University of Utah
Math 1220, Fall 2007
Name: Solutions

Midterm exam # 2
Time: 50 minutes

Please try to carefully explain the steps leading to your solutions.

Part 1: (8 points) Consider the function $f(x) = x^2 e^{-x}$.

1. What is the limit of $f(x)$ as $x \longrightarrow +\infty$? as $x \longrightarrow -\infty$?
2. Find an antiderivative of $f(x)$. **Bonus (1 point)**: Check your answer.
3. Does the integral $\int_0^{+\infty} x^2 e^{-x} dx$ converge? If it does, find its value.

1) $f(x) = x^2 \cdot e^{-x} = \frac{x^2}{e^x}$ As $x \to +\infty$, this is an indeterminate form of the type "$\frac{\infty}{\infty}$". We can use L'Hôpital's rule twice to find the limit:

$$\lim_{x\to+\infty} \frac{x^2}{e^x} \underset{(L)}{=} \lim_{x\to+\infty} \frac{2x}{e^x} \underset{(L)}{=} \lim_{x\to+\infty} \frac{2}{e^x} = 0$$

You could also just say that "e^x dominates any power of x".
As $x \to -\infty$, $f(x) \to +\infty$ because $x^2 \xrightarrow[x\to-\infty]{} +\infty$ and $e^{-x} \xrightarrow[x\to-\infty]{} +\infty$.

2) Integrate twice by parts: $\int x^2 e^{-x} dx = -x^2 \cdot e^{-x} + \int 2x \cdot e^{-x} dx$

$\boxed{u(x) = x^2 \to u'(x) = 2x \quad v'(x) = e^{-x} \to v(x) = -e^{-x}}$ $\boxed{u(x) = 2x \to u'(x) = 2 \quad v'(x) = e^{-x} \to v(x) = -e^{-x}}$

$= -x^2 e^{-x} - 2x e^{-x} - \int 2e^{-x} dx$

$= e^{-x}(-x^2 - 2x - 2) \; (+C)$ Check: $\left[e^{-x}(-x^2-2x-2)\right]' = e^{-x}(-2x-2) - e^{-x}(-x^2-2x-2)$
$= e^{-x} \cdot x^2$ ✓

3) $\int_0^{+\infty} x^2 e^{-x} dx = \lim_{b\to+\infty} \int_0^b x^2 \cdot e^{-x} dx = \lim_{b\to+\infty} \left[e^{-x}(-x^2-2x-2)\right]_0^b$ by question #2)
$= 2$ because $\lim_{x\to+\infty} e^{-x} \cdot x^n = 0$ for any n (see question 1)

Part 2: (3 points) Does the series $\Sigma_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ converge? If it does, find its value.
Bonus (2 points): Explain how to cut a cake into 3 identical pieces if you know how to cut any piece in 2.

* This is a geometric series of ratio $-\frac{1}{2}$; we know that it converges (because $\left|-\frac{1}{2}\right| < 1$) and that its sum is: $\sum_{n=0}^{+\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$

* We can get $\frac{2}{3}$ of the cake by using this series: $\frac{2}{3} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

Part 3: (5 points) Evaluate the integral:

$$\int_0^{\pi/2} \frac{\cos x}{\sin^2 x - 5\sin x + 6}\,dx$$

Start by using the substitution: $\begin{cases} u(x) = \sin x \\ du = \cos x \cdot dx \end{cases}$:

$$\int_0^{\pi/2} \frac{\cos x}{\sin^2 x - 5\sin x + 6} = \int_0^1 \frac{du}{u^2-5u+6}$$

Now we can factor the denominator as $u^2-5u+6 = (u-2)(u-3)$

and use partial fractions: $\dfrac{1}{u^2-5u+6} = \dfrac{A}{u-2} + \dfrac{B}{u-3}$

Find A and B: $1 = A\cdot(u-3) + B(u-2)$, evaluate at $u=2$: $A=-1$
and at $u=3$: $B=1$

Therefore: $$\int_0^1 \frac{du}{u^2-5u+6} = \int_0^1 \frac{du}{u-3} - \int_0^1 \frac{du}{u-2}$$

$$= \Big[\ln|u-3|\Big]_0^1 - \Big[\ln|u-2|\Big]_0^1$$ (because $\frac{1}{u-2}$ and $\frac{1}{u-3}$ are continuous on the interval $[0;1]$)

$$= \ln 2 - \ln 3 - \ln 1 + \ln 2 = 2\ln 2 - \ln 3$$

Part 4: (4 points) (a) Find the limit of $\frac{\ln(1+x)}{x}$ as $x \longrightarrow 0$. (b) Find the limit of the sequence $a_n = (1+\frac{1}{n})^n$. Hint: consider $\ln a_n$ and use part (a).

(a) This is an indeterminate form "$\frac{0}{0}$"; use L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

(b) Consider $b_n = \ln a_n = \ln\left(\left(1+\frac{1}{n}\right)^n\right) = n \cdot \ln\left(1+\frac{1}{n}\right) = \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$

As $n \longrightarrow +\infty$, $\frac{1}{n} \longrightarrow 0$ so we can use part (a) (think of $\frac{1}{n}$ as x)

and we get: $\ln a_n = \dfrac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow[n\to+\infty]{} 1$

Therefore: $a_n = e^{\ln a_n} \xrightarrow[n\to+\infty]{} e^1 = e$