Review problems for chapters 13 and 14:

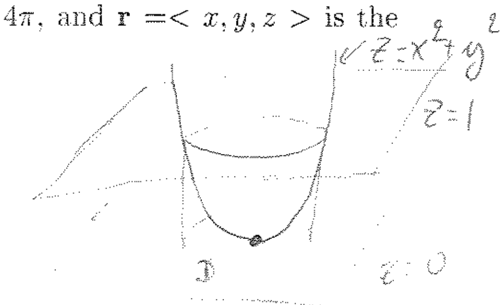
The 3 problems are independent.

(1) Consider the solid region R in 3-space above the surface S with equation $z = x^2 + y^2$ and below the plane P with equation $z = 1$. Evaluate the volume of R, by (a) using a double integral, and (b) using a triple integral. (c) Find the area of the portion of the surface S below P.

(2) Evaluate the following integral:

$$\int_0^{\pi/2} \int_0^{\sqrt{z}} \int_0^{2yz} \sin\left(\frac{x}{y}\right) dxdydz.$$

(3) Consider the vector field $\mathbf{F}(x, y, z) = < \cos x + 2yz, \sin y + 2xz, z + 2xy >$. (a) Is $\mathbf{F}$ conservative? (b) If yes, find a potential function f for $\mathbf{F}$ (recall that this means that $\nabla f = \mathbf{F}$). (c) Evaluate the integral $\int_C \mathbf{F}(\mathbf{r})d\mathbf{r}$, where C is the path parametrized by $\mathbf{r}(t) = < R\cos t, R\sin t, t >$ (for a fixed $R > 0$) as t varies from 0 to 4π, and $\mathbf{r} =< x, y, z >$ is the position vector in 3-space.



(1)(a) Note that R is above S; the volume of R can be obtained by integrating $f(x,y) = 1 - x^2 - y^2$ over the disk D centered at O with radius 1 (think of rotating the surface down by a half-turn). Equivalently, the volume of R can be obtained by subtracting $\iint_D (x^2+y^2)\,dxdy$ from the volume of the cylinder of height 1 above D.

In any case: $Vol(R) = \iint_D (1-x^2-y^2)\,dxdy$ is best computed in polar coordinates:

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \boxed{\frac{\pi}{2}}$$

(b) One could also do the following triple integration, again in cylindrical coordinates

$$Vol(R) = \iiint_R r\,dr\,d\theta\,dz = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dz.\,dr\,d\theta = 2\pi\int_0^1 r(1-r^2)\,dr = \boxed{\frac{\pi}{2}} \text{ again.}$$

(c) The area A of the portion of S (which is the graph of $g(x,y) = x^2+y^2$) above the disk D is given by: $A = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}.\,dxdy$ where $\frac{\partial g}{\partial x} = 2x$ and $\frac{\partial g}{\partial y} = 2y$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}.\,r\,dr\,d\theta \quad \text{(again, in polar coordinates)}$$

u-substitution
$u = 4r^2+1$
$du = 8r.dr$

$$= 2\pi\frac{1}{8}\int_1^5 \sqrt{u}\,du = \frac{\pi}{4}.\left[\frac{2}{3}.u^{3/2}\right]_1^5 = \frac{\pi}{6}(5^{3/2}-1) = \boxed{\frac{\pi}{6}(5.\sqrt{5}-1)}.$$

(2) This is a straightforward iterated integral:

$$\int_0^{\pi/2}\int_0^{\sqrt{z}}\int_0^{2yz} \sin\left(\frac{x}{y}\right)dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\sqrt{z}}\underbrace{\left[-y\cdot\cos\left(\frac{x}{y}\right)\right]_{x=0}^{x=2yz}}\,dy\,dz$$

$$= \int_0^{\pi/2}\int_0^{\sqrt{z}} y(1-\cos 2z)\,dy\,dz = \int_0^{\pi/2}\left[\frac{y^2}{2}(1-\cos 2z)\right]_{y=0}^{y=\sqrt{z}} dz$$

$$= \int_0^{\pi/2}\frac{z}{2}\,dz - \underbrace{\int_0^{\pi/2}\frac{z}{2}\cos 2z\,dz}$$

by parts: $\begin{cases} u=\frac{z}{2} \\ v'=\cos 2z\end{cases}$ $\begin{cases} u'=\frac{1}{2} \\ v=\frac{1}{2}\sin 2z\end{cases}$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)^2 - \cancel{\frac{1}{4}\left[z\sin 2z\right]_0^{\pi/2}} + \frac{1}{4}\int_0^{\pi/2}\sin 2z\,dz$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)^2 + \frac{1}{4}\left[-\frac{1}{2}\cos 2z\right]_0^{\pi/2} = \boxed{\frac{\pi^2}{16}+\frac{1}{4}}$$

(3) (a) Write $\vec{F}(x,y,z)$ as $\langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$. Then:

$\vec{F}$ is conservative $\iff \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, and $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

Here: $2z \quad 2z \quad 2y \quad 2y \quad 2x \quad 2x$

so $\boxed{\vec{F} \text{ is indeed conservative}}$

b) We want to find a function f such that:

$$\begin{cases} \frac{\partial f}{\partial x} = \cos x + 2yz & (1) \\ \frac{\partial f}{\partial y} = \sin y + 2xz & (2) \\ \frac{\partial f}{\partial z} = z + 2xy & (3)\end{cases}$$

Integrating (1) with respect to x tells us that:

$f(x,y,z) = \sin x + 2xyz + C_1(y,z)$ (4)

$\hookrightarrow$ "constant" of integration

Now take partial derivatives of (4) w.r.t. y and z and compare with (2) and (3):

$$\underbrace{\frac{\partial f}{\partial y} = 2xz + \frac{\partial C_1}{\partial y}}, \quad \frac{\partial f}{\partial z} = 2xy + \frac{\partial C_1}{\partial z}$$

$\frac{\partial C_1}{\partial y} = \sin y$, so $C_1(y,z) = -\cos y + C_2(z)$, then $\frac{\partial C_1}{\partial z} = \frac{dC_2}{dz} = z$ so $C_2(z) = \frac{z^2}{2}$ +

Putting these pieces together gives: $\boxed{f(x,y,z) = 2xyz + \sin x - \cos y + \frac{z^2}{2}}$ (+ constant).

c) Now that we know that $\vec{F} = \vec{\nabla} f$, the "Fundamental Theorem for Line Integrals" tells us that:

$$\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C \vec{\nabla} f(\vec{r})\cdot d\vec{r} = f(\vec{b}) - f(\vec{a}) \text{ where } \begin{cases}\vec{a} = (R,0,0) \\ \vec{b} = (R,0,4\pi)\end{cases}$$ are the endpoints of the path C (2 periods of a helix)

$$= \sin R - 1 + \frac{(4\pi)^2}{2} - (\sin R - 1) = \boxed{8\pi^2}.$$

As observed in class, it would be much longer and more difficult to directly evaluate

$$\int_0^{4\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

where $\vec{F}(\vec{r}(t)) = \langle \cos(R\cos t) + 2R\sin t.\,t,\ \sin(R\sin t) + 2R\cos t.\,t,\ t + 2R^2\cos t\sin t\rangle \ldots$

$\vec{r}(t) = \langle R\cos t, R\sin t, t\rangle$, $\vec{r}\,'(t) = \langle -R\sin t, R\cos t, 1\rangle$