University of Utah
Math 5210, Spring 2010

Key

Midterm Exam # 2

Problem 0: (2 points) Write the definitions of (a) the outer measure $m^*(A)$ of a subset $A \subset \mathbb{R}$, and (b) measurability of $A \subset \mathbb{R}$.

(a) $m^*(A) := \inf_{\left\{\substack{I_n \text{ open intervals} \\ A \subset \bigcup_{n=1}^{\infty} I_n}\right.} \left(\sum_{n=1}^{\infty} \ell(I_n) \right)$

(b) A is measurable $\iff (\forall E \subset \mathbb{R})\ m^*(E) = m^*(A \cap E) + m^*(A^c \cap E)$

Problem 1: (7 points) Prove or disprove (by counter-example) each of the following statements:

(a) For $E \subset \mathbb{R}$ measurable: $m(E) > 0 \iff E$ contains an open interval.

(b) If O is open (and non-empty) then $m(O) > 0$

(c) If C is closed then $m(C) = 0$

(d) If K is compact then $m(K) < \infty$

(e) If K is compact and infinite then $m(K) > 0$

(f) If D is totally disconnected then $m(D) = 0$ (recall that *totally disconnected* means that any subset containing more than a point is not connected).

State if each of the following 3 statements is true or false (you may quote results from homework). $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function.

(g) f is measurable $\iff$ $(\forall c \in \mathbb{R})\ f^{-1}(\{c\})$ is measurable

(h) f is measurable $\iff$ $(\forall a, b \in \mathbb{R})\ f^{-1}((a, b))$ is measurable

(i) f is measurable $\iff$ $(\forall \text{ measurable } M \subset \mathbb{R})\ f^{-1}(M)$ is measurable

(a) False. More precisely: $\Leftarrow$ is true:
If E contains an open interval I, then $m(E) \geq m(I) > 0$.
$\Rightarrow$ is false: consider for instance $E = \mathbb{R} \setminus \mathbb{Q}$ (or $(\mathbb{R} \setminus \mathbb{Q}) \cap [0;1]$ if you want $m(E) < \infty$).

(b) True: O open is a union of open intervals, so $m(O) > 0$.
(in fact, only need to say that O contains an open interval).

(c) Ridiculously false. Take $C = [a, b]$, $m(C) = b - a > 0$ (if $a < b$).

(d) True: K compact $\Rightarrow$ K bounded, say $K \subset [-M; M]$. Then $m(K) \leq 2M$.

(e) False: Take K = Cantor set; compact and infinite, but $m(K) = 0$.

(f) False: Take again $(\mathbb{R} \setminus \mathbb{Q})$ (or $(\mathbb{R} \setminus \mathbb{Q}) \cap [0;1]$).

(g) False: see Problem 3.18 of Royden.

(h) True (immediate consequence of definition).

(i) (Subtly) False: see Problem 3.28 of Royden.

Problem 2: (7 points) (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be measurable and bounded. Construct a sequence (ϕ_n) of simple functions which converges pointwise to f. (b) If f is only measurable, construct a sequence ψ_n of simple functions which converges pointwise to f ~~a.e.~~

(a) This is more or less Problem 3.23(b) of Royden (seen in class).

Let $M > 0$ be such that $|f(x)| \leq M$ for all x. Fix $n \geq 1$.

Subdivide $[-M; M]$ into $(2M)n$ pieces ("horizontal slices" of graph),

and let $E_{k,n} = \left\{ x \,/\, f(x) \in \Big[\underbrace{-M + \frac{k}{n}}_{t_{k,n}}, -M + \frac{(k+1)}{n}\Big] \right\}$ for $k = 0, \ldots, 2Mn - 1$.
(or $\lfloor 2Mn \rfloor$)

Let $\varphi_n = \sum_{k=0}^{\lfloor 2Mn \rfloor} t_{k,n} \chi_{E_{k,n}}$.

Then φ_n is simple (each $E_{k,n}$ is measurable because f is) and:

$(\forall x)\quad |\varphi_n(x) - f(x)| \leq \frac{1}{n}$ by construction.

Therefore: $\varphi_n(x) \longrightarrow f(x)$ for all x.

(b) Use part (a), for instance by considering $f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n \\ 0 & \text{otherwise} \end{cases}$

(in other words, $f_n = f \cdot \chi_{E_n}$, where $E_n = \{ x \,/\, |f(x)| \leq n \}$).

Then each f_n is bounded, and $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for all x.

By part (a), for a fixed n we have a sequence $(\varphi_{n,k})$ of simple functions s.t. $\varphi_{n,k}(x) \xrightarrow[k\to\infty]{} f_n(x)$ for all x.

Consider $\psi_n = \varphi_{n,n}$ ("diagonal trick" for double indices)

Then $\psi_n(x) \longrightarrow f(x)$ for all x. (Note: Don't need "a.e.". I was confused with the case where $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ and $m(\{x / f(x) = \pm\infty\}) = 0$).

Problem 3: (2 points) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be integrable. Prove that:

$$\int_{-\infty}^{\infty} f(x)e^{-x^2/n}dx \longrightarrow \int_{-\infty}^{\infty} f(x)dx \quad (\text{as } n \to \infty)$$

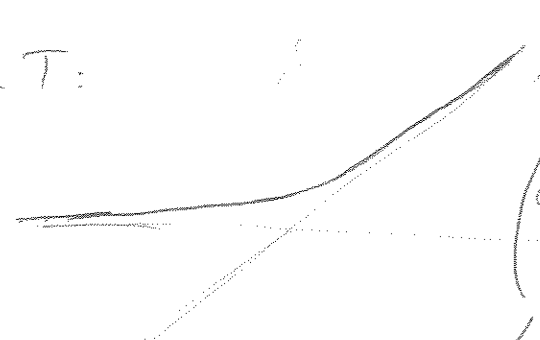
This is an immediate application of Lebesgue Dominated Convergence Theorem: indeed: $|f(x)e^{-x^2/n}| \leq |f(x)|$ for all n; $|f|$ is integrable (because f is) and for all x: $f(x)e^{-x^2/n} \xrightarrow[n\to\infty]{} f(x)$.

Therefore, by LDCT: $\int f(x)e^{-x^2/n}dx \xrightarrow[n\to\infty]{} \int f(x)dx$

Problem 4: (2 points) Prove that for any $\lambda \in (-1, 1)$, $\theta \in [0, 2\pi)$ and $v \in \mathbb{R}^2$, the transformation $D(\lambda) \circ R(\theta) \circ T(v)$ has a unique fixed point in $\mathbb{R}^2$ (where $D(\lambda)$ is a dilation of factor λ, $R(\theta)$ is a rotation through angle θ, and $T(v)$ is translation by v). More generally, if I is any isometry of $\mathbb{R}^n$ then $D(\lambda) \circ I = I \circ D(\lambda)$ has a unique fixed point.
Bonus: Find an example of a complete metric space X and a map $T : X \longrightarrow X$ such that $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$ but T has no fixed point in X.

Immediate application of Banach Fixed-Point Theorem (or, could do some linear algebra since finite dimension, but requires some work...).
Indeed, $R(\theta)$, $T(v)$ are isometries, and $|\lambda|<1$ so $D\circ R\circ T$ (or $D\circ I$) is a contraction.

Bonus: ~~Can~~ take $X=\mathbb{R}$, and T:



Note: there is a fixed point at ∞!

For a good reason, if X is also compact there are no such T.