F8 p.60 * Take $A=(0;1)$ and $B=(1;2)$ in $\mathbb{R}$. Then $d(A,B)=0$ but $A\cap B=\emptyset$. ①

#4 p.282 * Write: $\|x+y\|^2=\langle x+y,x+y\rangle=\langle x,x\rangle+2\langle x,y\rangle+\langle y,y\rangle$
and $\|x-y\|^2=\langle x-y,x-y\rangle=\langle x,x\rangle-2\langle x,y\rangle+\langle y,y\rangle$
and subtract the 2^d line from the 1^{st}: $\|x+y\|^2-\|x-y\|^2=4\langle x,y\rangle$.

#2 p.165 and #4 p.160 * We treat these together:

Th: (a) For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}\xrightarrow[p\to\infty]{}\underset{1\le i\le n}{\mathrm{Max}}|x_i|$, i.e.: $\|x\|_p\xrightarrow[p\to\infty]{}\|x\|_\infty$

(b) For f continuous on $[a,b]$ $(a,b\in\mathbb{R})$, $\left(\int_a^b|f(t)|^p dt\right)^{1/p}\xrightarrow[p\to\infty]{}\underset{a\le t\le b}{\mathrm{Sup}}|f(t)|$, i.e. $\|f\|_p\xrightarrow[p\to\infty]{}\|f\|_\infty$

(a) Fix $p\geq 1$ and $(x_1,\dots,x_n)\in\mathbb{R}^n$. Let $M=\underset{1\le i\le n}{\mathrm{Max}}|x_i|$. Then: $M^p\leq\sum_{i=1}^n|x_i|^p\leq n.M^p$

therefore: $M\leq\|x\|_p\leq n^{1/p}.M$

Now as $p\to\infty$, $n^{1/p}\to 1$ and the result follows.

(b) can be proved by using (a) and the "density of step functions".

A step function on $[a,b]$ is a function f for which there exists a subdivision of $[a,b]$, $c_0=a<c_1<c_2<\dots<c_n=b$ and numbers $x_1,\dots,x_n$ such that $f(x)=x_i$ on (c_{i-1},c_i).
(It doesn't really matter what value f takes at the endpoints c_i).

A convenient way to write this is: $f=\sum_{i=1}^n x_i\mathbb{1}_{(c_{i-1},c_i)}$ where: $\mathbb{1}_X(x)=\begin{cases}1 & \text{if } x\in X\\ 0 & \text{if } x\notin X\end{cases}$ ("characteristic fn. of X").

In other words, step functions are linear combinations of characteristic functions of intervals.

We can then associate to any $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ the step function f_x defined by $f_x=\sum_{i=1}^n x_i\mathbb{1}_{I_i}$ where the interval I_i is: $\left(a+\frac{(i-1)(b-a)}{n};\,a+\frac{i(b-a)}{n}\right)$ (we just divided $[a,b]$ into n equal pieces).

Note that: $\|f_x\|_p=\left(\frac{b-a}{n}\right)^{1/p}.\|x\|_p$ and $\|f_x\|_\infty=\|x\|_\infty$. (*)

In fact f_x is a special kind of step function with all the (c_{i-1},c_i) of equal length; call this an equal step function.

The main tool is the following result ("density of (equal) step functions"):

Prop: If f is continuous on $[a,b]$ $(a,b\in\mathbb{R})$, then $(\forall\varepsilon>0)\ \exists g$ equal step function s.t. $\|f-g\|_\infty\leq\varepsilon$.

Step 1: f is uniformly continuous on $[a,b]$, i.e.: $(\exists\delta>0)(\forall x,y\in[a,b])\left(|x-y|\leq\delta\Rightarrow|f(x)-f(y)|\leq\varepsilon\right)$.
Maybe you've seen this in 3210; if you haven't: exercise.

Step 2: Take $n\in\mathbb{N}$ such that $\frac{1}{n}\leq\delta$; subdivide $[a,b]$ into n equal pieces $(c_{i-1},c_i)=I_i$ and define the equal step function $g=\sum_{i=1}^n f(c_i).\mathbb{1}_{I_i}$. Then by construction $\|f-g\|_\infty\leq\varepsilon$.

Th. (b) then follows from the Prop. and (*) by taking $p\to\infty$ (note that $\left(\frac{b-a}{n}\right)^{1/p}\xrightarrow[p\to\infty]{}1$). □

University of Utah
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Homework # 2

Are the following functions continuous? (Prove your answer)

1. $F_1 : C([a,b]) \longrightarrow \mathbb{R}$ defined by $f \mapsto \int_a^b f(t)dt$, where $C([a,b])$ denotes the set of continuous functions on the bounded interval $[a,b]$, and is endowed with the norm $||.||_\infty$.

2. $F_2 : C([a,b]) \longrightarrow \mathbb{R}$ defined by $f \mapsto \text{Max}_{[a,b]}|f(t)|$, where $C([a,b])$ is endowed with the norm $||.||_1$.

3. $F_3 : D([a,b]) \longrightarrow C([a,b])$ defined by $f \mapsto f'$, where $D([a,b])$ denotes the set of differentiable functions on $[a,b]$ (note that $D([a,b]) \subset C([a,b])$). Consider the four cases where each space is endowed with the norm $||.||_1$ or $||.||_\infty$. (For simplicity you can produce examples which are only piecewise differentiable).

Note that F_1 and F_3 are linear maps between normed vector spaces
(F_2 is only sublinear $F_2(x+y) \leqslant F(x)+F(y)$, but it is linear when restricted to $\geqslant 0$ functions.)

~~We use the following fact:~~

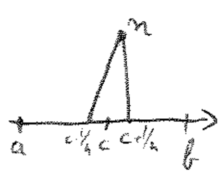
Prop: A linear map $f : (V_1, \|.\|_1) \longrightarrow (V_2, \|.\|_2)$ where $V_i, \|.\|_i$ are normed vector spaces, is continuous $\Longleftrightarrow (\exists C>0)(\forall x \in V_1)\ \|f(x)\|_2 \leqslant C.\|x\|_1$.

Note: The direction $\Leftarrow$ is true for any map (C-Lipschitz $\Rightarrow$ continuous).
We'll prove the other direction in class.

* F_1 is continuous: Indeed, $(\forall f \in C([a,b]))\ |F_1(f)| \leqslant \int_a^b |f(t)|dt \leqslant (b-a).\|f\|_\infty$
(so F_1 is $(b-a)$-Lipschitz and therefore continuous).

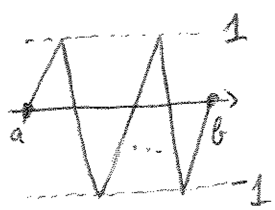
* F_2 and F_3 (all 4 cases) are not:

- for F_2 consider the sequence f_n:



then: $\|f_n\|_1 = 1$
but $F(f_n) = n$ is unbounded

- for F_3 consider the sequence: g_n



where the slope of each line is n or $-n$ (Exercise: write a formula for g_n)

(the derivative of g_n is equal to n or $-n$)

Then: $\|g_n\|_\infty = 1$ but $\|F_3(g_n)\|_\infty = n$
$\|g_n\|_1 \leqslant 1$ $\|F_3(g_n)\|_1 = n$