MATH 1090-2: PRACTICE FINAL[1]

December, 2007

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1. True or False:

(a) Every sequence of real numbers is either arithmetic or geometric.
(b) If $a_1, a_2, \ldots$ is a geometric sequence, then $a_1/a_2 = a_3/a_4$.
(c) One dollar invested for one year at a 6% annual interest compounded continuously will be worth more than one dollar invested at a 6% annual rate compounded quarterly.
(d) The sequence $1, 2, 4, 8, \ldots$ is arithmetic.
(e) If A and B are square matrices then their product AB exists.
(f) The graph of a quadratic equation always crosses the x-axis in two points.
(g) $\ln(a+b) = \ln(a) + \ln(b)$.
(h) $(x+1)^2 = x^2 + 1$.
(i) The domain of $f(x) = \sqrt{x}$ consists of all real numbers.
(j) Taking 1090 was a life-enhancing experience.

Solution.

(a) False.
(b) True.
(c) True.
(d) False.
(e) False. (If A and B have different sizes, AB doesn't make sense.)
(f) False.
(g) False. (The correct identity is $\ln(ab) = \ln(a) + \ln(b)$).
(h) False.
(i) False. (It consists only of positive real numbers.)
(j) True (obviously).

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2. Suppose you have your heart set on a \$15,000 car. If the interest rate on the loan is 12% (annually) compounded monthly, and you want to make monthly payments of \$400 for three years, how much must you have for a down payment?

Solution. Use the formula from §6.5 to find the value of the loan,

$$A = R \cdot \left[\frac{1 - (1+i)^{-n}}{i} \right]$$

[1] There are more problems on this practice final than on the actual final exam. To get a feel for the length of the actual exam, you should choose one problem numbered 1, one problem numbered 2, and so on.

with $i = 0.12/12$, $n = 3 \cdot 12 = 36$, and $R = 400$. Thus the down payment must be

$$15000 - 400 \cdot \left[\frac{1-(1.01)^{-36}}{0.01}\right],$$

or $2,957.

2*. Your parents are renovating their kitchen and they consult you for some mathematical advice. They plan on purchasing $20,000 worth of material for the renovation by making a $3,000 down payment and amortizing the rest with quarterly payments over the next five years. The 12% annual interest rate on the loan is compounded quarterly. Find

(a) the size of their quarterly payments;
(b) the total amount paid over the life of the loan; and
(c) the total interest paid over the life of the loan.

Solution.

(a) Use the formula from Section §6.5

$$R = A \cdot \left[\frac{i}{1-(1+i)^{-n}}\right]$$

with $A = 20000 - 3000$, $i = 0.12/4$, $n = 5 \cdot 4 = 20$ to find

$$R = 17000 \cdot \left[\frac{0.12/4}{1-(1+0.12/4)^{-20}}\right]$$

or about $1, 1142.67.

(b) There are 20 payments in five years, so the total amount paid over the life of the loan is

$$20 \cdot \$1142.67 = \$22,853.34.$$

(c) The interest paid is the total payments minus the principal:

$$\$22,853.34 - \$17000 = \$5,853.34.$$

2**. You wish to retire at age 65 and draw $4,000 at the end of each month for the net 25 years. If your money is invested in an account earning 9% annually which is compounded monthly, how much must you have in the bank at age 65?

Solution. Use the formula from Section §6.3:

$$A = R \cdot \left[\frac{1-(1+i)^{-n}}{i}\right]$$

with $R = 4000$, $i = 0.09/12$, $n = 25 \cdot 12 = 300$. So

$$A = 4000 \left[\frac{1-(1+0.09/12)^{-300}}{0.09/12}\right] = \$476,646,49.$$

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3. Find the sum of the first thirty terms of the geometric sequence

$$1, 0.99, (0.99)^2, (0.99)^3, \ldots.$$

Solution. Use the formula from §6.2

$$s_n = \frac{a_1(1-r^n)}{1-r} = \frac{1 \cdot (1-(0.99)^{30})}{1-0.99} \approx 26.03.$$

3*. Find the sum of the first 100 terms of the arithmetic sequence

$$2, 5, 8, 11, 14, \ldots.$$

Solution. Use the formula from §6.2:

$$s_n = \frac{n(a_1 + a_{100})}{2} = \frac{100(2+299)}{2} = 15050.$$

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4. How much is one dollar worth one year after investing it in an account earning 10% annually compounded monthly?

Solution. Use the formula from §6.2

$$S = P(1+i)^n = (1 + .1/12)^{12} = \$1.1047.$$

4*. How much is one dollar worth one year after investing it in an account earning 12% annually compounded continuously?

Solution. Use the formula from §6.2:

$$S = Pe^{rt} = 1e^{0.12 \cdot 1} = \$1.1275.$$

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5. Compute the interest for:

(a) an initial investment of \$1000 earning a simple annual interest rate of 12% for 10 years;
(b) an initial investment of \$1000 earning an annual interest rate of 8% compounded quarterly for 10 years.

Solution.

(a) Use the formula from §6.1:

$$I = Prt = 1000(0.12)(10) = 1200.$$

(b) Use the formula from §6.2:

$$I = S - P = P(1+i)^n - P = 1000(1+0.08/4)^{10\cdot 4} - 1000 = 1208.04.$$

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6. Determine if the following sequences are arithmetic, geometric, or neither:

(a) $1, 2, 3, 4, 5, \ldots$.
(b) $10, 2, \frac{2}{5}, \frac{2}{25}, \ldots$.
(c) $1, -1, 1, -1, \ldots$.
(d) $0, 2, 4, 8, 16, 32, \ldots$.

Solution.

(a) arithmetic.
(b) geometric.
(c) geometric.
(d) neither.

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7. What size of payments must be put into an account at the end of each quarter to establish an ordinary annuity that in 14 years will have a value of \$50,000 if the investment pays 12% compounded quarterly.

Solution. Use the §6.3 formula

$$S = R \cdot \left[\frac{(1+i)^n - 1}{i}\right]$$

with $S = 50000$, $i = 0.12/4 = 0.03$, $n = 14 \cdot 4 = 56$, and solve for R:

$$50000 = R \cdot \left[\frac{(1.03)^{56} - 1}{0.03}\right].$$

So

$$R = 50000 \cdot \left[\frac{0.03}{(1.03)^{56} - 1}\right],$$

or about \$354.

7*. You start investing \$125 at the end of each month into an ordinary annuity earning 7% annually compounded monthly. How many years will it take for the annuity to be worth

(a) \$100,000?
(b) \$200,000?

Solution.

(a) We use the §6.3 formula

$$A = R \cdot \left[\frac{(1+i)^n - 1}{i}\right]$$

with $R = 125$, $i = 0.07/12$, $A = 100000$, and solve for n:

$$100000 = 125 \cdot \left[\frac{(1 + 0.07/17)^n - 1}{0.07/12}\right]$$

and find

$$n = \frac{\ln\left(\frac{100000(0.07/12)}{125} + 1\right)}{\ln(1 + (0.07/12))} = 298.24.$$

So the number of years is $298.24/12 \approx 24.8$.

(b) The same reasoning gives

$$n = \frac{\ln\left(\frac{200000(0.07/12)}{125} + 1\right)}{\ln(1 + (0.07/12))} = 401.53,$$

about 33 and a half years.

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8. (a) Given that $\log_a(x) = 1.1$ and $\log_a(y) = -2$, find

$$\log_a\left(\frac{\sqrt{x^3}}{y^{2.4}}\right).$$

(b) If $f(x) = 4^x$, find $f(\log_2(5))$.

Solution. (a) Use the properties of logs to get $(3/2)(1.1) - (2.4)(-2) = 6.45$

(b)

$$f(\log_2(5)) = 4^{log_2(5)} = (2^2)^{\log_2(5)} = 2^{2\log_2(5)} = 2^{\log_2(5^2)} = 5^2 = 25.$$

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9. Because of a new advertising campaign, a company predicts that its sales will increase so that the yearly sales will be given by

$$N = 10000(0.3)^{(0.5^t)},$$

where t represents the number of years after the start of the campaign.

(a) What are the sales when the campaign begins?
(b) What are the maximum predicted sales?
(c) After how many years will sales reach 6000?

Solution.

(a) $10,000(0.3) = 3,000$.
(b) 10,000.
(c) Solve for t in terms of N to get

$$t = \frac{\ln\left(\frac{\ln(N/10000)}{\ln(0.3)}\right)}{\ln(0.5)},$$

and plug in $N = 6000$ to get $t = 1.237$ years.

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10. (a) Find the equation of the line that passes through the points $(0, 3)$ and $(3, 0)$. Write your answer in slope-intercept form.

(b) Find the equation of the line perpendicular to the line in (a) which passes through the origin. Write you answer in slope-intercept form

Solution. (a) $y = 3 - x$.

(b) $y = x$

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11. Solve the following equations:

(a) $(x-2)^2 - 5(x-2) - 24 = 0$.
(b) $5x^2 = 2x + 6$.

Solution.

(a) $x = 10$ or -1.
(b) $x = \frac{2 \pm \sqrt{124}}{10}$.

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12. Graph the function $y = 6 + x - x^2$. Be sure to label the coordinates of the vertex and the x- and y-intercepts (if there are any).

Solution. This is a parabola which opens downward, which crosses the x-axis at -2 and 3, and which has a vertex at $(1/2, 25/4)$. You can draw this.

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13. Solve the following system of equations by any method you wish:

$$2x - y + -z = 4$$
$$x + y + z = 8.$$
$$-x + 2y - z = -14.$$

Solution. $x = 4$; $y = -2$; $z = 6$.

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14. Maximize $f = 2x + 4y$ subject to the constraints

$$2x + 2y \geq 8$$
$$2x + y \leq 8$$
$$y \leq 4.$$

Solution. Maximum of 10 is attained at $(2,4)$. (Other corners are $(0,4)$ and $(4,0)$.)

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15. Suppsoe supply and demand functions are given by

$$(S) \quad p = 30q + 60$$
$$(D) \quad p = 240 - 6q.$$

Find the market equilibrium point.

Solution. $p = 210$; $q = 5$.

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16. Compute A^2 if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 3 & -2 & 1 \end{pmatrix}.$$

Solution.

$$A^2 = \begin{pmatrix} 14 & -4 & 10 \\ 8 & 0 & 8 \\ 2 & 4 & 6 \end{pmatrix}.$$

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17. Suppose

$$f(x) = \sqrt{x^3 + 1}$$
$$g(x) = x^2 + 5.$$

(a) Compute $g \circ f(x)$.
(b) Compute $g \circ g(x)$.
(c) What is the domain of $f(x)$?
(d) What is the domain of $g(x)$?
(e) What is the range of $f(x)$?
(f) What is the range of $g(x)$?

Solution.
(a) $x^3 + 6$.
(b) $x^4 + 10x^2 + 30$.
(c) All real numbers such that $x \geq -1$.
(d) All real numbers.
(e) All positive real numbers.
(f) All real numbers greater than or equal to 5.