University of Utah
Math 1220, Fall 2007

Solutions for the power series problems

Problem 1: (done in class) Find a power series expansion for $\frac{1}{1+x^2}$. What is its radius of convergence? Does it converge at the endpoints? From this, find a power series expansion for $\tan^{-1} x$. What is its radius of convergence? Does it converge at the endpoints?

We know that: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\dots$, and this converges $\Leftrightarrow |x|<1$ (so: radius of convergence $=1$)

Therefore: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1-x^2+x^4-x^6+x^8-\dots$

and this series converges $\Leftrightarrow |-x^2|<1 \Leftrightarrow |x|<1$ (so: radius of convergence 1 and it diverges at both endpoints).

Now we know that $\frac{1}{1+x^2}$ is the derivative of $\tan^{-1}x$, so we can integrate this power series to get a power series for $\tan^{-1}x$:

$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{2n+1}$ (integrate term by term).

We know that the radius of convergence is the same (1), but we need to check the endpoints:
- at $x=1$: the series is $\sum (-1)^n \cdot \frac{1}{2n+1}$: converges by Alt. Series Test (because $\frac{1}{2n+1}$ decreases to 0)
- at $x=-1$: Same series, converges.

Problem 2: Consider the power series $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}$.

1. For which values of x does it converge? (find the radius of convergence and determine what happens at the endpoints of the interval of convergence).

- Use Absolute Ratio Test to find radius of convergence: fix x and let $b_n = \frac{x^n}{n(n-1)}$

Then: $\left|\frac{b_{n+1}}{b_n}\right| = \frac{|x|^{n+1}}{(n+1)n} \times \frac{n(n-1)}{|x|^n} = |x| \cdot \frac{n-1}{n+1} \xrightarrow[n\to\infty]{} |x|$

Therefore the Abs. Ratio Test tells us that the series converges if $|x|<1$ and diverges if $|x|>1$. (so: radius of convergence $=1$)

What about $x=\pm 1$? (endpoints) - at $x=1$: the series is $\left(\sum_{n=2}^{\infty}\frac{1}{n(n-1)}\right)$: Converges (leading term: $\frac{1}{n^2}$ converging p-series)

- at $x=-1$: $\left(\sum \frac{(-1)^n}{n(n-1)}\right)$: Converges (because it is absolutely convergent, or Alt. Series)

2. Find a power series expansion for $f'(x)$. When does this series converge? (be careful about the endpoints). Take derivatives term by term:

$$f'(x) = \sum_{n=2}^{\infty} \frac{n \cdot x^{n-1}}{n(n-1)} = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1}$$

Radius of convergence $=1$ (same by theorem, or use Abs. Ratio Test again).

Endpoints:
- at $x=1$: $\left(\sum_{n=2}^{\infty}\frac{1}{n-1}\right)$ diverges (p-series $p=1$)
- at $x=-1$: $\left(\sum \frac{(-1)^n}{\dots}\right)$: converges by Alt. Series Test ($\frac{1}{n-1}$ decreases to 0).

3. Find a power series expansion for $f''(x)$. When does this series converge? (again, be careful about the endpoints). Again, take derivatives term by term:

$$f''(x) = [f'(x)]' = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n-1} = \sum_{n=2}^{\infty} x^{n-2} = 1 + x + x^2 + x^3 + \dots = \sum_{i=0}^{\infty} x^i$$

This is a geometric series of ratio x, so it converges $\Leftrightarrow |x|<1$ (in other words, radius of convergence $= 1$ and diverges at both endpoints).

4. Recognize this last power series to find expressions for $f''(x)$, $f'(x)$ and $f(x)$ using ordinary functions. You may need to use an antiderivative of $\ln x$ (Hint 1: integrate by parts; Hint 2: as we've seen in class, the result is $x \ln x - x$).

We recognize: $f''(x) = \sum_{n=2}^{\infty} x^{n-2} = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

Therefore, by integrating this:

$$f'(x) = \int \frac{1}{1-x}.dx = -\ln(1-x) + C = -\ln(1-x) \quad \left(\begin{array}{l} C=0 \text{ because} \\ f'(0)=0: \\ \text{look at series for } f' \end{array}\right)$$

Now we integrate this again:

$$f(x) = \int -\ln(1-x).dx = -\int \ln(1-x).dx \overset{\left\{\begin{array}{l} u=1-x \\ du=-dx \end{array}\right.}{=} \int \ln u.du \overset{\text{Hint 2}}{=} u\ln u - u + C$$

$$= (1-x)\ln(1-x) - (1-x) + C = (1-x)\ln(1-x) + x \quad \left(\begin{array}{l} C=+1 \text{ because} \\ f(0)=0=-1+C \end{array}\right)$$

For the hints: write $\ln x = 1.\ln x$ and integrate by parts:

$$\int \ln x \, dx \overset{\uparrow}{=} x.\ln x - \int x.\frac{1}{x} dx = x\ln x - x + C$$

$$\left\{\begin{array}{l} u(x) = \ln x \rightarrow u'(x) = \frac{1}{x} \\ v'(x) = 1 \rightarrow v(x) = x \end{array}\right.$$

5. Find the sum of the series $\left(\sum_{n=1}^{\infty} \frac{1}{n 2^n}\right)$ and $\left(\sum_{n=1}^{\infty} \frac{1}{n(n+1)2^n}\right)$. (Why do these series converge?)

What does this have to do with the previous questions? the $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ is obtained by evaluating things at $x = \frac{1}{2}$. More precisely:

$$\sum_{n=1}^{\infty} \frac{1}{n.2^n} = f'\left(\frac{1}{2}\right) = -\ln\left(1-\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \boxed{\ln 2}$$

The second series is "almost" $f\left(\frac{1}{2}\right)$. In fact: $f\left(\frac{1}{2}\right) = \sum_{n=2}^{\infty} \frac{1}{n(n-1).2^n} \overset{\left(\begin{array}{c}\text{with} \\ i=n-1\end{array}\right)}{=} \sum_{i=1}^{\infty} \frac{1}{i(i+1).2^{i+1}}$

therefore: $$\sum_{i=1}^{\infty} \frac{1}{i(i+1).2^i} = 2.f\left(\frac{1}{2}\right) = 2\left[\left(1-\frac{1}{2}\right)\ln\left(1-\frac{1}{2}\right) + \frac{1}{2}\right] = \ln\left(\frac{1}{2}\right) + 1 = \boxed{1 - \ln 2}$$

Both series converge because $\frac{1}{2}$ is inside the interval of convergence of the corresponding power series.