Review problem for Midterm # 2

Consider the function $f(x,y) = x.e^{-(x^2+y^2)}$ and its graph S. (1) Find the first partial derivatives and gradient of f. (2) (a) Find an equation for the tangent plane to S at the point $(1,1,e^{-2})$. (b) What is the steepest slope of a line in this plane? (c) Find a vector tangent to the level curve of f through $(1,1)$. (3) Find the critical points of f, then find the local maxima and minima of f. Bonus: are these also global maxima/minima? (4) If $x = r\cos\theta$ and $y = r\sin\theta$, find the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

(1) $\dfrac{\partial f}{\partial x} = e^{-(x^2+y^2)} - 2x.x.e^{-(x^2+y^2)} = (1-2x^2)e^{-(x^2+y^2)}$;

$\dfrac{\partial f}{\partial y} = -2xye^{-(x^2+y^2)}$; $\vec{\nabla} f(x,y) = \langle (1-2x^2)e^{-(x^2+y^2)}, -2xye^{-(x^2+y^2)} \rangle$

(2)(a) $\vec{\nabla} f(1,1) = \langle -e^{-2}, -2e^{-2} \rangle$ so an equation for the tagent plane is:

$$z - e^{-2} = -e^{-2}(x-1) - 2e^{-2}(y-1)$$

(b) We know that the directional derivative of f at a point (x,y) in the direction of a <u>unit</u> vector $\vec{u}$ is:

$$D_{\vec{u}} f(x,y) = \vec{u} \cdot \vec{\nabla} f(x,y)$$

This is maximal when $\vec{u}$ has the same direction as $\vec{\nabla} f(x,y)$; in that direction the slope of <u>the line tagent to the graph of</u> f is $\|\vec{\nabla} f(x,y)\|$.

Here: $\|\vec{\nabla} f(1,1)\| = \sqrt{(-e^{-2})^2 + (-2e^{-2})^2} = \sqrt{5e^{-4}} = \sqrt{5}.e^{-2}$

(c) This also tells us that level curves of f are everywhere perpendicular to the gradient of f. Any vector $\vec{v} = \langle v_1, v_2 \rangle$ perpendicular to $\vec{\nabla} f(1,1) = \langle -e^{-2}, -2e^{-2} \rangle$ will do, such as $\langle 2, -1 \rangle$.

$$(\langle 2,-1 \rangle \cdot \langle -e^{-2}, -2e^{-2} \rangle = 0).$$

(3) (x,y) is a critical point of $f \iff \vec{\nabla} f(x,y) = \vec{0} = \langle 0,0 \rangle$

(i.e. $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial y}(x,y) = 0$). Here the only critical points are $\left(\frac{1}{\sqrt{2}}, 0\right)$ and $\left(-\frac{1}{\sqrt{2}}, 0\right)$. $\rightarrow$

We use the second partial derivatives test to decide if these points are local max./min. for f:

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2} = -4xe^{-(x^2+y^2)} - 2x(1-2x^2)e^{-(x^2+y^2)} = (4x^3-6x)e^{-(x^2+y^2)} \\ \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = -2y(1-2x^2)e^{-(x^2+y^2)} \\ \dfrac{\partial^2 f}{\partial y^2} = -2xe^{-(x^2+y^2)} + 4xy^2 e^{-(x^2+y^2)} = -2x(1-2y^2)e^{-(x^2+y^2)} \end{cases}$$

Evaluating these 3 functions at $\left(\frac{1}{\sqrt{2}}, 0\right)$ then $\left(-\frac{1}{\sqrt{2}}, 0\right)$ gives:

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}\left(\frac{1}{\sqrt{2}}, 0\right) = -2\sqrt{2}\, e^{-1/2} \\ \dfrac{\partial^2 f}{\partial y \partial x}\left(\frac{1}{\sqrt{2}}, 0\right) = 0 \\ \dfrac{\partial^2 f}{\partial y^2}\left(\frac{1}{\sqrt{2}}, 0\right) = -\sqrt{2}\, e^{-1/2} \end{cases} \qquad \begin{cases} \dfrac{\partial^2 f}{\partial x^2}\left(-\frac{1}{\sqrt{2}}, 0\right) = 2\sqrt{2}\, e^{-1/2} \\ \dfrac{\partial^2 f}{\partial y \partial x}\left(-\frac{1}{\sqrt{2}}, 0\right) = 0 \\ \dfrac{\partial^2 f}{\partial y^2}\left(-\frac{1}{\sqrt{2}}, 0\right) = \sqrt{2}\, e^{-1/2} \end{cases}$$

"D" (ok) or "Δ" (class) $= \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 4e^{-1} > 0$ and $\dfrac{\partial^2 f}{\partial x^2} < 0$.

Therefore f has a local maximum at $\left(\frac{1}{\sqrt{2}}, 0\right)$ (value : $f\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{e^{-1/2}}{\sqrt{2}}$)

D or $\Delta = 4e^{-1} > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$

Therefore f has a local minimum at $\left(-\frac{1}{\sqrt{2}}, 0\right)$ (value $f\left(-\frac{1}{\sqrt{2}}, 0\right) = -\frac{e^{-1/2}}{\sqrt{2}}$)

__Bonus:__ These local min/max values are also global min/max values, because $f(x,y) \longrightarrow 0$ as $x^2+y^2 \longrightarrow \infty$ (meaning that $(x,y) \to \infty$ in any direction).

(4) Using the Chain Rule:

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial r} \\ \dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial \theta} \end{cases}$$

so:

$$\begin{cases} \dfrac{\partial f}{\partial r} = (1-2x^2)e^{-(x^2+y^2)} \cdot \cos\theta - 2xye^{-(x^2+y^2)} \cdot \sin\theta \\ \dfrac{\partial f}{\partial \theta} = -(1-2x^2)e^{-(x^2+y^2)} \cdot r\sin\theta - 2xye^{-(x^2+y^2)}\, r\cos\theta. \end{cases}$$