha(W_k)$ in (10.9) using (10.8). Notice that $\sum_k B(W_k, [\partial_i, \partial_j])W_k = [\partial_i, \partial_j]$, and that analogous facts are true for other commutators that appear, since they are all in $\mathfrak{g}_0$. This implies the following lemma, which will enable us to obtain a formula for the square of the Dirac operator.

Lemma 10.1.8. *The diagonal Casimir element from (10.9) can be written as*

$$\Omega_{\mathfrak{g}_{0\Delta}} = \sum_k W_k^2 \otimes 1 + 2\sum_{i,j}\big([\partial_i, \partial_j] \otimes x_i x_j + [x_i, x_j] \otimes \partial_i \partial_j - 2[\partial_i, x_j] \otimes x_i \partial_j\big)$$
$$-2\sum_i [\partial_i, x_i] \otimes 1 + C.$$

Here W_k form an orthonormal basis of $\mathfrak{g}_0$ with respect to B, and ∂_i and x_i form a basis of $\mathfrak{g}_1$, as in 10.1.3. The constant C is the same as above.

10.2 Dirac operator for $(\mathfrak{g}, \mathfrak{g}_0)$

We define the Dirac operator attached to the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ analogous to the Dirac operator attached to the Cartan decomposition of a reductive Lie algebra.

The *Dirac operator* D is an element of $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ given by

$$D = 2\sum_i (\partial_i \otimes x_i - x_i \otimes \partial_i). \tag{10.10}$$

where $\partial_1, \dots, \partial_n, x_1, \dots, x_n$ is a basis of $\mathfrak{g}_1$ introduced in 10.1.3.

As in the previous situations, we have the following lemma.

Lemma 10.2.1. *D is independent of the choice of basis in the following sense. If e_i is any basis for $\mathfrak{g}_1$ with dual basis f_i with respect to B, then $D = \sum e_i \otimes f_i$. Furthermore, D is $\mathfrak{g}_0$-invariant for the $\mathfrak{g}_0$-action on $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ induced by the adjoint action in both factors.*

Proof. The first claim is a routine calculation using the fact that if e_i is a basis for $\mathfrak{g}_1$ with dual basis f_i, then any $X \in \mathfrak{g}_1$ can be written as

$$X = \sum_i B(X, f_i)e_i = -\sum B_i(X, e_i)f_i.$$

(This is then applied for $X = \partial_j$ and $X = x_j$.)

The calculation also uses the following consequence of the above formula: for any $X, Y \in \mathfrak{g}_1$,

$$B(X, Y) = -\sum_i B(X, f_i)B(Y, e_i).$$

The second claim now follows from the first one. Namely, if X is in $\mathfrak{g}_0$, then

$$[X,D] = 2\sum_i ([X,e_i]\otimes f_i + e_i\otimes [X,f_i])$$
$$= 2\sum_{i,j} B([X,e_i],f_j)e_j\otimes f_i - 2\sum_{i,j} e_i\otimes B([X,f_i],e_j)f_j.$$

Exchanging i and j in the second sum, and noting that $B([X,e_i],f_j) = B([X,f_j],e_i)$, we see that the result is 0.

The following is formula for D^2 we have announced.

Proposition 10.2.2. *The Dirac operator $D \in U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ satisfies the equation*

$$D^2 = -\Omega_{\mathfrak{g}}\otimes 1 + \Omega_{\mathfrak{g}_{0\Delta}} - C,$$

where $\Omega_{\mathfrak{g}}$ is the Casimir element of $U(\mathfrak{g})$, $\Omega_{\mathfrak{g}_{0\Delta}}$ is the Casimir element of $U(\mathfrak{g}_{0\Delta})$, and C is the constant described above Theorem 10.1.7.

Proof. We begin by squaring the equation (10.10):

$$D^2 = 4\sum_{i,j}(\partial_i\otimes x_i - x_i\otimes\partial_i)(\partial_j\otimes x_j - x_j\otimes\partial_j)$$
$$= 4\sum_{i,j}(\partial_i\partial_j\otimes x_ix_j - \partial_ix_j\otimes x_i\partial_j - x_i\partial_j\otimes\partial_ix_j + x_ix_j\otimes\partial_i\partial_j). \quad (10.11)$$

Let us examine each of the four terms in (10.11) separately. Using $x_ix_j = x_jx_i$ in $W(\mathfrak{g}_1)$ and $\partial_i\partial_j + \partial_j\partial_i = [\partial_i,\partial_j]$ in $U(\mathfrak{g})$, we can write

$$4\sum_{i,j}\partial_i\partial_j\otimes x_ix_j = 2\sum_{i,j}\partial_i\partial_j\otimes x_ix_j + 2\sum_{i,j}\partial_j\partial_i\otimes x_jx_i = 2\sum_{i,j}[\partial_i,\partial_j]\otimes x_ix_j.$$

In the same way we see that the fourth term in (10.11) equals

$$4\sum_{i,j}x_ix_j\otimes\partial_i\partial_j = 2\sum_{i,j}[x_i,x_j]\otimes\partial_i\partial_j.$$

We rewrite the third term in (10.11) using $\partial_ix_j = x_j\partial_i$ in $W(\mathfrak{g}_1)$ for $i\neq j$, while $\partial_ix_i = x_i\partial_i + 1$. Thus the third term is

$$-4\sum_{i,j}x_i\partial_j\otimes\partial_ix_j = -4\sum_{i,j}x_i\partial_j\otimes x_j\partial_i - 4\sum_i x_i\partial_i\otimes 1.$$

Upon exchanging i and j, the first of these two sums combines with the second term of (10.11) to produce

$$-4\sum_{i,j}[\partial_i,x_j]\otimes x_i\partial_j.$$

Thus we have obtained

$$D^2 = 2\sum_{i,j}([\partial_i,\partial_j]\otimes x_i x_j + [x_i,x_j]\otimes \partial_i\partial_j - 2[\partial_i,x_j]\otimes x_i\partial_j) - 4\sum_i x_i\partial_i\otimes 1.$$

Comparing with the expression (10.3) for $\Omega_{\mathfrak{g}}$ and the expression for $\Omega_{\mathfrak{g}_{0\Delta}}$ from Lemma 10.1.8, we get the statement of the proposition.

10.3 Analog of Vogan's conjecture

In this section, we assume that the Lie algebra $\mathfrak{g}_0$ acts semisimply on the algebra $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ via the adjoint action in both factors. Since $U(\mathfrak{g})\cong S(\mathfrak{g}_0)\otimes\bigwedge(\mathfrak{g}_1)$ and $W(\mathfrak{g}_1)\cong S(\mathfrak{g}_1)$ as $\mathfrak{g}_0$-modules for the adjoint action, $\mathfrak{g}_0$ also acts semisimply on the graded versions of $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ with respect to filtrations by degree in each of the factors.

Our assumption is obviously satisfied if $\mathfrak{g}_0$ is semisimple. It is also satisfied if $\mathfrak{g}_0$ is reductive. In particular, $\mathfrak{g}$ can be any one of the basic classical Lie superalgebras described at the beginning of Section 10.4, so the assumption is not restrictive for what we are about to do.

10.3.1. A differential on $(U(\mathfrak{g})\otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$. We define a $\mathbb{Z}_2$-grading on $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ by using the $\mathbb{Z}_2$-grading of the superalgebra $U(\mathfrak{g})$. (So $W(\mathfrak{g}_1)$ is considered all to be in degree 0.). In this way $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ becomes a superalgebra. On this superalgebra we consider the operator of supercommuting with the Dirac operator D:

$$d(a) = [D,a] = Da - \epsilon_a aD,$$

where ϵ_a is 1 if a is even and -1 if a is odd. Since D is odd, $\epsilon_{Da}=\epsilon_{aD}=-\epsilon_a$ for any homogeneous $a\in U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$. So we see that

$$\begin{aligned} d^2(a) = d(Da-\epsilon_a aD) &= D^2a - \epsilon_a DaD - \epsilon_{Da} DaD + \epsilon_a\epsilon_{aD} aD^2 \\ &= D^2 a - aD^2 \end{aligned}$$

for any homogoenous a. It follows that d induces a differential on the centralizer of D^2 in $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$. Since $D^2 = -\Omega_{\mathfrak{g}}\otimes 1 + \Omega_{\mathfrak{g}_{0\Delta}} - C$ by Proposition 10.2.2, and since $-\Omega_{\mathfrak{g}}\otimes 1 - C$ is in the center of $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$, we see that $a\in U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ commutes with D^2 if and only if a commutes with $\Omega_{\mathfrak{g}_{0\Delta}}$. In particular, this is true if a commutes with all of $U(\mathfrak{g}_{0\Delta})$, i.e., if a is a $\mathfrak{g}_0$-invariant element of $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ with respect to the adjoint action. So we see that d induces a differential on the algebra $(U(\mathfrak{g})\otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$ of $\mathfrak{g}_0$-invariants in $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$. We denote this differential again by d.

As D is $\mathfrak{g}_0$-invariant and odd, the operator d on $U(\mathfrak{g})\otimes W(\mathfrak{g}_1)$ is $\mathfrak{g}_0$-equivariant and odd. To summarize, we have the following lemma.

Lemma 10.3.2. *The operator d on $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ is $\mathfrak{g}_0$-equivariant and odd. It defines a differential on the algebra $(U(\mathfrak{g}) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$ of $\mathfrak{g}_0$-invariants in $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$.*

The fact that D is $\mathfrak{g}_0$-invariant also implies that it commutes with $U(\mathfrak{g}_{0\Delta})$. In particular, it follows that $Z(\mathfrak{g}_{0\Delta}) = U(\mathfrak{g}_{0\Delta}) \cap (U(\mathfrak{g}) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$ is contained in the kernel of the differential d on $(U(\mathfrak{g}) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$. Furthermore, we have the following result.

Theorem 10.3.3. *Let d be the differential on $(U(\mathfrak{g}) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$ introduced above. Then*

$$\operatorname{Ker} d = Z(\mathfrak{g}_{0\Delta}) \oplus \operatorname{Im} d.$$

In particular, the cohomology of d is isomorphic to $Z(\mathfrak{g}_{0\Delta})$.

Proof. The proof is very similar to the proof of Theorem 3.3.2. We first introduce a filtration on $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$, using the filtration by degree in the first factor mentioned at the end of 10.1.2. The associated graded superalgebra is then

$$\operatorname{Gr} U(\mathfrak{g}) \otimes W(\mathfrak{g}_1) = S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1).$$

This filtration is clearly compatible with the $\mathbb{Z}_2$-gradation and the action of $\mathfrak{g}_0$. In particular, it induces a filtration of $A = (U(\mathfrak{g}) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$.

Since d raises the filtration degree by 1, it induces an operator

$$\bar{d} : \operatorname{Gr}^{\,i} U(\mathfrak{g}) \otimes W(\mathfrak{g}_1) \to \operatorname{Gr}^{\,i+1} U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$$

for any i. Let us calculate the action of $\bar{d}$ on a monomial of the form $P \otimes \lambda \otimes x^I \partial^J$, where $P \otimes \lambda \in S^l(\mathfrak{g}_0) \otimes \bigwedge^k(\mathfrak{g}_1)$, and $x^I \partial^J$ is the usual multiindex notation for $x_1^{i_1} \dots x_n^{i_n} \partial_1^{j_1} \dots \partial_n^{j_n}$. Here $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ is the basis of $\mathfrak{g}_1$ from 10.1.3, used in the definition of the Dirac operator. We denote the image of the Dirac operator D in $\operatorname{Gr}^{\,i} U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ again by D. It is again given by the same expression (10.10).

$$\begin{aligned}
\bar{d}(P \otimes \lambda \otimes x^I \partial^J) &= D(P \otimes \lambda \otimes x^I \partial^J) - (-1)^k (P \otimes \lambda \otimes x^I \partial^J) D \\
&= 2 \sum_{r=1}^{n} \big(\partial_r (P \otimes \lambda) \otimes x_r x^I \partial^J - x_r (P \otimes \lambda) \otimes \partial_r x^I \partial^J \\
&\quad - (-1)^k (P \otimes \lambda) \partial_r \otimes x^I \partial^J x_r + (-1)^k (P \otimes \lambda) x_r \otimes x^I \partial^J \partial_r \big) \\
&= 2(-1)^k \sum_r \big(P \otimes \lambda \partial_r \otimes [x_r, x^I \partial^J]_W - P \otimes \lambda x_r \otimes [\partial_r, x^I \partial^J]_W \big).
\end{aligned}$$

Here we used $\partial_r (P \otimes \lambda) = (-1)^k (P \otimes \lambda) \partial_r$ and $x_r (P \otimes \lambda) = (-1)^k (P \otimes \lambda) x_r$ in $S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1)$.

To calculate the commutators in $W(\mathfrak{g}_1)$ appearing in the above formula, let us denote $\hat{I}_r = (i_1, \dots, i_r - 1, \dots, i_n)$ (if $i_r > 0$), and analogously $\hat{J}_r = (j_1, \dots, j_r - 1, \dots, j_n)$. Then using (10.4) we get

$$[x_r, x^I \partial_J]_W = -j_r x^I \partial^{\hat{J}_r}; \quad [\partial_r, x^I \partial^J]_W = i_r x^{\hat{I}_r} \partial^J$$

in $W(\mathfrak{g}_1)$. (Note that it does not matter that $\hat{I}_r$ is defined only for $i_r > 0$, as the corresponding term is 0 in any case.) So we see that $\bar{d} = -2 \,\mathrm{id} \otimes d_{\mathfrak{g}_1}$, where

$$d_{\mathfrak{g}_1} : \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1) \to \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1)$$

is given by

$$d_{\mathfrak{g}_1}(\lambda \otimes x^I \partial^J) = \epsilon_\lambda \sum_r (j_r \lambda \partial_r \otimes x^I \partial^{\hat{J}_r} + i_r \lambda x_r \otimes x^{\hat{I}_r} \partial^J).$$

(As before, ϵ_λ denotes the parity of λ.)

From this expression we see that if we compose $d_{\mathfrak{g}_1}$ with $\mathrm{id} \otimes \sigma$, where σ is the symmetrization map from $S(\mathfrak{g}_1)$ to $W(\mathfrak{g}_1)$, we are getting exactly the polynomial de Rham differential for $\mathfrak{g}_1^*$. This differential appeared as the homotopy h in the proof of Proposition 3.3.5. As we remarked at the end of proof of Proposition 3.3.5, that same proof proves the polynomial Poincaré lemma, i.e.,

$$\mathrm{Ker}\, d_{\mathfrak{g}_1} = \mathbb{C}\, 1 \otimes 1 \oplus \mathrm{Im}\, d_{\mathfrak{g}_1}.$$

Knowing this immediately gives an analogous statement about the operator $\bar{d}$ on $S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1)$:

$$\mathrm{Ker}\, \bar{d} = S(\mathfrak{g}_0) \otimes 1 \otimes 1 \oplus \mathrm{Im}\, \bar{d}. \tag{10.12}$$

(Note that since $\bar{d} = -2 \,\mathrm{id} \otimes d_{\mathfrak{g}_1}$, we see that $\bar{d}$ is actually a differential on the whole algebra $S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1)$, it is not necessary to pass to $\mathfrak{g}_0$-invariants to obtain a differential.)

Since $\bar{d}$ and the decomposition (10.12) are $\mathfrak{g}_0$-equivariant, and since we assumed $\mathfrak{g}_0$-action on $S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1)$ is semisimple, we can pass to $\mathfrak{g}_0$-invariants and conclude that for $\bar{d}$ on $(S(\mathfrak{g}_0) \otimes \bigwedge(\mathfrak{g}_1) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$ we have

$$\mathrm{Ker}\, \bar{d} = S(\mathfrak{g}_0)^{\mathfrak{g}_0} \otimes 1 \otimes 1 \oplus \mathrm{Im}\, \bar{d}.$$

The rest of the proof consists of going back to the filtered setting by induction on degree. This part is identical to the corresponding part of the proof of Theorem 3.3.2.

Since any $z \in Z(\mathfrak{g})$ is clearly in the kernel of the differential d on $(U(\mathfrak{g}) \otimes W(\mathfrak{g}_1))^{\mathfrak{g}_0}$, we get the following result.

Corollary 10.3.4. *For any $z \in Z(\mathfrak{g})$, there is a unique $\zeta(z) \in Z(\mathfrak{g}_{0\Delta})$ and a $\mathfrak{g}_0$-invariant odd $a \in U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$, such that*

$$z \otimes 1 = \zeta(z) + Da + aD.$$

10.4 Dirac cohomology for Lie superalgebras

10.4.1. Basic classical Lie superalgebras. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called *classical* if $\mathfrak{g}_0$ is reductive. In that case one can show that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is completely reducible.

If a classical Lie superalgebra $\mathfrak{g}$ is also of Riemannian type, i.e., it admits a nondegenerate supersymmetric invariant bilinear form, then $\mathfrak{g}$ is called a *basic classical Lie superalgebra* . Kac classified all simple Lie superalgebras in [Ka1]. Besides the ordinary simple Lie algebras, the list of simple basic classical Lie superalgebras in [Ka1] includes $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1,\alpha)$, $F(4)$ and $G(3)$.

In order to apply our results about the Dirac operator to representation theory of Lie superalgebras, we first recall some fundamental results of Kac [Ka2] about structure and representations of basic classical Lie superalgebras.

10.4.2. Cartan subalgebras and roots. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical Lie superalgebra. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. Then $\mathfrak{h}_0$ is automatically a Cartan subalgebra of $\mathfrak{g}$. In other words, the supersymmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is always of equal rank; an analogous statement is not true for ordinary symmetric pairs.

As usual, roots of $\mathfrak{g}$ with respect to $\mathfrak{h}_0$ are defined as functionals on $\mathfrak{h}_0$ describing the nonzero eigenvalues of the operators $\operatorname{ad} X$, $X \in \mathfrak{h}_0$, on $\mathfrak{g}$. The set of all roots Δ decomposes as $\Delta_0 \cup \Delta_1$, where Δ_0 and Δ_1 denote the sets of all even roots respectively odd roots. Here of course a root $\alpha \in \Delta$ is even (respectively odd) if the corresponding root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_0$ (respectively in $\mathfrak{g}_1$). Clearly, Δ_0 is the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$.

We will fix a system of positive roots $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$. The corresponding Borel subalgebra of $\mathfrak{g}$ will be denoted by $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$, and its nilradical by $\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{n}_1^+$. The opposite Borel subalgebra and its nilradical will be denoted by $\mathfrak{b}^-$ respectively $\mathfrak{n}^-$. Clearly, both $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are invariant under the adjoint action of $\mathfrak{h}_0$. Moreover,

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^-, \qquad \text{and} \qquad \mathfrak{b} = \mathfrak{h}_0 \oplus \mathfrak{n}^+.$$

We will also use the notation

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha, \qquad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha, \qquad \text{and} \qquad \rho = \rho_0 - \rho_1.$$

10.4.3. Representations of Lie superalgebras. Let $V = V_0 \oplus V_1$ be a superspace, i.e., a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. Then the space $\operatorname{End} V$ of all linear endomorphisms of V is an associative superalgebra in a natural way. Namely, we define a $\mathbb{Z}_2$-grading

$$\operatorname{End}(V) = \operatorname{End}_0(V) \oplus \operatorname{End}_1(V)$$

by letting $\mathrm{End}_0(V)$ consist of all linear endomorphisms which preserve V_0 and V_1, and letting $\mathrm{End}_1(V)$ consist of all linear endomorphisms which exchange V_0 and V_1. We can then also view $\mathrm{End}(V)$ as a Lie superalgebra as in 10.1.2.

A *representation* of a Lie superalgebra $\mathfrak{g}$ is a superspace V together with a homomorphism

$$\pi\colon \mathfrak{g} \to \mathrm{End}(V)$$

of Lie superalgebras. Such V will also be called a $\mathfrak{g}$-module. By 10.1.2, V is then also a $U(\mathfrak{g})$-module.

If $\mathfrak{g}$ is a basic classical Lie superalgebra, then there is a good theory of highest weight $\mathfrak{g}$-modules. The definitions and basic properties are parallel to the highest weight theory over an ordinary Lie algebra. We will denote by $V(\Lambda)$ the unique irreducible $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}_0^*$.

10.4.4. Infinitesimal characters. Any element z of the center $Z(\mathfrak{g})$ of the enveloping superalgebra $U(\mathfrak{g})$ can be written in the form

$$z = u_z + \sum_i u_i^- u_i^0 u_i^+,$$

for some uniquely determined $u_z, u_i^0 \in U(\mathfrak{h}_0)$ and $u_i^\pm \in \mathfrak{n}^\pm U(\mathfrak{n}^\pm)$. The map $z \mapsto u_z$ gives a monomorphism

$$\beta\colon Z(\mathfrak{g}) \to U(\mathfrak{h}_0) = S(\mathfrak{h}_0).$$

Recall the ρ-shift automorphism s_ρ of $S(\mathfrak{h}_0)$ from 1.4.8, given on the generators $X \in \mathfrak{h}_0$ by $s_\rho(X) = X - \rho(X) \cdot 1$. Then, like in the ordinary case, $s_\rho(\beta(Z(\mathfrak{g})))$ is contained in the algebra $S(\mathfrak{h}_0)^W$ of W-invariants in $S(\mathfrak{h}_0)$. Here W is the (ordinary) Weyl group of the pair $(\mathfrak{g}_0, \mathfrak{h}_0)$.

The Harish-Chandra monomorphism is the composition

$$\gamma = s_\rho \circ \beta\colon Z(\mathfrak{g}) \to S(\mathfrak{h}_0)^W.$$

In contrast with the ordinary case, γ is typically not an isomorphism. In fact, the algebra $Z(\mathfrak{g})$ is far more complicated than in the ordinary case. This is a source of complications in the representation theory of $\mathfrak{g}$.

On the other hand, the subalgebra $\gamma(Z(\mathfrak{g}))$ of $S(\mathfrak{h}_0)^W$ is not too small: the fields of fractions of $\gamma(Z(\mathfrak{g}))$ and $S(\mathfrak{h}_0)^W$ coincide.

Any $\lambda \in \mathfrak{h}_0^*$ defines a character

$$\chi_\lambda\colon Z(\mathfrak{g}) \to \mathbb{C};$$

for $z \in Z(\mathfrak{g})$, $\chi_\lambda(z)$ is equal to the evaluation of $\gamma(z)$ at λ. We say that a $\mathfrak{g}$-module V has infinitesimal character λ if $Z(\mathfrak{g})$ acts on V via the character χ_λ.

Clearly, $\chi_\lambda = \chi_{w\lambda}$ for any $w \in W$. Moreover, like for ordinary Lie algebras, if V is an irreducible highest weight $\mathfrak{g}$-module with highest weight Λ, then the infinitesimal character of V is $\Lambda + \rho$ (or any element in $W \cdot (\Lambda + \rho)$).

10.4.5. The Weil representation. Since we want our Dirac operator $D \in U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$ to act, we need to tensor the $\mathfrak{g}$-module V with a module for the Weyl algebra $W(\mathfrak{g}_1)$. This module should be an analog of the spin representation of the Clifford algebra.

Unlike the Clifford algebra, the Weyl algebra $W(\mathfrak{g}_1)$ has a vast collection of irreducible modules; they are the subject of the theory of D-modules on $\mathbb{C}^n$. We could in principal choose any one of these modules. The analog of the spin module, with a similar ("dual") definition is however the *Weil (or metaplectic, or oscillator) module* $M(\mathfrak{g}_1)$.

To describe the module $M(\mathfrak{g}_1)$, recall first from 10.1.4 that $W(\mathfrak{g}_1)$ can be identified with the algebra of differential operators in the variables $x_1, \dots, x_n$ with polynomial coefficients. The module $M(\mathfrak{g}_1)$ is then the space of all complex polynomials in $x_1, \dots, x_n$, with the natural action of differential operators.

It is clear now that $V \otimes M(\mathfrak{g}_1)$ is a module over the algebra $U(\mathfrak{g}) \otimes W(\mathfrak{g}_1)$; in particular, D acts on it. Moreover, if we write $M^+(\mathfrak{g}_1)$ and $M^-(\mathfrak{g}_1)$ for the submodules of $M(\mathfrak{g}_1)$ spanned by homogeneous polynomials of even and odd degrees respectively, we see that

$$D \colon V \otimes M^{\pm}(\mathfrak{g}_1) \to V \otimes M^{\mp}(\mathfrak{g}_1).$$

Definition 10.4.6. Let V be a $\mathfrak{g}$-module. The *Dirac cohomology* $H_D(V)$ of V is defined to be the $\mathfrak{g}_0$-module

$$\operatorname{Ker} D / \operatorname{Ker} D \cap \operatorname{Im} D.$$

We can now formulate and prove the following analog of the last part of Vogan's conjecture.

Theorem 10.4.7. *The map $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{g}_{0\Delta})$ given by Corollary 10.3.4 is a homomorphism of algebras, which fits into the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{g}_0) \\ {\scriptstyle H.C.\ hom}\downarrow & & \downarrow{\scriptstyle H.C.\ isom} \\ S(\mathfrak{h}_0)^W & \xrightarrow{\mathrm{id}} & S(\mathfrak{h}_0)^W \end{array}$$

Here the bottom horizontal map is the identity map while the two vertical maps are the Harish-Chandra monomorphism and isomorphism respectively.

Proof. It is easy to see that the map ζ is a homomorphism; this is analogous to Lemma 3.4.1.

It remains to show that ζ is determined by the above commuting diagram. It suffices to test this for all irreducible highest weight $\mathfrak{g}$-modules. Let $V(\Lambda)$ be an irreducible highest weight $\mathfrak{g}$-module with highest weight Λ and a highest weight vector v_Λ.

Then $v_\Lambda \otimes 1$ is killed by D. Namely, recall that $D = 2\sum_i(\partial_i \otimes x_i - x_i \otimes \partial_i)$. Assuming we chose the positive root system so that ∂_i are the positive odd root vectors, $\partial_i \in U(\mathfrak{g})$ kills the highest weight vector v_Λ. On the other hand, $\partial_i \in W(\mathfrak{g}_1)$ kills the constant polynomial 1.

We claim that $v_\Lambda \otimes 1$ generates a highest weight $\mathfrak{g}_{0\Delta}$-module with highest weight $\Lambda - \rho_1$. To prove this claim we need to understand the image of $\mathfrak{h}_0$ under the map α of 10.1.6.

Note first that for any $h \in \mathfrak{h}_0$, the matrix of the operator ad (h) restricted to $\mathfrak{g}_1$ is the diagonal matrix

$$\sum_{i=1}^{n} \alpha_i(h)(E_{i\,i} - E_{i+n\;i+n}),$$

where $\alpha_1, \cdots, \alpha_n \in \Delta_1^+$ are all odd positive roots. In view of (10.7) and (10.6), it follows that

$$\alpha(h) = \sum_{i=1}^{n} \alpha_i(h)(-\partial_i x_i + \frac{1}{2}) = \sum_{i=1}^{n} \alpha_i(h)(-x_i\partial_i - \frac{1}{2}).$$

Hence $\alpha(h)$ acts on the constant polynomial 1 by $-\frac{1}{2}\sum_{i=1}^{n}\alpha_i(h) = -\rho_1(h)$.

Moreover, note that the operators $x_i\partial_i$ act by nonnegative constants on all monomials in $M(\mathfrak{g}_1)$. It follows that the action of $\mathfrak{g}_{0\Delta}$ on $v_\Lambda \otimes 1$ generates a highest weight $\mathfrak{g}_0$-module of highest weight $\Lambda - \rho_1$. It is clear that $v_\Lambda \otimes 1$ cannot be in the image of D on $V(\Lambda) \otimes M^-(\mathfrak{g}_1)$. Since D is $\mathfrak{g}_{0\Delta}$-invariant, $v_\Lambda \otimes 1$ generates a $\mathfrak{g}_0$-module in $H_D(V)$ with infinitesimal character

$$\lambda = (\Lambda - \rho_1) + \rho_0 = \Lambda + (\rho_0 - \rho_1) = \Lambda + \rho,$$

which coincides with the infinitesimal character of the highest weight $\mathfrak{g}$-module $V(\Lambda)$.

Finally, we obtain the following corollary, analogous to Theorem 3.2.5. The proof is the same as the proof of Theorem 3.2.5.

Corollary 10.4.8. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra and let V be a $\mathfrak{g}$-module with infinitesimal character χ. If the Dirac cohomology $H_D(V)$ contains a nonzero $\mathfrak{g}_0$-module with infinitesimal character $\lambda \in \mathfrak{h}_0^*$, then χ is determined by the W-orbit of λ. More precisely, for any $z \in Z(\mathfrak{g})$, $\chi(z) = \chi_\lambda(\zeta(z))$.*

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