University of Utah
Math 1210, Spring 2008
Name: Solutions

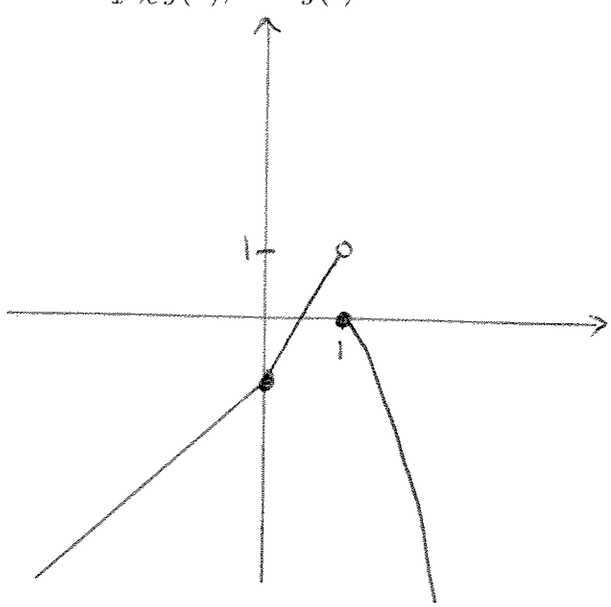
Midterm Exam # 1

Time: 50 minutes

No books, notes, calculators. Show all work.

Part 1: (6 points) Consider the function g defined by $g(x) = x - 1$ if $x \leqslant 0$, $g(x) = 2x - 1$ if $0 < x < 1$, and $g(x) = -x^2 + 1$ if $x \geqslant 1$. (a) Sketch the graph of g. (b) For $c = 0$ then $c = 1$ find each of the following (or state that it does not exist): $\lim_{x\to c^-} g(x)$, $\lim_{x\to c^+} g(x)$, $\lim_{x\to c} g(x)$, and $g(c)$.

(a)



(b) From the graph:

$\lim_{x\to 0^-} g(x) = -1$

$\lim_{x\to 0^+} g(x) = -1$

$\lim_{x\to 0} g(x) = -1$

$g(0) = -1$

$\lim_{x\to 1^-} g(x) = 1$

$\lim_{x\to 1^+} g(x) \quad 0$

$\lim_{x\to 1} g(x)$: does not exist (because limit on the left and limit on the right are different)

$g(1) = 0$

Part 2: (4 points) Find the following limits: $\lim_{x\to 0} \frac{1-\cos 3x}{x}$, $\lim_{x\to 0^+} \frac{\sin x}{x^2}$, and $\lim_{x\to 0^-} \frac{\sin x}{x^2}$.

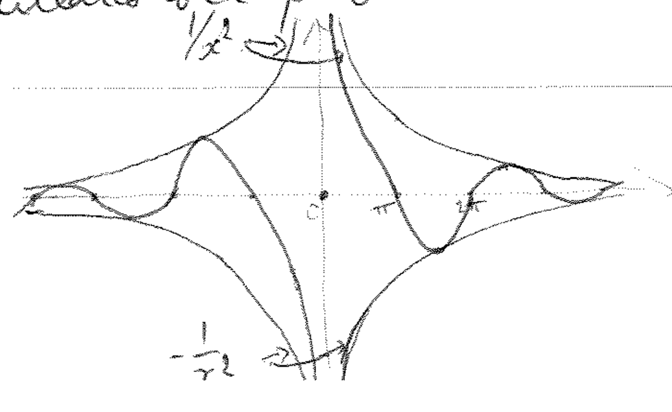
Bonus question: (2 points) Sketch the graph of $h(x) = \frac{\sin x}{x^2}$.

* $\lim_{x\to 0} \frac{1-\cos 3x}{x} = \lim_{x\to 0} 3 \cdot \frac{1-\cos 3x}{3x} = 3 \cdot 0 = 0$

* $\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \frac{1}{x} = 1 \cdot (+\infty) = +\infty$

* $\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\sin x}{x} \cdot \frac{1}{x} = 1 \cdot (-\infty) = -\infty$

Bonus: The function $\frac{\sin x}{x^2}$ oscillates with "period" 2π between the graphs of $\frac{1}{x^2}$ and $-\frac{1}{x^2}$:



Part 3: (10 points) Consider the function $f(x) = \frac{3x}{x^2-x-2}$. The goal of this problem is to give a reasonable sketch for the graph of f. The following steps should help: (a) Find the domain of f. (b) Is f even? odd? (c) Find the $x-$ and $y-$intercepts of f. (d) Find horizontal and vertical asymptotes, if any. (e) At $\pm\infty$, determine if the graph is above or below the horizontal asymptote. Find the limits to the left and right of each vertical asymptote. (f) Sketch the graph of f. **Bonus question:** (2 points) Sketch the graph of $j(x) = \frac{1}{f(x)}$.

$f(x) = \dfrac{3x}{x^2-x-2} = \dfrac{3x}{(x+1)(x-2)}$

(a) Domain of f is $\{x / x \neq -1 \text{ and } 2\}$ $= (-\infty;-1) \cup (-1;2) \cup (2;\infty)$

b) $f(-x) = \dfrac{-3x}{(-x)^2-(-x)-2} = \dfrac{-3x}{x^2+x-2} \neq f(x)$ and $\neq -f(x)$

so f is not even nor odd.

(c) y-intercept $= f(0) = 0$; x-intercepts $= \{x$ such that $f(x) = 0\} = \{0\}$.

(d) The denominator is 0 for $x = -1$ and 2 (and the numerator is not 0), so there are 2 vertical asymptotes: $(x=-1)$ and $(x=2)$

In order to find a horizontal asymptote if there is one, one must find $\lim\limits_{x\to\pm\infty} f(x)$: $\lim\limits_{x\to\pm\infty} \dfrac{3x}{x^2-x-2} = \lim\limits_{x\to\pm\infty} \dfrac{3x}{x^2}$ (highest degree terms)

$= 0$

Therefore there is a horizontal asymptote: $(y=0)$.

(e) At $+\infty$ (for large positive x), $f(x) > 0$ so the graph is above the asymptote.

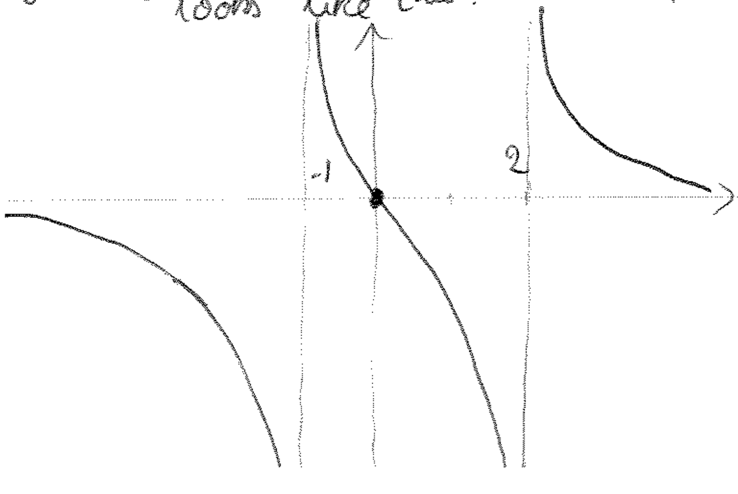
At $-\infty$ (for large negative x), $f(x) < 0$ so ——— below ———.

To find these and the following limits, you need only to analyze the sign of $f(x)$:

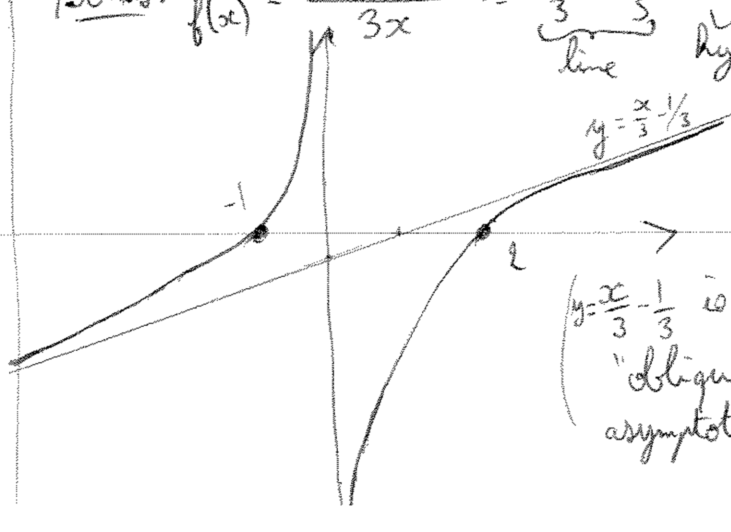
x	$-\infty$		-1		0		2		$+\infty$
$f(x)$		$-$	$\|\|$	$+$	0	$-$	$\|\|$	$+$	

Therefore: $\lim\limits_{x\to -1^-} f(x) = -\infty$, $\lim\limits_{x\to -1^+} f(x) = +\infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, $\lim\limits_{x\to 2^+} f(x) = +\infty$

(f) Using all this information, the graph looks like this:



Bonus: $\dfrac{1}{f(x)} = \dfrac{x^2-x-2}{3x} = \underbrace{\dfrac{x}{3} - \dfrac{1}{3}}_{\text{line}} - \underbrace{2/3x}_{\text{hyperbola}}$



($y = \dfrac{x}{3} - \dfrac{1}{3}$ is an "oblique asymptote").