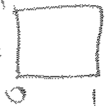
#1 p.81: (a) If $x, y \in B(a,r)$ then $d(x,y) \leqslant d(x,a) + d(a,y) < 2r$ so $\mathrm{Diam}(B(a,r)) \leq 2r$.

In general, $\mathrm{Diam}\, B(a,r) = 2r$ (if $\exists x_n, x'_n \in B(a,r)$ with $d(x_n, x'_n) \longrightarrow 2r$),

however $\mathrm{Diam}\, B(a,r)$ could also be anything between 0 and $2r$

(think of a discrete set / metric space).

(b) $E =$  $\mathrm{Diam}(E) = \sqrt{2}$ $\left(\begin{array}{l} \text{need to observe: (a) } (\forall x, y \in E)\ d(x,y) < \sqrt{2} \\ \text{and} \quad \text{(b) } (\exists x_n, x'_n \in E) \text{ s.t. } d(x_n, x'_n) \longrightarrow \sqrt{2} \end{array}\right)$.

(c) $\mathrm{Diam}((\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]) = 1$ (by density). (Same).

#2 p.81: (Seen in class). Here: "bounded" means "having finite diameter"!

So: If E is bounded: $(\exists M)$ $d(x,y) \leq M < M+1$ for all $x, y \in E$. Fix $x \in E$. Then $E \subset B(x, M+1)$.

If $E \subset B(a,r)$: for $x, y \in E$, $d(x,y) \leq d(x,a) + d(a,y) \leq 2r$ so $\mathrm{Diam}(E) \leq 2r$.

#4 p.81: $E = \{1/n \mid n \in \mathbb{N}\}$ is not closed (0 is a limit point of E) so not compact.

$E \cup \{0\}$ is (bounded and) closed (because $\frac{1}{n}$ converges to 0) so compact.

#7 p.81: If F is finite, any open cover of F contains a finite subcover (just keep one open set containing each point of F), so F is compact.

#8 p.81: This is an important special case of the theorem:

Th: If X is compact and f is continuous, then $f(X)$ is compact.

(Recall that in $\mathbb{R}$, compact = closed and bounded).

#1 p.84: ~~If~~ (a,b) is not sequentially compact because for instance the sequence $\left(a + \frac{1}{n}\right)$ has no subsequence converging in (a,b) (take $n \geqslant N$ with $\frac{1}{N} < b-a$).

#2 p.84: Totally bounded $\Rightarrow$ bounded: seen in class.

Example: (also seen in class) (e_n) in $\ell^\infty(\mathbb{R}) = \left(\left\{\begin{array}{l}\text{bounded sequences} \\ \text{of real numbers}\end{array}\right\}, \|\cdot\|_\infty\right)$, $e_n = (0, \ldots, 0, 1, 0, \ldots)$
n^{th} place.

#3 p.84: Observe that for $n \geqslant 11$, $\left(\frac{1}{n+1}, \frac{1}{n-1}\right) \subset \left[0; \frac{1}{10}\right)$, so we only need to consider the first $2+10$ intervals, $[0; 1/10)$, $(1/2, 1)$ and $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ for $2 \leq n \leq 11$.

Now a Lebesgue number for that finite cover is $\frac{1}{10} - \frac{1}{11} = \frac{1}{110}$ (or anything smaller).

Note that any interval $\left(\frac{1}{11}, \frac{1}{10} + \varepsilon\right)$ is not contained in any of the open sets of the cover.

#7: follows from #4)(a) p.75.

#8 p.84: Suppose that (x_n) has a limit point x but does not converge to x.

Then: $(\exists \varepsilon > 0)(\forall N > 0)(\exists n \geqslant N)$ ~~s.t.~~ $d(x_n, x) \geqslant \varepsilon$. This allows us to find a subsequence $(x_{n_k})_k$ such that $(\forall k)$ $d(x_{n_k}, x) \geqslant \varepsilon$. But X is compact, so (x_{n_k}) has a subsequence which converges, say to x_1. Then $d(x_1, x) \geqslant \varepsilon$ so $x_1 \neq x$ and (x_n) has at least 2 limit points.