University of Utah
Math 1220, Fall 2007
Name: Solutions

Midterm exam # 1
Time: 50 minutes

Please try to carefully explain the steps leading to your solutions.

Part 1: (2 points) Simplify the expression $f(x) = \ln(x^x)$ $(x > 0)$ and find its derivative.

* $f(x) = \ln(x^x) = x \cdot \ln(x)$ because $\ln(a^n) = n \cdot \ln a$
(you can also see this by noting that $x^x = e^{x \cdot \ln x}$)
* $f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$ (product rule)
$= \ln x + 1$

Part 2: (8 points) Find all functions $y(x)$ satisfying the differential equation:

$$xy' + 2y = e^{2x^2}$$

Bonus (2 points): Check your answer.

This is a first-order linear differential equation, which we can solve with the method of the "integrating factor".
First, rewrite the equation as: $y' + \underbrace{\frac{2}{x}}_{p(x)} \cdot y = \frac{e^{2x^2}}{x}$

The integrating factor is $e^{2\ln x} = x^2$ $\left(p(x) = \frac{2}{x},\ H(x) = \int p(x)\,dx = 2\ln x\right)$, so we multiply the whole equation by x^2:

$$\underbrace{x^2 y' + 2xy}_{} = x e^{2x^2}$$

$[x^2 y]' = x e^{2x^2}$ which we solve by integrating the right-hand side:

$\begin{cases} u = 2x^2 \\ du = 4x\,dx \end{cases}$

$$x^2 y = \int x \cdot e^{2x^2} dx = \frac{1}{4} \int e^u du = \frac{1}{4} e^u + C = \frac{1}{4} e^{2x^2} + C$$

therefore: $\boxed{y(x) = \frac{e^{2x^2}}{4x^2} + \frac{C}{x^2}}$ is the general solution.

Check: With this expression for $y(x)$, we evaluate $y'(x)$ and plug y and y' into the equation:

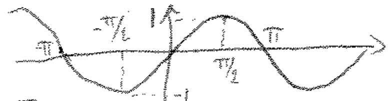
$$y'(x) = \left[\frac{e^{2x^2}}{4x^2} + \frac{C}{x^2}\right]' = \frac{4xe^{2x^2} \cdot 4x^2 - 8xe^{2x^2}}{16x^4} - \frac{2C}{x^3} \text{ (quotient rule)}$$

$$= \frac{e^{2x^2}}{x} - \frac{e^{2x^2}}{2x^3} - \frac{2C}{x^3}$$

therefore: $x \cdot y'(x) + 2y(x) = x\left[\frac{e^{2x^2}}{x} - \frac{e^{2x^2}}{2x^3} - \frac{2C}{x^3}\right] + 2\left[\frac{e^{2x^2}}{4x^2} + \frac{C}{x^2}\right] = e^{2x^2}$ ✓

Part 3: (4 points) Consider the function $g(x) = \sin^{-1}(x)$. (a) What are its domain and range? (b) Sketch the graph of g. (c) What is the derivative of g? (d) What is $g(\sin(4\pi))$?

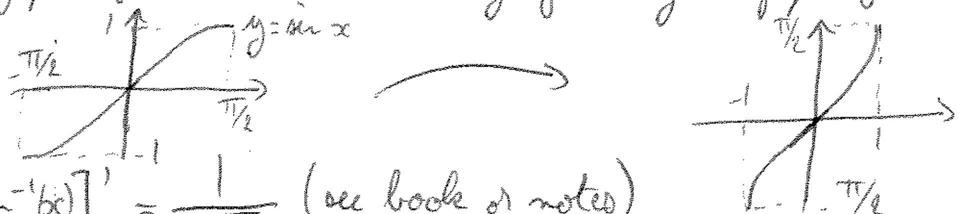
The graph of $\sin(x)$ looks like:



so we restrict $\sin(x)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to obtain a one-to-one (invertible) function.

(a) The domain of $\sin^{-1} x$ is $[-1;1]$; its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) The graph of $\sin^{-1}(x)$ is obtained by reflecting the graph of $\sin(x)$ along the diagonal



Note the tangents: vertical at endpoint slope 1 at 0.

(c) $\left[\sin^{-1}(x)\right]' = \frac{1}{\sqrt{1-x^2}}$ (see book or notes)

(d) $g(\sin(4\pi)) = \sin^{-1}(\sin(4\pi)) = \sin^{-1}(0) = 0$ (Not 4π! This must be a value in $[-\pi/2, \pi/2]$)

Part 4: (6 points) Evaluate the integral:

$$\int \frac{e^x}{2 + 8e^{2x}} dx$$

Bonus (1 point): Check your answer.

Recall that $e^{2x} = (e^x)^2$, so we can use the substitution: $\begin{cases} u = e^x \\ du = e^x \cdot dx \end{cases}$

$$\int \frac{e^x}{2+8e^{2x}} dx = \int \frac{du}{2+8u^2}$$

make this into $\int \frac{dv}{1+v^2}$ by the substitution $\begin{cases} v = 2u \\ dv = 2du \end{cases}$

$$= \frac{1}{2}\int \frac{du}{1+4u^2} = \frac{1}{4}\int \frac{dv}{1+v^2}$$

$$= \frac{1}{4}\tan^{-1}(v) + C = \boxed{\frac{1}{4}\tan^{-1}(2e^x) + C}$$

Check: $\left[\frac{1}{4}\tan^{-1}(2e^x)\right]' = \frac{1}{4} \cdot \frac{1}{1+(2e^x)^2} \cdot 2e^x$ (chain rule)

$$= \frac{e^x}{2 + 8e^{2x}} \quad \checkmark$$