#3: If f is measurable and $f \geq 0$, then: $\int f = 0 \Rightarrow f = 0$ a.e.

(seen in class). Let $E_n = \{x \mid f(x) \geq \frac{1}{n}\}$. Then: $F = \{x \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$.

Moreover: $\int f \geq \frac{1}{n} \cdot m(E_n)$ for all n, so $m(E_n) = 0$. Therefore $m(F) \leq \sum_{n=1}^{\infty} m(E_n) = 0$.

#4 (a) Let f be measurable with $f \geq 0$. From Problem 3.23(b) (proved in class), applied to $f_n = f \cdot \chi_{[-n,n]}$, there exists for all $n \geq 1$ a simple function φ_n such that: $|f_n(x) - \varphi_n(x)| < \frac{1}{n}$ except where $|f_n(x)| \geq n$. (and we may also choose: $\varphi_n \geq 0$, $\varphi_n = 0$ outside $[-n, n]$ because $f_n \geq 0$ and $f_n = 0$ ——).

To get rid of "except where $|f_n(x)| \geq n$", consider $\tilde{\varphi}_n = \varphi_n + n \chi_{\{f_n \geq n\}}$. Then $\tilde{\varphi}_n(x) \longrightarrow f(x)$ for all x.

To make the sequence increasing, let $\varphi_n = \mathrm{Max}\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_n\}$. [Could also use proof of Prop. 3, #3 and #6 of this page ...]

(b) * For any fn. $\varphi \leq f$, $\int \varphi \leq \int f$ so $\sup_{\{\varphi \text{ simple},\ \varphi \leq f\}} \int \varphi \leq \int f$.

* Take a sequence (φ_n) as in part (a). Then $\int \varphi_n \longrightarrow \int f$ by Monotone Convergence Th. (proves reverse inequality).

Therefore: $\int f = \sup_{\{\varphi \text{ simple},\ \varphi \leq f\}} \int \varphi$.

#5: $f \geq 0$ and integrable $\Longrightarrow F(x) = \int_{-\infty}^{x} f$ is continuous.

Proof (using MCT as indicated). Fix x, and let (x_n) be a sequence of points increasing to x. Let $f_n = f \cdot \chi_{[-\infty, x_n]}$ and $\tilde{f} = f \cdot \chi_{[-\infty, x]}$. Then (f_n) increases to $\tilde{f}$, so by MCT $\int f_n \longrightarrow \int \tilde{f}$, i.e. $F(x_n) \longrightarrow F(x)$. If (x_n) decreases to x, let $f_n = f \cdot \chi_{[x_n, \infty)}$ and $\hat{f} = f \cdot \chi_{[x, \infty)}$. Again, by MCT, $\int f_n \longrightarrow \int \hat{f}$ i.e. $\int_{-\infty}^{\infty} f - F(x_n) \longrightarrow \int_{-\infty}^{\infty} f - F(x)$ and $F(x_n) \to F(x)$ (use integrability of f here).

Therefore F is sequentially continuous, therefore continuous.

Note: If f is bounded the proof is much easier: If $|f| \leq M$ a.e. then $|F(x) - F(x_0)| = \left|\int_{x_0}^{x} f\right| \leq M \cdot |x - x_0|$.

#7: (a) As indicated let $f_n(x) = \begin{cases} 1 & \text{if } x \in [n, n+1) \\ 0 & \text{otherwise} \end{cases}$. Then for all x $f_n(x) \longrightarrow 0$, but $\int f_n = 1$ ($\forall n$).

Therefore $\int (\lim f_n) = 0 < 1 = \underline{\lim} \int f_n$. (We saw another example in class: f_n: )

(b) As indicated let $f_n(x) = \begin{cases} 0 & \text{if } x < n \\ 1 & \text{if } x \geq n \end{cases}$. Then for all x $f_n(x)$ decreases to 0, but ($\forall n$) $\int f_n = \infty$, so $\int (\lim f_n) = 0 < \infty = \lim \int f_n$

#15: (a) is straightforward by applying Problem 4 to f^+ and f^- (as indicated)
(b) & (c) also: combine (a) with Prop. 22 (proven in class).

#16: Riemann-Lebesgue Th: f integrable $\Rightarrow \int_{-\infty}^{\infty} f(x)\cdot\cos(nx)\,dx \xrightarrow[n\to\infty]{} 0$

(In particular, if f is periodic and integrable over a period, the Fourier coefficients of f tend to 0).

Proof: (a) Suppose $f = \chi_{(a,b)}$.

Then: $\int f(x)\cos(nx)\,dx = \int_a^b \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_a^b$ so $\left|\int f(x)\cos nx\,dx\right| \leq \frac{2}{n} \xrightarrow[n\to\infty]{} 0$.

(b) If f is a step function, by (a) and linearity $\int f(x)\cos nx\,dx \to 0$.

(c) Use density of step functions (#15(b) is this density for the L^1-norm).

Let f be integrable. Fix $\varepsilon > 0$, let ψ be a step function s.t. $\int_{\mathbb{R}} |f-\psi| < \varepsilon$

Then: $\left|\int f(x)\cos nx - \int \psi(x)\cos nx\right| = \left|\int (f(x)-\psi(x))\cos nx\right| \leq \int |f-\psi|\cdot|\cos nx| < \varepsilon$

As $\int \psi(x)\cos nx \xrightarrow[n\to\infty]{} 0$ by (b), $\int f(x)\cos nx \to 0$.