University of Utah
Math 2210, Fall 2008
Name: Solutions

Midterm Exam # 1
Time: 50 minutes

No calculators, notes, books. Show all work.

Part 1: A point $P(t)$ is moving in 3-space with position vector $\vec{r}(t) = (t\cos 2\pi t, t\sin 2\pi t, t)$.

1) Find the velocity and acceleration vectors and the speed of the point at any time t, then at time $t = 0$. (8 points)

2) Find parametric equations for the tangent line to the curve at $t = 0$. (2 points)

3) Write the formulae for the tangential and normal components of acceleration (a_T and a_N) and the curvature κ in terms of the vectors $\vec{r'}(t)$ and $\vec{r''}(t)$. Find a_T, a_N, and κ at $t = 0$. (6 points)

Bonus question (2 points): What does the curve look like? (sketch and/or describe it)

1) $\vec{v}(t) = \vec{r}\,'(t) = (\cos 2\pi t - 2\pi t\sin 2\pi t, \sin 2\pi t + 2\pi t\cos 2\pi t, 1)$

$\vec{v}(0) = (1,0,1)$

$\vec{a}(t) = \vec{r}\,''(t) = (-2\pi\sin 2\pi t - 2\pi\sin 2\pi t - 4\pi^2 t\cos 2\pi t, 2\pi\cos 2\pi t + 2\pi\cos 2\pi t - 4\pi^2\sin 2\pi t, 0)$

$= (-4\pi\sin 2\pi t - 4\pi^2 t\cos 2\pi t, 4\pi\cos 2\pi t - 4\pi^2 t\sin 2\pi t, 0)$

$\vec{a}(0) = (0, 4\pi, 0)$

Speed: $v(t) = \|\vec{r}\,'(t)\| = \sqrt{(\cos 2\pi t - 2\pi t\sin 2\pi t)^2 + (\sin 2\pi t + 2\pi t\cos 2\pi t)^2 + 1}$

$= \sqrt{\underbrace{\cos^2 2\pi t + \sin^2 2\pi t}_{=1} + 4\pi^2 t^2(\underbrace{\sin^2 2\pi t + \cos^2 2\pi t}_{=1}) \cancel{- 2\pi t\cos 2\pi t\sin 2\pi t} \cancel{+ 2\pi t\cos 2\pi t\sin 2\pi t} + 1}$

$= \sqrt{2 + 4\pi^2 t^2}$ $\qquad v(0) = \sqrt{2}$

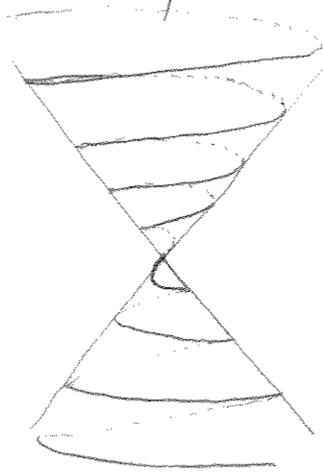
2) At $t=0$: $\vec{r}(0) = (0,0,0)$, $\vec{v}(0) = (1,0,1)$, so the tangent line has parametric equations: $x(t) = t$, $y(t) = 0$, $z(t) = t$.

3) We know that: $a_T = \dfrac{\vec{r}\,' \cdot \vec{r}\,''}{\|\vec{r}\,'\|}$ $\quad a_N = \dfrac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|}$ $\quad \kappa = \dfrac{a_N}{v^2} = \dfrac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$

Evaluating these at $t=0$: $a_T = 0$ $\quad a_N = \dfrac{\|(-4\pi, 0, 4\pi)\|}{\sqrt{2}} = \dfrac{\sqrt{32\pi^2}}{\sqrt{2}} = 4\pi$

and $\kappa = \dfrac{4\pi}{\sqrt{2}^2} = 2\pi$.

Bonus: the curve is a "helix" whose radius increases with t, and where the vertical distance between two successive coils is constant (2π). (It is contained in the regular cone with equation $z^2 = x^2+y^2$).
It looks something like this:



Part 2: (4 points) Consider the vectors $\vec{u} = (1,2,1)$ and $\vec{v} = (2,-1,1)$. (a) Are $\vec{u}$ and $\vec{v}$ orthogonal? (b) Find a vector orthogonal to both $\vec{u}$ and $\vec{v}$. (c) Find an equation for the plane through the origin containing $\vec{u}$ and $\vec{v}$.

(a) $\vec{u}\cdot\vec{v} = 1\times 2 + 2\times(-1) + 1\times 1 = 1$ so no, $\vec{u}$ and $\vec{v}$ are not orthogonal.

(b) We can either take a general vector $\vec{w} = (x,y,z)$ and write down the 2 equations saying that $\vec{u}\cdot\vec{w} = 0 = \vec{v}\cdot\vec{w}$, or, much quicker, remember that $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v} = \begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}2+1\\-(1-2)\\-1-4\end{pmatrix} = \begin{pmatrix}3\\1\\-5\end{pmatrix}$$

(c) This plane can be described as the set of all vectors $\vec{w} = (x,y,z)$ which are orthogonal to $\vec{u}\times\vec{v}$, so it has equation:

$$3x + y - 5z = 0.$$