F4 p.88: If $(\exists M>0)(\forall f\in E)(\forall x\in[a,b]), |f'(x)|\leq M$, then E is equicontinuous ①

Note that by the Mean Value Theorem, for any $f\in E$ and $x_1,x_2\in[a,b]$ there exists $c\in\,]x_1,x_2[$ such that: $f'(c)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$. In particular: $(\forall f\in E)(\forall x_1,x_2\in[a,b])\ \left|\dfrac{f(x_2)-f(x_1)}{x_2-x_1}\right|\leq M$. Therefore, if $\varepsilon>0$ is fixed, let $\delta=\dfrac{\varepsilon}{M}$. Then $(\forall f\in E)(\forall x_1,x_2\in[a,b])\big(|x_2-x_1|<\delta\Rightarrow|f(x_2)-f(x_1)|<\delta M=\varepsilon\big)$.

#1 p.91: If $E_1,\dots,E_n$ are connected and $\bigcap_{i=1}^n E_i\neq\emptyset$, then $\bigcup_{i=1}^n E_i$ is connected.

Let $f:\bigcup_{i=1}^n E_i\longrightarrow\{0;1\}$ be a continuous function. Since each E_i is connected, $f|_{E_i}$ is constant on each E_i. But $\bigcap_{i=1}^n E_i\neq\emptyset$ so all the constants must be the same, so f is constant. Therefore $\bigcup_{i=1}^n E_i$ is connected. (Characterization seen in class).

#2 p.91: If E is connected and $E\subset F\subset\overline{E}$, then F is connected. (In particular: $\overline{E}$ is connected).

Let $f:\overline{E}\to\{0;1\}$ be a continuous function. Since E is connected, f is constant on E. Since every point of $\overline{E}$ is a limit point of E, f is also constant on $\overline{E}$ (in particular, on F) by continuity. Therefore F, $\overline{E}$ are connected.

#3 p.91: Connected components: $xRy\Leftrightarrow\exists$ connected subset of X containing x and y.

(a) R is an equivalence relation: reflexive, symmetric are obvious. Transitive: if xRy and yRz, $\exists C_1,C_2$ connected with $x,y\in C_1$, $y,z\in C_2$. Then $C_1\cup C_2$ is connected (by #1 above, $y\in C_1\cap C_2$)

(b) If $C(x)=$ equivalence class of x, then $C(x)=\bigcup_{\substack{C_\alpha\text{ connected}\\ x\in C_\alpha}} C_\alpha$:

Indeed, if C_α is connected and contains x then $C_\alpha\subset C(x)$ by def. of R, so $\bigcup C_\alpha\subset C(x)$. Conversely, if $y\in C(x)$, $\exists C_\alpha$ connected containing x and y.

c) $C(x)$ is a maximal connected set: $*$ $C(x)$ is connected by (b) and #1 $\left(x\in\bigcap C_\alpha\right)$.

$*$ If E is connected and contains $C(x)$, it contains x and so is one of the C_α above.

d) $\overline{C(x)}$ is connected by #2 above and contains $C(x)$, so $C(x)=\overline{C(x)}$. So $C(x)$ is closed.

e) If E is connected and $E\cap C(x)\neq\emptyset$ then $E\subset C(x)$: Let $y\in E\cap C(x)$. Then E is a connected set containing y, so $E\subset C(y)$. But $C(x)=C(y)$ because $y\in C(x)$.

f) X connected $\Leftrightarrow$ X has only one connected component: obvious by (c).

#4 p.92: $B(a,r)\subset\mathbb{R}^n$ is convex: by convexity of $\|.\|$ (continuous version of triangle inequality).

More precisely, let $x, y \in B(a,r)$. Recall that: $[x,y]=\{(1-t)x+ty \,/\, t\in[0;1]\}$.

$\forall t\in[0;1]$ $\|(1-t)x+ty-a\| = \|(1-t)x+ty-(1-t)a-ta\| \leqslant \|(1-t)(x-a)+t(y-a)$

Δ-ineq. $\rightarrow$ $\leqslant \|(1-t)(x-a)\|+\|t(y-a)\|$

$= |1-t|\cdot\|x-a\|+|t|\cdot\|y-a\|$

$\leqslant (1-t).r+tr = r.$

#6 p.92: Any open set in $\mathbb{R}$ is the union of countably many disjoint open intervals.

Let O be open in $\mathbb{R}$ and let (C_α) be its connected components.

* Each C_α is open because O is open.
* Each C_α is an interval because it is connected. (th. in class).
* The C_α are disjoint.
* There are (at most...) countably many C_α's because any (nonempty...) open interval contains a rational, and $\mathbb{Q}$ is countable.

#8 p.92 (optional): $E=\underbrace{\left\{(x,y)\in\mathbb{R}^2 \,/\, y=\sin\frac{1}{x},\ x>0\right\}}_{E_1}\cup\underbrace{[(0;-1),(0;1]}_{E_2}$ is connected but not path-connected.

* E is connected:

Indeed, E_1 is connected because it's the graph of a continuous function (so, image of $\mathbb{R}_+^*$, connected, under the continuous map $F(x)=(x,\sin 1/x)$). Moreover, every point of E_2 is a limit point of E_1 (exercise: write an explicit sequence $(x_n,\sin 1/x_n)\longrightarrow(0,t)$), so $E_1\subset E\subset\overline{E_1}$ (in fact, $E=\overline{E_1}$). By #2 above, E is connected.

* E is not path-connected: More precisely, we'll prove that there is no continuous path in E connecting a point of E_1 to a point of E_2, by contradiction.

So suppose $\varphi:[0;1]\longrightarrow E$, $t\longmapsto(\varphi_1(t),\varphi_2(t))$ is a continuous path joining $p_1=(x_0,\sin 1/x_0)$ to $p_2=(0,t_0)$.

Suppose WLOG that $\varphi_1(t)>0$ for all $t\in[0,1)$ (if not, consider $t=\inf\{t\in[0;1]\,/\,\varphi_1(t)=0\}$, >0)

Therefore, for $0\leq t<1$: $\varphi(t)=(x(t),\sin 1/x(t))$ with $x(t)$ continuous, $x(t)\xrightarrow[t\to 1]{}0$.

But then φ can't be continuous at $t=1$, because $\sin\frac{1}{x}$ has unbounded variation (i.e. oscillates wildly).

More precisely, if $x_n=\frac{1}{\pi/2+n\pi}$, $\sin 1/x_n=(-1)^n$ but $x_n-x_{n+1}=\frac{\pi}{(\pi/2+n\pi)(\pi/2+(n+1)\pi)}\xrightarrow[n\to\infty]{}0$.]

#11 p.92: If X connected contains at least 2 points, then X is uncountable.

We follow the hint: Let x_0, x_1 be 2 distinct points in X, and consider $f: X\longrightarrow\mathbb{R}$, $x\longmapsto d(x,x_0)$.

Then f is continuous, so $f(X)$ is connected, i.e. an interval, containing 0 and $d=d(x_0,x_1)>0$. Therefore, $\forall t\in[0,d]$, $f^{-1}(\{t\})\neq\emptyset$, and X is uncountable. (because $[0,1]$ is