Fundamental domains for finite subgroups of $U(2)$ and Configurations of Lagrangians

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Abstract

We show that every finite subgroup of $U(2)$ is contained with index two in a group generated by involutions fixing Lagrangian planes. We describe fundamental domains for their action on $\mathbb{C}^2$ related to the configuration of these Lagrangian planes.

1 Introduction

Finite subgroups of $U(2)$, of the group $\widehat{U(2)}$ obtained by adjoining complex conjugation to $U(2)$, and more generally of $O(4)$ have been enumerated (see [CS, C, D] for lists and discussions on that problem). It contains two important classes of subgroups, namely, those generated by complex reflections and those which are index two subgroups of groups generated by involutions fixing a Lagrangian subspace of $\mathbb{C}^2$. We show in fact that all finite subgroups of $U(2)$ are of the second type (th. 2.1). This will follow from the fact that almost all finite subgroups of $U(2)$ have two generators (see th. 2.3), and the main goal in this paper is to obtain a description of these groups in terms of Lagrangian reflections. In particular we describe fundamental domains for those groups which are adapted to the Lagrangian decomposition and seem much simpler than previous fundamental domains (sections 4.4 and 5.1.3). Previous fundamental domains for some of those finite groups were obtained by Mostow in [M] in view of applications in complex hyperbolic geometry. Fundamental domains in the hyperbolic case for some groups generated by Lagrangian reflections were studied in [FZ].

It is interesting to note that, in our construction, the boundary of the fundamental domain is made out of pieces which are foliated by complex lines or Lagrangian planes and sometimes both and they show that recent constructions of fundamental domains in complex hyperbolic geometry (cf. [S]) appear naturally in the realm of finite groups.

2 Lagrangian subspaces and $\mathbb{R}$-reflections

Let $\mathbb{C}^n$ be the n-dimensional complex vector space which we will consider together with its standard Hermitian metric given by $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. We write the real and imaginary decomposition of the Hermitian metric

$$\langle z, w\rangle = g(z, w) - i\omega(z, w)$$

thus defining the standard symplectic form ω on $\mathbb{C}^n$ compatible with the Hermitian metric. The complex structure $J : \mathbb{C}^n \to \mathbb{C}^n$ defined by $Jz = iz$ satisfies the following relation

$$g(z, w) = \omega(z, Jw).$$

Let $U(n)$ be the unitary group of $\mathbb{C}^n$ relative to $\langle\,,\rangle$, $O(2n)$ the orthogonal group of $\mathbb{C}^n$ relative to g and $Sp(n)$ the symplectic group of $\mathbb{C}^n$ relative to ω. Then by definition of $\langle\,,\rangle$, we have $U(n) = O(2n) \cap Sp(n)$. It follows that J is a map that is both orthogonal and symplectic.

Definition 2.1 *A real subspace L of $\mathbb{C}^n$ is said to be* Lagrangian *if its orthogonal $L^{\perp_\omega}$ with respect to ω is L itself.*

Equivalently, L is isotropic with respect to ω and of maximal dimension with respect to this property. A real subspace W of $\mathbb{C}^n$ is said to be *totally real* if $\langle u, v\rangle \in \mathbb{R}$ for all $u, v \in W$. A real subspace L of $\mathbb{C}^n$ is therefore Lagrangian if and only if it is totally real and of maximal dimension with respect to this property. Thus Lagrangians in $\mathbb{C}^n$ are none other than $\mathbb{R}^n$-planes (see [G]). Equivalently again, L is Lagrangian if and only if its orthogonal with respect to g is $L^{\perp_g} = JL$. In particular, given a Lagrangian subspace L of $\mathbb{C}^n$, we have the following g-orthogonal decomposition $\mathbb{C}^n = L \oplus JL$. That decomposition enables us to associate to every Lagrangian L an $\mathbb{R}$-linear map

$$\begin{array}{cccc} \sigma_L : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & x + Jy & \longmapsto & x - Jy \end{array}$$

Definition 2.2 *A Lagrangian reflection (or $\mathbb{R}$-reflection) is an involution in $\mathbf{O}(2n)$ which fixes a Lagrangian subspace L and acts as $-I$ on $J(L)$.*

Observe that

1. A Lagrangian reflection is a skew-symplectic ($\omega(\sigma_L(z), \sigma_L(w)) = -\omega(z, w)$) and anti-holomorphic map ($\sigma_L \circ J = -J \circ \sigma_L$).

2. Reciprocally, if an anti-holomorphic map fixes a Lagrangian space L, it acts on $J(L)$ by $-I$.

3. If n is even the Lagrangian reflections are in $\mathbf{SO}(2n)$,

4. In complex dimension 1, Lagrangian reflections coincide with real reflections in lines,

5. Lagrangian reflections are in $\widehat{\mathbf{U}}(n) \subset \mathbf{O}(2n)$, the group generated by $\mathbf{U}(n)$ and the conjugation $z \mapsto \bar{z}$.

Denoting by $L(n)$ the $\frac{n(n+1)}{2}$-dimensional manifold of all Lagrangian subspaces of $\mathbb{C}^n$ (the *Lagrangian grassmannian* of $\mathbb{C}^n$), we have defined a map

$$\begin{array}{ccc} L(n) & \to & \widehat{\mathbf{U}}(n) \\ L & \to & \sigma_L. \end{array}$$

Proposition 2.1 (cf. [N, FMS]) *Given two Lagrangian subspaces L_1 and L_2 there exists a unique $g_{12} \in U(n)$ such that $L_2 = g_{12}L_1$ satisfying the following conditions:*

1. *the eigenvalues are unitary complex numbers $e^{i\lambda_j}$ verifying $\pi > \lambda_1 \geq \cdots \lambda_n \geq 0$*
2. *there exists an orthonormal basis for L_1 which is a basis of eigenvectors for g_{12}.*

A corollary of this proposition is given by the following classification of pairs of Lagrangians:

Proposition 2.2 ([N]) *Pairs of Lagrangian subspaces in $\mathbb{C}^n$ are classified up to unitary transformations by a list $(\lambda_1, \cdots, \lambda_n)$, with $\pi > \lambda_1 \geq \cdots \lambda_n \geq 0$.*

The vector $(\lambda_1, \cdots, \lambda_n)$ will be called *angle between* L_1 and L_2. Note that the angle between L_2 and L_1 is $(\pi - \lambda_n, \cdots, \pi - \lambda_1)$ (in the case $\lambda_i > 0$). Sometimes we will write the angle between L_1 and L_2 without fixing a particular convention for the eigenvalues. To relate the angle with the inversions associated to the Lagrangians we use the following proposition which is a consequence of proposition 1. We refer to [FMS] for a proof.

Proposition 2.3 *Let g_{12} be the unique unitary map sending L_1 to L_2 defined in the above proposition. Then:*

$$g_{12}^2 = \sigma_2 \circ \sigma_1$$

Let $L_0 = \mathbb{R}^n$ be the standard real plane in $\mathbb{C}^n$ and σ_0 its corresponding $\mathbb{R}$-reflection.

Definition 2.3 *Let $L \subset \mathbb{C}^n$ be a Lagrangian subspace and σ its corresponding $\mathbb{R}$-reflection. The matrix of $\sigma \circ \sigma_0$ with respect to the canonical basis is called the Souriau matrix of L (or of σ).*

We introduce the following terminology:

Definition 2.4 *If a unitary transformation U is expressed as a product $\sigma_2\sigma_1$ of two $\mathbb{R}$-reflections in $\mathbb{R}^n$-planes L_1 and L_2, we say that L_1 (or L_2)* decomposes *U.*

Note that the second $\mathbb{R}^n$-plane is then uniquely determined.

Proposition 2.4 *Let U be a unitary transformation of $\mathbb{C}^n$, and C_1,...,C_k its eigenspaces with dimensions n_1,...,n_k. Then an $\mathbb{R}^n$-plane L decomposes $U \iff L \cap C_i$ is a Lagrangian subspace in C_i for all i. The space of such $\mathbb{R}^n$-planes is isomorphic to $U(n_1)/O(n_1) \times ... \times U(n_k)/O(n_k)$. In particular:*

- *If U is a multiple of identity, then any $\mathbb{R}^n$-plane decomposes U.*
- *If U has n distinct eigenvalues, let C_1,..., C_n be its eigenspaces. Then an $\mathbb{R}^n$-plane L decomposes $U \iff L \cap C_i$ is a real line for all i. There exists an n-torus of such $\mathbb{R}^n$-planes.*

This proposition follows from Nicas' diagonalization lemma (prop. 1). We use this result to simultaneously decompose two unitary elements (in view of dimension 2, we are only concerned with the two particular cases above):

Proposition 2.5 *Let U_1, U_2 be unitary transformations of $\mathbb{C}^n$, each with distinct eigenvalues, and let C_1,..., C_n, C'_1,..., C'_n be their respective eigenspaces. Then an $\mathbb{R}^n$-plane L simultaneously decomposes U_1 and $U_2 \iff L \cap C_i$ and $L \cap C'_i$ are real lines for all i.*

The case of dimension 2 is special because any pair of unitary transformations can be simultaneously decomposed into $\mathbb{R}$-reflections,as we will see below. We begin by recalling a few facts about the behavior of $\mathbb{R}$-planes of $\mathbb{C}^2$ under projectivisation, for which the reader can refer to [FMS].

Let:

$$\pi : \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{C}P^1$$

denote projectivisation, inducing a surjective morphism:

$$\tilde{\pi} : U(2) \longrightarrow SO(3).$$

If C_i is a $\mathbb{C}$-plane (i. e. a complex line) and L_j an $\mathbb{R}$-plane, then $\pi(C_i)$ is a point in $\mathbb{C}P^1$ which we denote c_i, and $\pi(L_j)$ is a great circle in $\mathbb{C}P^1$ which we denote l_j. Moreover their relative position is as follows:

- $c_i \in l_j \iff C_i \cap L_j$ is a real line
- $c_i \notin l_j \iff C_i \cap L_j = \{0\}$

We can state now the

Theorem 2.1 *Every finite subgroup of $U(2)$ admits an index 2 supergroup in $\widehat{U(2)}$ generated by $\mathbb{R}$-reflections. In fact:*

- *Every two-generator subgroup of $U(2)$ admits an index 2 supergroup in $\widehat{U(2)}$ generated by 3 $\mathbb{R}$-reflections. There exists generically a circle of such groups, and a 2-torus in the degenerate cases.*
- *The exceptional finite subgroups of $U(2)$ which are not generated by 2 elements ($\langle 2q, 2, 2\rangle_m$ with m even, see theorem 2.3) admit an index 2 supergroup in $\widehat{U(2)}$ generated by 4 $\mathbb{R}$-reflections.*

Proof. We will only prove the first part for now; the case of the exceptional family $\langle 2q, 2, 2\rangle_m$ (m even), will be handled in the last section where we give an explicit decomposition into $\mathbb{R}$-reflections.

The case of two generators can be seen in $\mathbb{C}P^1$ as follows. Assuming that the generators U_1, U_2 each have distinct eigenvalues, their eigenspaces project to two pairs of antipodal points in $\mathbb{C}P^1$, and there exists a great circle l containing these points (l is unique if the four points are distinct, that is if U_1 and U_2 have distinct eigenspaces). This means that any $\mathbb{R}$-plane L above

l intersects the four eigenspaces along a real line, and thus simultaneously decomposes U_1 and U_2. $\square$

Remark: Theorem 4.1 in [F] gives an incomplete classification. In the course of the proof the imprimitive groups were neglected. Also, the primitive family $(\mathbf{C}_{6m}/\mathbf{C}_{2m}; \mathbf{T}/\mathbf{V})$ was excluded because $\mathbf{T}$ was not generated by pure quaternions. In fact, $\mathbf{T}$ is a subgroup of index 2 of $\mathbf{O}$, a group generated by pure quaternions and this is admissible in the proof.

It remains to see which finite subgroups of $U(2)$ can be generated by two elements; to this effect, we recall Du Val's classification of finite subgroups of $U(2)$ among those of $SO(4)$ (they comprise the 9 first types of his list, [D] p. 57, see also [C] p. 98). We will use Du Val's notations, writing elements of $U(2)$ as pairs (l, r) with $l \in U(1)$ and $r \in SU(2)$ (which we write as a unit quaternion under an implicit isomorphism), with the relations $(l, r) = (-l, -r)$; and subgroups of $U(2)$ as $\Gamma = (L/L_K; R/R_K)$ where $L_K \lhd L \subset U(1)$ and $R_K \lhd R \subset SU(2)$ are defined as follows:

$$L_K = \{l \in U(1) \,|\, (l, 1) \in \Gamma\}$$

$$R_K = \{r \in SU(2) \,|\, (1, r) \in \Gamma\}$$

$$L = \{l \in U(1) \,|\, \exists\, r \in SU(2), (l, r) \in \Gamma\}$$

$$R = \{r \in SU(2) \,|\, \exists\, l \in U(1), (l, r) \in \Gamma\}$$

Γ is then recovered from L_K, L, R_K, R, fixing an isomorphism $\phi : L/L_K \longrightarrow R/R_K$, as:

$$\Gamma = \{(l, r) \in L \times R \,|\, \phi(\bar{l}) = \bar{r}\}$$

The boldface letters $\mathbf{C}_k$, $\mathbf{D}_k$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$, $\mathbf{V}$ indicate subgroups of $SU(2)$ which are double covers of the corresponding groups in $SO(3)$ (the cyclic, dihedral, tetrahedral, octahedral, icosahedral groups and the classical quaternion group of order 8). The subscript notation, such as in $\langle p, 2, 2\rangle_m$, is made explicit in the table below and indicates a group where $L = L_K$ and $R = R_K$ (its double cover is a direct product).

These notations established, the classification is:

Theorem 2.2 (see [D], [C]) *The finite subgroups of $U(2)$ are*

1. $(\mathbf{C}_{2m}/\mathbf{C}_f; \mathbf{C}_{2n}/\mathbf{C}_g)_d$
2. $\langle p, 2, 2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m}; \mathbf{D}_p/\mathbf{D}_p)$
3. $(\mathbf{C}_{4m}/\mathbf{C}_{2m}; \mathbf{D}_p/\mathbf{C}_{2p})$ *and* $(\mathbf{C}_{4m}/\mathbf{C}_m; \mathbf{D}_p/\mathbf{C}_p)$*, m and p odd*
4. $(\mathbf{C}_{4m}/\mathbf{C}_{2m}; \mathbf{D}_{2p}/\mathbf{D}_p)$
5. $\langle 3, 3, 2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m}; \mathbf{T}/\mathbf{T})$
6. $(\mathbf{C}_{6m}/\mathbf{C}_{2m}; \mathbf{T}/\mathbf{V})$
7. $\langle 4, 3, 2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m}; \mathbf{O}/\mathbf{O})$

8. $(\mathbf{C}_{4m}/\mathbf{C}_{2m};\mathbf{O}/\mathbf{T})$

9. $\langle 5,3,2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m};\mathbf{I}/\mathbf{I})$

Type 1 consists of the reducible groups (those that fix a decomposition $\mathbb{C}^2 = V_1 \oplus V_2$), types 2-4 of the irreducible imprimitive groups (those that stabilize a decomposition $\mathbb{C}^2 = V_1 \oplus V_2$, see [ST]), and types 5-9 of the irreducible primitive groups.
We can now state which of the above groups are generated by two elements:

Theorem 2.3 *All finite subgroups of $U(2)$ can be generated by two elements, except for one exceptional family. More precisely:*

- *Any finite subgroup of $U(2)$ whose projection in $SO(3)$ is different from an "even" dihedral group $(2q,2,2)$ is a 2-generator subgroup.*
- *Among those groups which project to $(2q,2,2)$ (types 2, 3, and 4 in Du Val's classification), all are 2-generator subgroups but one exceptional family in type 2, namely $\langle 2q,2,2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{2q})$ with m even.*

Proof. We use the projection $\tilde{\pi}$ introduced above, which gives us the short exact sequence:

$$1 \longrightarrow U(1) \longrightarrow U(2) \xrightarrow{\tilde{\pi}} SO(3) \longrightarrow 1$$

or for a finite subgroup Γ of $U(2)$:

$$1 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \xrightarrow{\tilde{\pi}} G \longrightarrow 1$$

Now the finite subgroups G of $SO(3)$ are well known: they are either cyclic or triangular of type $(p,2,2)$, $(3,3,2)$, $(4,3,2)$ or $(5,3,2)$, all generated by two rotations. The main result is the following:

Lemma 2.1 *If G is generated by g_1 and g_2 such that $g_1^{r_1} = g_2^{r_2} = 1$ with r_1 and r_2 relatively prime, then Γ is generated by two elements.*

Indeed, choose at first γ_1 and γ_2 to be arbitrary preimages of g_1 and g_2 under $\tilde{\pi}$. Since $g_1^{r_1} = g_2^{r_2} = 1$, there exist $l_1, l_2 \in \mathbb{Z}$ such that $\gamma_1^{r_1} = z^{l_1}$ and $\gamma_2^{r_2} = z^{l_2}$, where z is a generator of the cyclic center $Z(\Gamma)$.
Now, for different preimages $\gamma_1' = z^{a_1}.\gamma_1$ and $\gamma_2' = z^{a_2}.\gamma_2$, we write:

$$(z^{a_1}.\gamma_1)^{r_1}.(z^{a_2}.\gamma_2)^{r_2} = z^{a_1r_1+a_2r_2+l_1+l_2}$$

Then, r_1 and r_2 being relatively prime, there exist $a_1, a_2 \in \mathbb{Z}$ such that $a_1r_1 + a_2r_2 = 1 - l_1 - l_2$; for such a choice of a_1 and a_2, we have $z = (\gamma_1')^{r_1}.(\gamma_2')^{r_2}$ so that the group generated by γ_1' and γ_2', containing z and projecting to G, is the entire Γ. This proves the lemma, and settles all cases but that where G is the even dihedral group $(2q,2,2)$, which we now examine separately.
We start by giving generating triples for the 3 types in question (we write elements of $SU(2)$ as unitary quaternions, the quaternion q corresponding to right multiplication by q in $\mathbb{C}^2$):

Lemma 2.2 • $\langle 2q,2,2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{2q})$ *(type 2) is generated by* $a=(e^{i\pi/m},1)$, $b=(1,e^{i\pi/2q})$ *and* $c=(1,j)$.

- $(\mathbf{C}_{4m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{C}_{4q})$ *(type 3) is generated by* $a=(e^{i\pi/m},1)$, $b=(1,e^{i\pi/2q})$ *and* $c=(e^{i\pi/2m},j)$.
- $(\mathbf{C}_{4m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{q})$ *(type 4) is generated by* $a=(e^{i\pi/m},1)$, $b=(e^{i\pi/2m},e^{i\pi/2q})$ *and* $c=(1,j)$.

The lemma is easily proven: the first type is obvious, and in the two other cases the subgroups L_K, R_K are of index 2 in L, R. We then notice that in type 3 a and b generate the direct product $L_K\times R_K$ so that we only need to add any element of $\Gamma\setminus L_K\times R_K$ such as c to obtain Γ. The last case is analogous except that $L_K\times R_K$ is generated by a, c and b^2a^{-1}.
Now in most cases these generating triples can be reduced to 2 elements:

Lemma 2.3 • $\langle 2q,2,2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{2q})$ *(type 2) is generated by ac and b <u>when m is odd</u>.*

- $(\mathbf{C}_{4m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{C}_{4q})$ *(type 3) is generated by b and c.*
- $(\mathbf{C}_{4m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{q})$ *(type 4) is generated by ac and b.*

We simply show that the previous 3 generators can be written in terms of the new pair.
In the first case, we write: $(ac)^m=a^mc^m=c^{m+2}$ so that <u>when m is odd</u>, this is c or c^{-1} (c has order 4).
In the second case, we compute: $c^{2m+2}=(e^{i(\pi+\pi/m)},(-1)^{m+1})$ which is equal to a if m is even or $a.(-1,1)=ab^{2q}$ if m is odd.
In the third case, we compute:

$$b.(ac)^{-1}.b.ac=(e^{i\pi/m},e^{i\pi/2q}.(-j).e^{i\pi/2q}.j)=(e^{i\pi/m},1)=a$$

The only case that remains is the family $\langle 2q,2,2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{2q})$ with m even, which behaves differently:

Lemma 2.4 *None of the groups $\langle 2q,2,2\rangle_m = (\mathbf{C}_{2m}/\mathbf{C}_{2m};\mathbf{D}_{2q}/\mathbf{D}_{2q})$ with m even can be generated by two elements.*

We prove this first for the smallest group $\langle 2,2,2\rangle_2=(\mathbf{C}_4/\mathbf{C}_4;\mathbf{D}_2/\mathbf{D}_2)$, then extend the result to the other groups in the family by showing that they all project onto this group.
We have seen in a straightforward manner that the group $\langle 2,2,2\rangle_2$ (of order 16, whose elements are in $\{\pm1,\pm i\}\times\{\pm1,\pm i,\pm j,\pm k\}$ modulo $(-1,-1)$) cannot be generated by a pair of its elements, by testing different pairs (in fact, this group has many symmetries and the only relevant candidates are of the type $(1,i),(i,j)$ or $(i,i),(i,j)$, which both generate a subgroup of order 8).
We then define, for m even, an application $s:\langle 2q,2,2\rangle_m\longrightarrow\langle 2,2,2\rangle_2$ by sending $a=(e^{i\pi/m},1)$ to $a^{m/2}=(i,1)$, $b=(1,e^{i\pi/2q})$ to $b^q=(1,i)$ and $c=(1,j)$ to itself. In fact this only defines an application $\tilde{s}$ from the free group over $\{a,b,c\}$ to $\langle 2,2,2\rangle_2$; that s is well-defined and is a morphism comes from the fact that we know a presentation of $\langle 2q,2,2\rangle_m$ in terms of the

generators a, b, c and that $\tilde{s}$ is compatible with all the relations involved. These are (with redundancy):

$$a^{2m} = b^{4q} = c^4 = 1$$

$$a^m = b^{2q} = c^2 \text{ (the central (1,-1))}$$

$$cbc^{-1} = b^{-1} \text{ and } [a, b] = [a, c] = 1$$

The relations are easy to write in this case because a, b, c are of the form $(1, r)$ or $(l, 1)$. This concludes the proof of the lemma and the theorem, for if $\langle 2q, 2, 2\rangle_m$ were generated by two elements g_1 and g_2, then $\langle 2, 2, 2\rangle_2$ would be generated by $s(g_1)$ and $s(g_2)$.
We finish by making the remark that our proof relies on Du Val's classification, which is slightly incomplete as pointed out in [C] and [CS]. However the missing groups do not interfere with our result, because Coxeter's type 3' (p. 98) projects to $(p, 2, 2)$ with p odd, whereas Conway and Smith add subgroups in $SO(4) \setminus U(2)$. $\square$

3 Lagrangian pairs

3.1 Groups generated by two $\mathbb{R}$-reflections

In order to classify finite groups generated by two $\mathbb{R}$-reflections we observe first that the angle between the two Lagrangians should be of the form

$$(\frac{\pi q_1}{p_1}, \cdots, \frac{\pi q_n}{p_n})$$

where $\frac{q_i}{p_i}$ are reduced fractions satisfying $1 > \frac{q_1}{p_1} \geq \cdots \frac{q_n}{p_n} \geq 0$. If $n = 1$ it is clear that we can find generators with angle of the form $\frac{\pi}{p}$. The normal form for the angle between two generators in higher dimensions is essentially given by the chinese remainder theorem.

Suppose the Lagrangians L_1 and L_2 form an angle $(\frac{\pi q_1}{p_1}, \cdots, \frac{\pi q_n}{p_n})$. We define the following map

$$(L_1, L_2) \mapsto \frac{\pi q_1}{p_1} + \cdots + \frac{\pi q_n}{p_n} \in [0, n\pi[$$

Different generators are obtained by taking a power of g_{12}^2; computing its eigenvalues we obtain

$$g_{12}^{2k} \mapsto [\frac{\pi k q_1}{p_1}] + \cdots + [\frac{\pi k q_n}{p_n}] \in [0, n\pi[$$

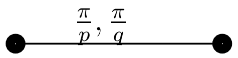
where $[\frac{\pi k q_1}{p_1}]$ is the representative of $\frac{\pi k q_1}{p_1} \mod \pi\mathbb{Z}$ in $[0, \pi[$.

We choose k to be one with minimal image under the above map. Clearly, if the p_i are pairwise prime the chinese remainder theorem implies that the minimal choice (unique in that case) is a generator and is given precisely by $(\frac{2\pi}{p_1}, \cdots, \frac{2\pi}{p_n})$.

In dimension 2 we can list all possibilities:

1. $\theta_1 = 2\pi/p_1$ and $\theta_2 = 2\pi/p_2$, with $(p_1, p_2) = 1$.

2. $\theta_1 = 2\pi/p_1$ and $\theta_2 = 2\pi q_2/p_2$, with $(p_1, p_2) = d \neq 1$ and $(q_2, p_2) = 1$.



This follows from the chinese remainder theorem, that is, the congruence $x \equiv a_i \ (p_i)$ (with $(p_1, p_2) = d$) has a solution if and only if $a_1 \equiv a_2 \ (d)$. Having obtained $\theta_1 = 2\pi/p_1$ by some power of the generator we ask whether another power by x satisfies $x \equiv 1 \ (p_1)$ and $xq_2 \equiv a_2 \ (p_2)$. This has a solution if and only if $q_2 \equiv a_2 \ (d)$. So one can suppose $q_2 < d$.

Given r $\mathbb{R}$-reflections, it is convenient to construct a diagram with r nodes corresponding to the generators σ_i and edges joining each pair of nodes labeled by $m(i,j)$. Here $m(i,j)$ is the angle between the given Lagrangians. For example, in the two dimensional case if two Lagrangians L_1 and L_2 have an angle $(\frac{\pi}{p}, \frac{\pi}{q})$ we can represent them by the following diagram:

3.2 Representations of an $\mathbb{R}$-reflection by matrices

We use throughout this section the canonical basis of $\mathbb{C}^n$. Recall that we have in $\mathbb{C}^n$ a standard $\mathbb{R}^n$-plane L_0:

$$L_0 = \{(x_1, ..., x_n) \in \mathbb{C}^n | x_i \in \mathbb{R}\}$$

whose $\mathbb{R}$-reflection is in coordinates:

$$\sigma_0 : (z_1, ..., z_n) \longmapsto (\overline{z_1}, ..., \overline{z_n})$$

This allows us to define for a single $\mathbb{R}^n$-plane L_i the invariants of a pair by considering the pair (L_0, L_i). We thus call *standard direct image matrix* for L_i any $U_i \in U(n)$ such that: $L_i = U_i L_0$ (the element U_i is defined up to the stabilizer of L_0 which is $O(2)$); we define the *Souriau map* of L_i which is the unitary map $\phi_i = \sigma_i \circ \sigma_0$ as well as the *Souriau matrix* A_i of L_i, the matrix of ϕ_i **in the canonical basis**. We will often characterize an $\mathbb{R}^n$-plane L_i by its Souriau matrix A_i, because as we will now see these matrices behave nicely under composition.

Indeed, rewrite the relation defining the Souriau map $\phi_i = \sigma_i \circ \sigma_0$ as: $\sigma_i = \phi_i \circ \sigma_0$. This is written in coordinates $z = (z_1, ..., z_n) : \sigma_i(z) = A_i \overline{z}$. Concretely, this allows us on one hand to obtain a parametrization of the $\mathbb{R}^n$-plane L_i as the set of solutions of the system $A_i \overline{z} = z$ (solved for instance by diagonalization in a real basis), and on the other hand to compose $\mathbb{R}$-reflections in matrix form. Indeed, if L_1 and L_2 are two $\mathbb{R}^n$-planes, we write in coordinates:

$$\sigma_2 \circ \sigma_1(z) = \sigma_2(A_1.\overline{z}) = A_2.(\overline{A_1.\overline{z}}) = A_2.\overline{A_1}.z$$

Thus the Souriau matrix of the pair (L_1, L_2) is $A_2.\overline{A_1}$; we will often use this for the decomposition of unitary (or complex hyperbolic) groups into $\mathbb{R}$-reflections. This also shows, $\mathbb{R}$-reflections being involutory, that every Souriau matrix A_i of a single $\mathbb{R}^n$-plane L_i (that is a Souriau matrix of the pair (L_0, L_i)) verifies the condition $A_i.\overline{A_i} = Id$.

Finally, it will be useful for our purposes to obtain the Souriau matrix A_i from a standard direct image matrix U_i. The relation we use is simply: $A_i = U_i.\overline{U_i^{-1}} = U_i.U_i^T$, which expresses the fact that σ_i and σ_0 are conjugated by U_i.

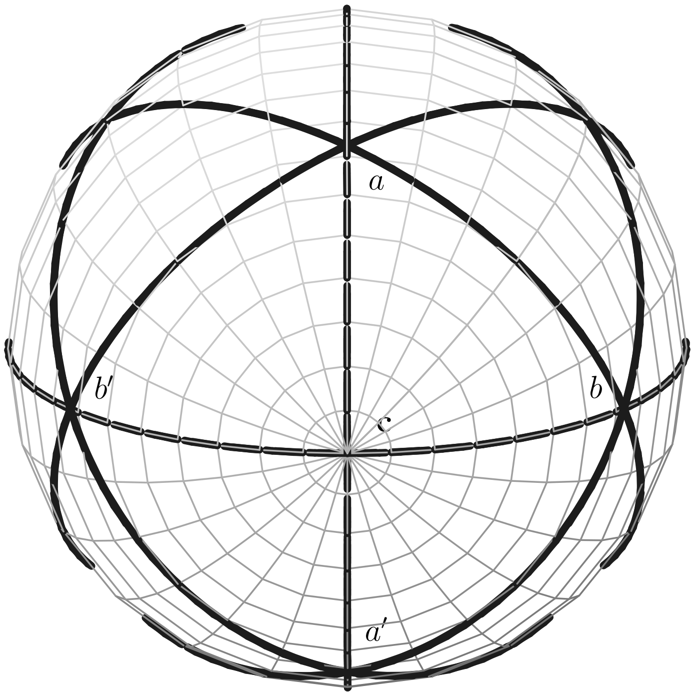


Figure 1: Projected Lagrangians for the group 3[3]3

4 An example: the group 3[3]3

The group in which we are now interested, denoted 3[3]3 by Coxeter and $(\mathbf{C}_6/\mathbf{C}_2;\mathbf{T}/\mathbf{V})$ by Du Val (type 6 in his classification), is a finite subgroup Γ of $U(2)$, of order 24, generated by two $\mathbb{C}$-reflections R_1 and R_2 in terms of which it has the presentation:

$$\Gamma =< R_1, R_2 \mid R_1^3 = R_2^3 = 1, R_1R_2R_1 = R_2R_1R_2 >$$

This group appears in Mostow as the finite subgroup fixing a point in $\mathbb{C}H^2$. We will use (especially in the determination of a fundamental domain) the fact that $Z(\Gamma)$ is cyclic of order 2, generated by $(R_1R_2)^3$.

4.1 Action on $\mathbb{C}P^1$

As we will now see, the generators R_1 and R_2 project onto two rotations $r_1, r_2 \in SO(3)$ with angles $-\frac{2\pi}{3}$ about the vertices b and b' of a spherical triangle abb' having angles $\frac{\pi}{3}$ at b and b', and $\frac{2\pi}{3}$ at the remaining vertex a (see figure 2.1). Thus, in Coxeter's notations:

$$\tilde{\pi}(3[3]3) = (3, 3, 2)$$

the latter being the group of orientation-preserving isometries of the tetrahedron (the corresponding reflection group is generated by reflections in the sides of a spherical triangle with angles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2})$).

The three vertices b, b' and a are projections in $\mathbb{C}P^1$ of eigenspaces of the matrices R_1, R_2, and R_2R_1 respectively. Our generators, slightly different from Coxeter's (his are above a and a' whereas ours are above b and b'), are explicitly:

$$R_1 = \frac{e^{\frac{i\pi}{3}}}{2}\begin{pmatrix} 1+i & 1+i \\ i-1 & 1-i \end{pmatrix}$$

$$R_2 = \frac{e^{\frac{i\pi}{3}}}{2}\begin{pmatrix} 1+i & -1-i \\ 1-i & 1-i \end{pmatrix}$$

The eigenvectors we have chosen, and whose images in $\mathbb{C}P^1$ are drawn in figure 2.1, are:

$$B^+ = \begin{bmatrix} \frac{1}{2}(1+\sqrt{3})(1-i) \\ 1 \end{bmatrix}$$

$$B'^+ = \begin{bmatrix} \frac{1}{2}(1+\sqrt{3})(i-1) \\ 1 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} \frac{1}{2}(1+\sqrt{3})(1+i) \\ 1 \end{bmatrix}$$

4.2 $\mathbb{R}$-reflections

We will now determine two triples of $\mathbb{R}$-planes which decompose the group $3[3]3$: the former corresponds to writing each of the generators R_1 and R_2 as a product of $\mathbb{R}$-reflections, whereas for the latter we use the generators R_1 and R_1R_2. Explicitly, we will find $\mathbb{R}$-planes L_1, L_2, L_3, and L_4 such that:

$$\sigma_2 \circ \sigma_1 = R_1 \ , \ \sigma_1 \circ \sigma_3 = R_2$$

$$\sigma_2 \circ \sigma_1 = R_1 \ , \ \sigma_4 \circ \sigma_2 = R_1R_2$$

The second triple of $\mathbb{R}$-planes is more canonical geometrically in the sense that it projects onto the Möbius triangle abc (type (233)) instead of the Schwarz triangle abb' (type $(3\frac{3}{2}3)$), thus possessing a kind of minimality property (among the triangles in $\mathbb{C}P^1$ bounded by the projective lagrangians which are great circles).

The idea for the explicit determination of these $\mathbb{R}$-planes is very simple: their images in $\mathbb{C}P^1$ are given, we have a free S^1-parameter in the choice of the first $\mathbb{R}$-plane, the others being completely determined by the pairwise products.

We begin with L_1, above the great circle (bb'). We need only notice that with our choice of vectors B^+ and B'^+, the hermitian product $\langle B^+, B'^+\rangle$ is real; thus $L_1 := Span_{\mathbb{R}}(B^+, B'^+)$ is an $\mathbb{R}$-plane projecting to (bb'). We will however also need the unitary matrices associated as earlier with this $\mathbb{R}$-plane, which requires a bit more calculation. Recall that we have a standard

$\mathbb{R}$-plane $L_0 = \{(x,y) \in \mathbb{C}^2 | x, y \in \mathbb{R}\}$ with associated $\mathbb{R}$-reflection $\sigma_0 : (z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$. The two (families of) unitary matrices U_1 and A_1 associated with L_1 (and its $\mathbb{R}$-reflection σ_1) are such that:

$$L_1 = U_1.L_0$$

$$\sigma_1 = A_1 \circ \sigma_0$$

As we have seen, the matrix U_1 is simply obtained by the choice of an orthonormal basis for L_1. Since we already have the basis (B^+, B'^+), we only need to orthonormalize it, for instance by a Gram-Schmidt procedure. We thus obtain:

$$U_1 = \begin{pmatrix} \frac{(1+\sqrt{3})(1-i)}{2(3+\sqrt{3})^{1/2}} & \frac{(1+\sqrt{3})(i-1)}{2(9+5\sqrt{3})^{1/2}} \\ (3+\sqrt{3})^{-1/2} & \frac{2+\sqrt{3}}{(9+5\sqrt{3})^{1/2}} \end{pmatrix}$$

We then obtain an admissible A_1 as $U_1.U_1^T$, which gives us here:

$$A_1 = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

This latter matrix is surprisingly simpler than U_1, which means that our choice of an orthonormal basis for L_1 was not very good; however we obtain the following nice parametrization:

$$L_1 = \{(e^{-\frac{i\pi}{4}}.x, y) \in \mathbb{C}^2 | x, y \in \mathbb{R}\}$$

We now proceed to find L_2 such that $\sigma_2 \circ \sigma_1 = R_1$. Rewriting this as $\sigma_2 = R_1 \circ \sigma_1$ (and applying this to coordinate-vector z as $A_2.\overline{z} = R_1.(A_1.\overline{z})$), we see that we can choose $A_2 = R_1.A_1$. Explicitly:

$$A_2 = \frac{e^{\frac{i\pi}{3}}}{2} \begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$$

We will then obtain L_2 as the set of solutions in $\mathbb{C}^2$ of $A_2.\overline{z} = z$. This can be done by diagonalizing A_2 in a real basis. We compute the eigenvalues of A_2 which are $e^{\frac{i\pi}{3}}$ and $e^{-\frac{i\pi}{6}}$, with corresponding eigenvectors:

$$V_2^{\pm} = \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix}$$

We thus obtain the parametrization:

$$L_2 = \{x.e^{\frac{i\pi}{6}}.V_2^+ + y.e^{-\frac{i\pi}{12}}.V_2^- | x, y \in \mathbb{R}\}$$

We find in the same way L_3 such that $\sigma_1 \circ \sigma_3 = R_2$. We rewrite this as $\sigma_3 = \sigma_1 \circ R_2$ and see that we can choose $A_3 = A_1.\overline{R_2}$. Explicitly:

$$A_3 = \frac{e^{-\frac{i\pi}{3}}}{2} \begin{pmatrix} -1-i & 1+i \\ 1+i & 1+i \end{pmatrix}$$

We then obtain L_3 as the set of solutions in $\mathbb{C}^2$ of $A_3.\overline{z} = z$, by diagonalizing as above A_3 in a real basis. The eigenvalues of A_3 are $e^{-\frac{i\pi}{12}}$ and $e^{\frac{5i\pi}{12}}$, with corresponding eigenvectors:

$$V_3^{\pm} = \begin{bmatrix} -1 \pm \sqrt{2} \\ 1 \end{bmatrix}$$

We obtain the parametrization:

$$L_3 = \{x.e^{-\frac{i\pi}{24}}.V_3^+ + y.e^{\frac{5i\pi}{24}}.V_3^- | x, y \in \mathbb{R}\}$$

The last $\mathbb{R}$-plane L_4 is obtained in the same way, asking that:
$\sigma_4 \circ \sigma_1 = R_1 R_2 R_1$, which gives us $A_4 = R_1.R_2.R_1.A_1$. Explicitly:

$$A_4 = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}$$

We then obtain L_4 as the set of solutions in $\mathbb{C}^2$ of $A_4.\overline{z} = z$, in the easy case where A_4 is diagonal. This gives us the following parametrization :

$$L_4 = \{(i.x, e^{\frac{i\pi}{4}}.y) \in \mathbb{C}^2 | x, y \in \mathbb{R}\}$$

4.3 Presentations and diagrams

Recall the above presentation for the unitary group 3[3]3:

$$\Gamma = \langle R_1, R_2 \mid R_1^3 = R_2^3 = 1, R_1 R_2 R_1 = R_2 R_1 R_2 \rangle$$

Two presentations for $\widehat{\Gamma}$, the index 2 supergroup of Γ generated by three $\mathbb{R}$-reflections, given by the two systems of generators above, are easily derived (we simply rewrite the relations in terms of the $\mathbb{R}$-reflections, and add that these are of order 2). We obtain explicitly in the first case:

$$\widehat{\Gamma} = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_i^2 = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_3)^3 = 1, \sigma_1.\sigma_2\sigma_3\sigma_2.\sigma_1 = \sigma_3\sigma_2\sigma_3 \rangle$$

which implies that $\sigma_2\sigma_3$ has order 6, and in the second case :

$$\widehat{\Gamma} = \langle \sigma_1, \sigma_2, \sigma_4 \mid \sigma_i^2 = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_4\sigma_2)^3 = 1, \sigma_1\sigma_4\sigma_1 = \sigma_2.\sigma_4\sigma_2\sigma_4.\sigma_2 \rangle$$

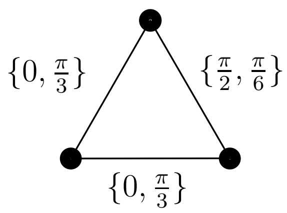
which implies as above that $\sigma_2\sigma_4$ has order 6.

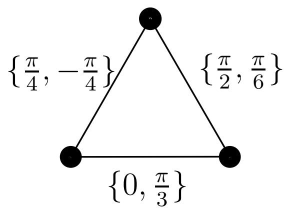
We now write down for each of the Lagrangian triples the Coxeter-style diagrams we have introduced earlier, which contain the information characterizing each of the three intersecting pairs, that is a pair of angles (written on the edge joining the generators). Recall that these pairs are characterized by the product of the two $\mathbb{R}$-reflections, in our case:

$$\sigma_2 \circ \sigma_1 = R_1 \ , \ \sigma_1 \circ \sigma_3 = R_2$$

$$\sigma_2 \circ \sigma_1 = R_1 \ , \ \sigma_4 \circ \sigma_2 = R_1 R_2$$

Computing the eigenvalues of the four corresponding matrices, we find the following diagrams for our two Lagrangian triples:





4.4 Fundamental domains

We now determine fundamental domains for the action of $\widehat{\Gamma}$ (and thus Γ) on $\mathbb{C}^2$, or rather on its unit ball $B^2_{\mathbb{C}}$ (in the perspective of complex hyperbolic geometry). We use the action of these groups on $\mathbb{C}P^1$ to determine these fundamental domains as (parts of) circle bundles over spherical triangles ; we thus obtain domains essentially different from Mostow's construction, the faces of our "polyhedra" not being bisectors.

Our groups Γ and $\widehat{\Gamma}$ are (finite) subgroups of $U(2)$ (resp. $\widehat{U(2)}$), so they fix the origin in $\mathbb{C}^2$ and we can choose a fundamental domain D^4 for their action on $B^2_{\mathbb{C}}$ as a cone (in half $\mathbb{R}$-lines) over a fundamental domain D^3 for their action on $\partial B^2_{\mathbb{C}} = S^3$, which we like to look at as the Heisenberg group H^3 for the same reason as above. We then determine D^3 using the Hopf fibration $H : S^3 \longrightarrow \mathbb{C}P^1$ induced by complex projectivisation; the actions of $\widehat{\Gamma}$ and Γ on S^3 and $\mathbb{C}P^1$ being equivariant with respect to H, a domain D^3 is obtained as a part of the Hopf bundle over a spherical triangle D^2, a fundamental domain in $\mathbb{C}P^1$: $D^3 \subset H^{-1}(D^2)$. We have seen that $\widehat{\Gamma}$ acts on $\mathbb{C}P^1$ as the reflection group of type (3,3,2) (with for instance the Möbius triangle abc for D^2), and that its index 2 subgroup Γ acts on $\mathbb{C}P^1$ as the rotation group of type (3,3,2) (with for instance the Schwarz triangle abb' for D^2).

It now remains to see exactly which part of this triangular prism we want to keep. Each fiber is a $\mathbb{C}$-circle (the intersection of a complex line with $\partial B^2_{\mathbb{C}}$), and the fiberwise action which remains is the one which was invisible in $\mathbb{C}P^1$, that is the action of the kernel of $\tilde{\pi} : \Gamma \longrightarrow SO(3)$, which is none other than the order 2 center $Z(\Gamma)$, acting on each fiber by a half-turn. The domain D^3 will thus be a union of half-circles obtained by cutting in half the triangular prism $H^{-1}(D^2)$, forming a sort of polyhedron with five faces (compare with the ten bisector faces obtained by Mostow). Three of these faces are (contained in something) canonical, that is the union of $\mathbb{C}$-circles above the boundary of the spherical triangle. It remains to choose how to perform the cutting to obtain the two remaining faces in a reasonable fashion (the least we can ask is that they be smooth). The best would be to use surfaces that are geometrically natural in $\partial B^2_{\mathbb{C}}$, such as bisectors [G], however in the perspective of assembling different copies of these domains it can be useful to bear in mind the flexibility that we have in the choice of these faces.

5 Other two-generator subgroups of $U(2)$

The method used here relies on the fact that the unitary group which we want to decompose admits a system of two generators, and in that case is a straightforward generalization of the example of 3[3]3. We will describe this method in a general manner, but will apply it explicitly only in two cases where the pairs of generators are given by Coxeter (in quaternionic form) namely:

- the binary polyhedral groups (that is all non-cyclic finite subgroups of $SU(2)$, [C] pp. 74-78)
- the finite groups generated by two $\mathbb{C}$-reflections ([C] pp. 98-102 and table p. 176)

These groups seem sufficiently representative, as they cover all but type 4 (and type 1 which has cyclic action on $\mathbb{C}P^1$) of Du Val's 9 types.

5.1 The strategy

We start off with a finite subgroup Γ of $U(2)$ with two explicit generators G_1 and G_2, and proceed to find three $\mathbb{R}$-reflections σ_i, σ_j, and σ_k such that $\widehat{\Gamma} = \langle \sigma_i, \sigma_j, \sigma_k \rangle$ contains Γ with index two.

5.1.1 Action on $\mathbb{C}P^1$

We begin by calculating the eigenvectors of G_1 and G_2, which give us the two (pairs of antipodal) fixed points in $\mathbb{C}P^1$; these and the eigenvalues tell us the type of action on $\mathbb{C}P^1$ (that is dihedral, tetrahedral, octahedral, or icosahedral). We only need to do this once for each type, for instance only for the binary polyhedral groups $\langle p, 2, 2 \rangle, \langle 3, 3, 2 \rangle, \langle 4, 3, 2 \rangle$ and $\langle 5, 3, 2 \rangle$.

5.1.2 $\mathbb{R}$-reflections

Given the two (pairs of antipodal) fixed points, we know from Prop. 5 that any $\mathbb{R}$-plane which decomposes simultaneously G_1 and G_2 projects to the only great circle in $\mathbb{C}P^1$ containing these points. Such an $\mathbb{R}$-plane is thus determined up to multiplication by $U(1)$, and the two remaining $\mathbb{R}$-planes are uniquely determined by this choice.

5.1.3 Fundamental domains

We have thus obtained three $\mathbb{R}$-planes whose projections to $\mathbb{C}P^1$ comprise a triangle which is part of a triangular tessellation of the sphere. Now this triangle is either an elementary triangle of this tessellation (if it is a Möbius triangle, i.e. with angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$ with $p, q, r \in \mathbb{N}$), or a union of two or more of such triangles (it is then a Schwarz triangle, with angles $(\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r})$ where one of p, q, r is in $\mathbb{Q} \setminus \mathbb{N}$). This can be seen graphically for instance. If the triangle is of Möbius type, it is a fundamental domain D^2 for the action of $\widehat{\Gamma}$ on $\mathbb{C}P^1$ (and two adjacent copies form a fundamental domain for the action of Γ on $\mathbb{C}P^1$). If not, we change a well-chosen reflection by conjugation, as in the example of 3[3]3, to obtain a Möbius triangle (or "minimal" triple of $\mathbb{R}$-planes).

We then proceed exactly as in the example, constructing the triangular prism $H^{-1}(D^2)$ in S^3 and cutting it into $|Z(\Gamma)|$ "slices" according to the cyclic action of $Z(\Gamma)$ on each fiber ($\mathbb{C}$-circle); any of these slices is then a fundamental domain D^3 for the action of Γ (resp. $\widehat{\Gamma}$) on S^3. The final fundamental domain D^4 in $\mathbb{C}^2$ (or in the unit ball of $\mathbb{C}^2$) is then simply a cone (in half-$\mathbb{R}$-lines) over D^3 based at the origin.

We have thus obtained D^4 as a 5-sided polyhedral cone over D^3, whose 5 faces are as follows:

- three "vertical" faces are parts of canonical objects, the union of $\mathbb{C}$-circles over a geodesic segment in $\mathbb{C}P^1$. Such an object is foliated both by $\mathbb{C}$-circles and $\mathbb{R}$-circles, and is part of a "Clifford torus", see [G]. These faces are four-sided.
- two "horizontal" faces (bisectors, see [G], are possible choices or other surfaces, see [S]). These are three-sided.

5.2 Explicit decomposition of reflection groups

We now find triples of $\mathbb{R}$-planes for the finite subgroups of $SU(2)$ and the two-generator reflection groups in $U(2)$; we proceed according to type of action on $\mathbb{C}P^1$. The results are summarized in the following tables, where we use the digit notation for words in the $\mathbb{R}$-reflections σ_i; for instance "1321" denotes the unitary elment $\sigma_1 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1$. For the $\mathbb{R}$-plane L_7, τ denotes the golden ratio $\frac{1+\sqrt{5}}{2}$.

The details for all these groups can be found in the appendix; for the sake of clarity and completeness, we will write the details for the exceptional family stated on the last line of table 2, and for this we need the decomposition of the binary dihedral group $\langle p, 2, 2\rangle$ given in the first line of the same table.

For this group $\langle p, 2, 2\rangle$, Coxeter gives the two quaternion generators $e^{i\pi/p}$ and j, corresponding to the following matrices in $SU(2)$:

$$G_1 = \begin{pmatrix} e^{i\pi/p} & 0 \\ 0 & e^{-i\pi/p} \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We apply the method described earlier to simultaneously decompose these two generators: the fixed points in $\mathbb{C}P^1$ of these elements (the projection of their eigenvectors) are, in the notation of the figure on page 13, c and $A^{-1}(c)$; the great circle containing these points is $l_{101} = \sigma_1(l_0)$, so that an $\mathbb{R}$-plane decomposing simultaneously G_1 and G_2 can only be some multiple of $L_{101} = \sigma_1(L_0)$. We thus need the Souriau matrix A_{101} of $\mathbb{R}$-reflection $\sigma_1\sigma_0\sigma_1$ (denoted 101), which is simply:

$$A_{101} = A_1 \overline{A_0} A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbb{R}$-plane	Souriau matrix	Orthogonal basis for the $\mathbb{R}$-plane
L_0	$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
L_1	$A_1 = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$	$e^{-\frac{i\pi}{4}}.\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
L_2	$A_2 = \frac{e^{\frac{i\pi}{3}}}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$	$e^{\frac{i\pi}{6}}.\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-\frac{i\pi}{12}}.\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
L_3	$A_3 = \frac{e^{-\frac{i\pi}{3}}}{2}\begin{pmatrix} -1-i & 1+i \\ 1+i & 1+i \end{pmatrix}$	$e^{-\frac{i\pi}{24}}.\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}, e^{\frac{5i\pi}{24}}.\begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$
L_4	$A_4 = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}$	$i.\begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^{\frac{i\pi}{4}}.\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
L_5	$A_5 = e^{-\frac{i\pi}{12}}.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$e^{-\frac{i\pi}{24}}.\begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^{-\frac{i\pi}{24}}.\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
L_6	$A_6 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$e^{\frac{i\pi}{4}}.\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-\frac{i\pi}{4}}.\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
L_7	$A_7 = \frac{1}{2}\begin{pmatrix} \tau+\frac{i}{\tau} & i \\ i & \tau-\frac{i}{\tau} \end{pmatrix}$	$e^{\frac{i\pi}{6}}.\begin{bmatrix} \tau-1+2\sin\frac{\pi}{5} \\ 1 \end{bmatrix}, e^{-\frac{i\pi}{6}}.\begin{bmatrix} \tau-1-2\sin\frac{\pi}{5} \\ 1 \end{bmatrix}$
$L_8(p)$	$A_8(p) = \begin{pmatrix} -e^{-\frac{i\pi}{p}} & 0 \\ 0 & e^{\frac{i\pi}{p}} \end{pmatrix}$	$-e^{-\frac{i\pi}{2p}}.\begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^{\frac{i\pi}{2p}}.\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
L_9	$A_9 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, i.\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Figure 2: $\mathbb{R}$-planes for finite groups of $U(2)$

We thus have, for suitably chosen $\mathbb{R}$-planes L_8 and L_9:

$$\begin{cases} G_1 = 101.8 \\ G_2 = 101.9 \end{cases}$$

We sum this up in the last column of the table by the notation $(101; 8; 9)$. Now we almost have the decomposition of the exceptional family $\langle 2q, 2, 2\rangle_{2m'}$, because it is obtained from $\langle 2q, 2, 2\rangle$ by simply adding the generator $G_3 = e^{i\pi/2m'}.Id$, which we denoted earlier $a = (e^{i\pi/2m'}, 1)$. Since we have in an obvious manner: $G_3 = e^{i\pi/2m'}.\sigma_8.\sigma_8$ (for instance), we obtain as claimed a decomposition of the exceptional family with 4 $\mathbb{R}$-reflections $(101; 8; 9; e^{i\pi/2m'}.8)$.

References

[CS] J. H. Conway, D. A. Smith ; On Quaternions and Octonions. K. Peters. 2003.

[C] H. S. M. Coxeter ; Regular Complex Polytopes. Cambridge University Press (1991).

Γ	Action on $\mathbb{C}P^1$	Du Val type	Order; order of center	Generators of $\widehat{\Gamma}$
$\mathbf{D}_p = \langle p, 2, 2\rangle$	D_p	2	$4p$; 2	$(101; 8; 9)$
$\mathbf{T} = \langle 3, 3, 2\rangle$	T	5	24; 2	$(1; e^{-i\pi/3}.2; -4)$
$\mathbf{O} = \langle 4, 3, 2\rangle$	O	7	48; 2	$(5; e^{-5i\pi/6}.4; i.121)$
$\mathbf{I} = \langle 5, 3, 2\rangle$	I	9	120; 2	$(0; 6; 7)$
$p[4]2$	D_p	3 or 4	$2p^2$; p	$(101; e^{-i\pi/p}.8; -i.9)$
$3[4]3 = \langle 3, 3, 2\rangle_3$	T	5	72; 6	$(121; e^{-i\pi/3}.131; 1)$
$3[3]3$	T	6	24; 2	$(1; 2; 3)$, $(1; 2; 4)$
$3[6]2$	T	6	48; 4	$(4; e^{2i\pi/3}.121; i.1)$
$4[6]2 = \langle 4, 3, 2\rangle_4$	O	7	192; 8	$(5; e^{-7i\pi/12}.4; 121)$
$4[3]4$	O	8	96; 4	$(5; e^{-7i\pi/12}.4; e^{i\pi/12}.12121)$
$3[8]2$	O	8	144; 6	$(121; e^{-2i\pi/3}.4; 5)$
$4[4]3$	O	8	288; 12	$(4; e^{7i\pi/12}.5; e^{2i\pi/3}.121)$
$3[5]3 = \langle 5, 3, 2\rangle_3$	I	9	360; 6	$(6; e^{-i\pi/3}.7; e^{2i\pi/3}.070)$
$5[3]5 = \langle 5, 3, 2\rangle_5$	I	9	600; 10	$(0; e^{i\pi/5}.7; e^{4i\pi/5}.676)$
$3[10]2 = \langle 5, 3, 2\rangle_6$	I	9	720; 12	$(6; e^{-i\pi/3}.7; -i.0)$
$5[6]2 = \langle 5, 3, 2\rangle_{10}$	I	9	1200; 20	$(0; e^{i\pi/5}.7; i.6)$
$5[4]3 = \langle 5, 3, 2\rangle_{15}$	I	9	1800; 30	$(7; e^{-i\pi/5}.0; e^{i\pi/3}.6)$
$\langle 2q, 2, 2\rangle_{2m'}$	D_{2q}	2	$32m'q$; $4m'$	$(101; 8; 9; e^{i\pi/2m'}.8)$

Figure 3: Finite groups and $\mathbb{R}$-reflections. We list the irreducible subgroups of $SU(2)$, then the complex reflection groups generated by two reflections and finally the imprimitive exceptional family which is a subgroup of index two of a group generated by four Lagrangian reflections.

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