University of Utah
Math 2210, Fall 2008
Name: Key

Midterm Exam # 2
Time: 50 minutes

No calculators, notes, books. Show all work.

Consider the function $f(x,y) = (2x + y^2).e^{-x^2}$ and its graph S. **(1) (5 points)** Find the first partial derivatives and gradient of f. **(2) (4 points)** (a) Find an equation for the tangent plane to S at the point $(0,-1,1)$. (b) What is the directional derivative of f at $(0,-1)$ in the direction of the vector $\mathbf{v} =< 3,4 >$? (c) What is the maximal directional derivative of f at $(0,-1)$? **(3) (9 points)** Find the critical points of f, then find the local maxima and minima of f. **Bonus (2 points)**: are these also global maxima/minima? **(4) (2 points)** If $x = st$ and $y = s + t$, find the partial derivatives $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

(1) $\dfrac{\partial f}{\partial x} = 2e^{-x^2} - 2x(2x+y^2)e^{-x^2} = (2-4x^2-2xy)e^{-x^2}$; $\dfrac{\partial f}{\partial y} = 2ye^{-x^2}$

Therefore: $\vec{\nabla} f(x,y) = \langle (2-4x^2-2xy)e^{-x^2}, 2ye^{-x^2} \rangle$.

(2)(a) $\vec{\nabla} f(0,-1) = \langle 2,-2 \rangle$ so an equation for the tangent plane is:

$$z-1 = 2x - 2(y+1) \quad \text{or} \quad z = 2x-2y-1$$

(b) Recall that for a unit vector $\vec{u}$, the directional derivative of f at (x,y) in the direction of $\vec{u}$ is: $D_{\vec{u}} f(x,y) = \vec{u} \cdot \vec{\nabla} f(x,y)$.

We first rescale $\vec{v}$ to have unit length: $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$; then:

$$D_{\vec{v}} f(0,-1) = D_{\vec{u}} f(0,-1) = \vec{u} \cdot \vec{\nabla} f(0,-1) = 2\times\frac{3}{5} - 2\times\frac{4}{5} = -\frac{2}{5}.$$

(c) We know that the maximal directional derivative is $\|\vec{\nabla} f\|$ (in the direction of $\vec{\nabla} f$). Here: $\|\vec{\nabla} f\| = \sqrt{4+4} = 2\sqrt{2}$

(3) For the critical points, solve $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial y}(x,y) = 0$ for x and y.

This gives: $(y=0)$ and $(2-4x^2=0)$.

Therefore f has two critical points: $(\frac{1}{\sqrt{2}}, 0) = P^+$ and $(-\frac{1}{\sqrt{2}}, 0) = P^-$

We then use the second partial derivative test to determine whether or not f has a local maximum or minimum at P^+ and P^-.

We start by computing the second partial derivatives of f:

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2} = (-8x-2y^2)e^{-x^2} - 2x(2-4x^2-2xy^2)e^{-x^2} = (-12x-2y^2+8x^3+4x^2y^2)e^{-x^2} \\ \dfrac{\partial^2 f}{\partial x\partial y} = \dfrac{\partial^2 f}{\partial y\partial x} = -4xy\,e^{-x^2} \\ \dfrac{\partial^2 f}{\partial y^2} = 2e^{-x^2} \end{cases}$$

Evaluating these 3 functions at $p^+ = \left(\frac{1}{\sqrt{2}}, 0\right)$ and $p^- = \left(-\frac{1}{\sqrt{2}}, 0\right)$ gives:

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2}\left(\frac{1}{\sqrt{2}}, 0\right) = 4x(2x^2-3)e^{-x^2} = -4\sqrt{2}e^{-1/2} \\ \dfrac{\partial^2 f}{\partial x\partial y}\left(\frac{1}{\sqrt{2}}, 0\right) = 0 \\ \dfrac{\partial^2 f}{\partial y^2}\left(\frac{1}{\sqrt{2}}, 0\right) = 2e^{-1/2} \end{cases} \qquad \begin{cases} \dfrac{\partial^2 f}{\partial x^2}\left(-\frac{1}{\sqrt{2}}, 0\right) = 4\sqrt{2}e^{-1/2} \\ \dfrac{\partial^2 f}{\partial x\partial y}\left(-\frac{1}{\sqrt{2}}, 0\right) = 0 \\ \dfrac{\partial^2 f}{\partial y^2}\left(-\frac{1}{\sqrt{2}}, 0\right) = 2e^{-1/2} \end{cases}$$

Determinant "D" (book) or "Δ" (class) $= \dfrac{\partial^2 f}{\partial x^2}\cdot\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x\partial y}\right)^2 = -8\sqrt{2}e^{-1} < 0$

f does not have a local min. or max at p^+ ("Saddle point").

"D" or "Δ" $= 8\sqrt{2}e^{-1} > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0 \Rightarrow f$ has a local minimum at $p^- = \left(-\frac{1}{\sqrt{2}}, 0\right)$.

Min. value $= f(p^-) = -\sqrt{2}e^{-1/2}$

<u>Bonus</u>: Along vertical lines ($x =$ constant), $f(x,y) \xrightarrow[y\to\pm\infty]{} \infty$.

In all other directions (think lines $y = mx+p$), $f(x,y) \longrightarrow 0$ (e^{-x^2} dominates).

Therefore the local min. at p^- is also a global minimum for f.

4) Write the Chain Rule:

$$\begin{cases} \dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial s} \\ \dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial t} \end{cases}$$

So:

$$\begin{cases} \dfrac{\partial f}{\partial s} = (2-4x^2-2xy^2)e^{-x^2}\cdot t + 2ye^{-x^2}\cdot 1 = (2-4s^2t^2-2st(s+t)^2)e^{-s^2t^2}\cdot t + 2(s+t)e^{-s^2t^2} \\ \dfrac{\partial f}{\partial t} = (2-4x^2-2xy^2)e^{-x^2}\cdot s + 2ye^{-x^2}\cdot 1 = (2-4s^2t^2-2st(s+t)^2)e^{-s^2t^2}\cdot s + 2(s+t)e^{-s^2t^2} \end{cases}$$