Math 1010 — Fall 2010 — Departmental Final Exam Answers

-1- Simplify:

$$\frac{\frac{3}{5}-\frac{2}{3}}{\frac{1}{3}+\frac{3}{5}} = \frac{\frac{9-10}{15}}{\frac{5+9}{15}} = \frac{-1}{15} \div \frac{14}{15} = \frac{-1}{15} \times \frac{15}{14} = -\frac{1}{14}. \qquad (1)$$

-2- Solve the equation

$$4x - 3 = 10 - 2(x-1) \qquad (2)$$

Proceeding as usual we obtain

$$\begin{array}{rcl|l} 4x-3 &=& 10-2(x-1) & \text{distribute} \\ 4x-3 &=& 10-2x+2 & \text{collect like terms} \\ 4x-3 &=& 12-2x & +2x+3 \\ 6x &=& 15 & \div 6 \\ x &=& \frac{15}{6} & \text{the solution} \\ &=& \frac{5}{2} & \text{simplified} \end{array} \qquad (3)$$

Of course we check by substituting $x = \frac{5}{2}$ in the original equation that we have the right answer!

-3- Find all solutions of the equation

$$x^2 - x - 20 = 0 \qquad (4)$$

This equation can be solved factoring, by the quadratic formula, or by completing the square. Factoring gives

$$x^2 - x - 20 = (x-5)(x+4) \qquad (5)$$

which implies that the solution is

$$x = -4 \quad \text{or} \quad x = 5 \qquad (6)$$

Completing the square gives:

$$\begin{array}{rcl|l} x^2-x-20 &=& 0 & +20+\frac{1}{4} \\ x^2-x+\frac{1}{4} &=& \frac{81}{4} & \text{perfect square} \\ \left(x-\frac{1}{2}\right)^2 &=& \frac{81}{4} & \sqrt{} \\ x-\frac{1}{2} &=& \pm\frac{9}{2} & +\frac{1}{2} \\ x &=& \frac{1}{2}\pm\frac{9}{2} & \text{the solution} \end{array} \qquad (7)$$

which is the same result.
As for all quadratic equations, you can also apply the quadratic formula if you can remember it reliably. The solution of the quadratic equation

$$ax^2 + bx = c = 0 \qquad (8)$$

is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this case, $a = 1$, $b = -1$, and $x - 20$. We obtain

$$x = \frac{1 \pm \sqrt{1 - 4 \times (-20)}}{2} = \frac{1 \pm \sqrt{81}}{2} = \frac{1 \pm 9}{2} \qquad (9)$$

which of course also is the same answer.

-4- Find all solutions of the equation

$$x^2 - 2x - 5 = 0 \tag{10}$$

Completing the square as in the preceding problem we obtain:

$$\begin{array}{rcl|l} x^2 - 2x - 5 &=& 0 & +6 \\ x^2 - 2x + 1 &=& 6 & \text{perfect square} \\ (x-1)^2 &=& 6 & \sqrt{\ } \\ x - 1 &=& \pm\sqrt{6} & +1 \\ x &=& 1 \pm \sqrt{6} & \text{the solution} \end{array} \tag{11}$$

-5- Find all solutions of

$$\frac{8}{x-2} - \frac{5}{x-3} + 1 = 0 \tag{12}$$

This equation can be converted into a quadratic equation by multiplying on both sides of the equation by the common denominator $(x-2)(x-3)$. We obtain:

$$8(x-3) - 5(x-2) + (x-2)(x-3) = 3x - 14 + x^2 - 5x + 6 = x^2 - 2x - 8 = 0. \tag{13}$$

The last expression can be factored:

$$x^2 - 2x - 8 = (x+2)(x-4) \tag{14}$$

Thus

$$x = -2 \qquad \text{or} \quad x = 4. \tag{15}$$

Substituting in the original equation shows that this is indeed the correct solution.

-6- Write the following polynomial expression in standard form. What is its degree and its leading coefficient?

$$(x^2 - 1)(x + 3) + 2x + 4 \tag{16}$$

We obtain

$$(x^2 - 1)(x + 3) + 2x + 4 = x^3 + 3x^2 - x - 3 + 2x + 4 = x^3 + 3x^2 + x + 1. \tag{17}$$

This is a polynomial expression of degree 3, with leading coefficient 1.

-7- Precisely describe the (natural) domain of the function f defined by $f(x) = \ln(2x - 1)$.

The domain of the natural logarithm is the set of all positive real numbers. Therefore we must have $2x-1 > 0$ which is equivalent to

$$x > \frac{1}{2}. \tag{18}$$

In interval notation the domain is $\left(\frac{1}{2}, \infty\right)$.

-8- For $f(x) = \ln(2x - 1)$ find $f\left(\frac{1}{2}(e^x + 1)\right)$ and express it as simply as possible.

We get

$$f\left(\frac{1}{2}(e^x + 1)\right) = \ln\left(2 \times \frac{1}{2}(e^x + 1) - 1\right) = \ln e^x = x. \tag{19}$$

-9- Evaluate

$$\log_3 54 - \log_3 2 = \log_3 \frac{54}{2} = \log_3 27 = 3. \tag{20}$$

-10- Solve the equation

$$\sqrt{x+4} + \sqrt{x+11} = 7. \tag{21}$$

We isolate the square roots and square one at a time:

$$\begin{array}{rcll} \sqrt{x+4}+\sqrt{x+11} & = & 7 & \mid \quad -\sqrt{x+11} \\ \sqrt{x+4} & = & 7-\sqrt{x+11} & \mid \quad ()^2 \\ x+4 & = & 49\ -14\sqrt{x+11}+x+11 & \mid \quad +14\sqrt{x+11}-x-4 \\ 14\sqrt{x+11} & = & 56 & \mid \quad \div 14 \\ \sqrt{x+11} & = & 4 & \mid \quad ()^2 \\ x+11 & = & 16 & \mid \quad -11 \\ x & = & 5 & \mid \quad \text{the answer} \end{array} \tag{22}$$

-11- Simplify (i.e., write with only positive exponents, such that x and y occur only once) the expression

$$\frac{(x^2y^{-3})^2}{(x^{-1}y)^{-3}} \tag{23}$$

We apply the Rules

$$a^m a^n = a^{m+n}, \qquad (a^m)^n = a^{mn}, \qquad (ab)^n = a^n b^n, \quad \text{and} \quad a^{-n} = \frac{1}{a^n}. \tag{24}$$

This gives

$$\frac{\left(x^2y^{-3}\right)^2}{\left(x^{-1}y\right)^{-3}} = \frac{x^4y^{-6}}{x^3y^{-3}} = x^{4-3}y^{-6+3} = xy^{-3} = \frac{x}{y^3}. \tag{25}$$

-12- Simplify

$$\frac{\sqrt[3]{16x^4y^6z^7}}{\sqrt[3]{2x^7z}} = \left(8x^{-3}y^6z^6\right)^{\frac{1}{3}} = 2x^{-1}y^2z^2 = \frac{2y^2z^2}{x}. \tag{26}$$

Either of the last two expressions would be a correct answer. The important part is that in the simplified version each variable occurs just once.

-13- Solve the system

$$\begin{array}{rrcl} 4x & -y & = & 1 \\ 2x & +y & = & 0 \end{array} \tag{27}$$

Show all your work, don't just give the answer.

Adding the two equations gives $6x = 1$ which implies

$$x = \frac{1}{6}. \tag{28}$$

Substituting in the first equation gives $\frac{4}{6} - y = 1$ which implies

$$y = \frac{4}{6} - 1 = -\frac{1}{3}. \tag{29}$$

To summarize, we get

$$x = \frac{1}{6} \quad \text{and} \quad y = -\frac{1}{3}, \qquad \text{or} \qquad (x, y) = \left(\frac{1}{6}, -\frac{1}{3}\right).$$

Either of these notations would be acceptable as an answer.
Substituting the values of x and y in the original equations shows that we have the right solution.

-14- Solve the linear system

$$\begin{array}{rrrcl} x & +y & +z & = & 5 \\ x & +2y & -z & = & 3 \\ 2x & +3y & -z & = & 5 \end{array} \tag{30}$$

Again, state how you solve it, don't just give the answer.

This is a little more elaborate. We write equation numbers in square brackets and obtain:

$$\begin{array}{rrrcr|l} x & +y & +z & = & 5 & [1] \\ x & +2y & -z & = & 3 & [2] \\ 2x & +3y & -z & = & 5 & [3] \\ & y & -2z & = & -2 & [4] = [2] - [1] \\ & y & -3z & = & -5 & [5] = [3] - 2 \times [1] \\ & & z & = & 3 & [6] = [4] - [5] \end{array} \tag{31}$$

Thus $z = 3$, and we obtain from equation [4] that $y = 2z - 2 = 4$. Finally, we obtain from equation [1] that $x = 5 - y - z = -2$. Thus

$$x = -2, \qquad y = 4, \qquad z = 3. \tag{32}$$

-15- Find the slope-intercept form of the equation of the line that passes through (2,1) and has slope -1/2.

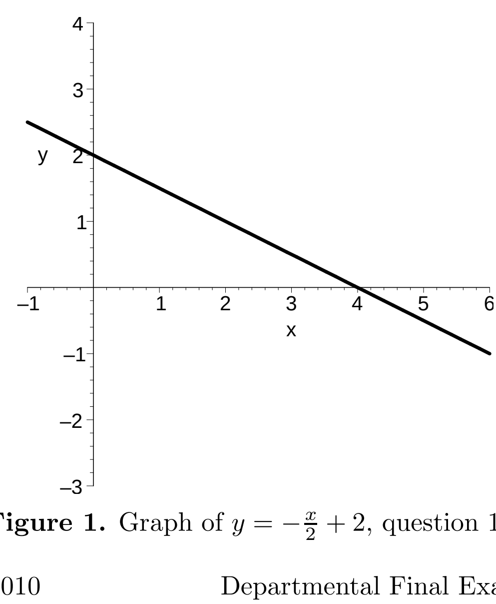

Figure 1. Graph of $y = -\frac{x}{2} + 2$, question 15.

The point slope form of the equation of a straight line gives

$$\frac{y-1}{x-2} = -\frac{1}{2}. \tag{33}$$

The slope-intercept form of that equation is

$$y = 1 - \frac{1}{2}(x-2) = -\frac{1}{2}x + 2. \tag{34}$$

The graph is shown in Figure 1.

-16- Find the distance between the points $(-1, 3)$ and $(2, 4)$.

We apply the distance formula, giving:

$$d = \sqrt{(-1-2)^2 + (3-4)^2} = \sqrt{10}. \tag{35}$$

-17- Express $\frac{2}{x+1} + \frac{1}{x-2} + \frac{2}{x-3}$ as a single rational expression. Write the numerator of your answer as a polynomial in standard form. You may leave the denominator factored into linear factors.

We get

$$\begin{aligned} \frac{2}{x+1} + \frac{1}{x-2} + \frac{2}{x-3} &= \frac{2(x-2)(x-3) + (x+1)(x-3) + 2(x+1)(x-2)}{(x+1)(x-2)(x-3)} \\ &= \frac{2(x^2-5x+6) + x^2 - 2x - 3 + 2(x^2 - x - 2)}{(x+1)(x-2)(x-3)} \\ &= \frac{2x^2 - 10x + 12 + x^2 - 2x - 3 + 2x^2 - 2x - 4}{(x+1)(x-2)(x-3)} \\ &= \frac{5x^2 - 14x + 5}{(x+1)(x-2)(x-3)} \end{aligned} \tag{36}$$

-18- Macy's is having their annual white sale where all sheets are discounted 20%. A set of sheets is on sale for \$48. What was the original price of the sheets?

If x is the original price of the set of sheets then \$48 is 80% of x, i.e., $48 = 0.8x$. Dividing on both sides by 0.8 gives the original price:

$$x = \frac{48}{0.8} = \$60. \tag{37}$$

-19- You and your brother working together take 3 hours to dig a hole. By yourself it would take you five hours. How long would it take your brother to dig the hole by himself?

Suppose it takes your brother x hours to dig the hole by himself. In one hour he can dig $1/x$ of the hole, the two of you together can dig $1/3$ of the hole, and by yourself you can dig $1/5$ of the hole. Thus

$$\frac{1}{x} + \frac{1}{5} = \frac{1}{3}. \tag{38}$$

Multiplying with $15x$ on both sides gives the linear equation

$$15 + 3x = 5x \tag{39}$$

which has the solution

$$x = \frac{15}{2} \text{ hours.} \tag{40}$$

(It's OK to say that the solution is 7.5 hours, or seven and a half hours.)

-20- Carbon 14 (^{14}C) has a half life of 5730 years. If you start with 16 milligrams of Carbon 14, how much is left after $4 \times 5730 = 22920$ years?

The original amount of Carbon 14 gets divided by 2 every 5730 years. During four such periods of 5730 years each we divide by $2^4 = 16$. Thus after 22920 years we are left with 1 mg of Carbon 14.