Math 2280-1   Notes of August 24, 2022

- Let's compare two population models.
- The solution of the initial value problem

$$P' = kP, \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}.$$

- The solution of the initial value problem

$$P' = kP^2, \qquad P(0) = P_0$$

is

$$P(t) = \frac{1}{\frac{1}{P_0} - kt} = \frac{P_0}{\underbrace{1 - kP_0 t}_{=0}}$$

- The crucial difference between the two solutions is that while the first population grows forever the second reaches infinity in finite time

$$t_0 = \frac{1}{kP_0}$$

and self-destructs.

$$\frac{dP}{dt} = kP^2$$

$$\frac{dP}{P^2} = k\,dt \quad \Big| \int$$

$$-\frac{1}{P} = kt + c$$

$$P = \frac{-1}{kt + c}$$

$$P(0) = P_0 = -\frac{1}{c}$$

$$c = -\frac{1}{P_0}$$

$$t_0 = \frac{1}{kP_0}$$

$$P(t) = \frac{-1}{kt - \frac{1}{P_0}}$$

$$= \frac{P_0}{1 - ktP_0}$$

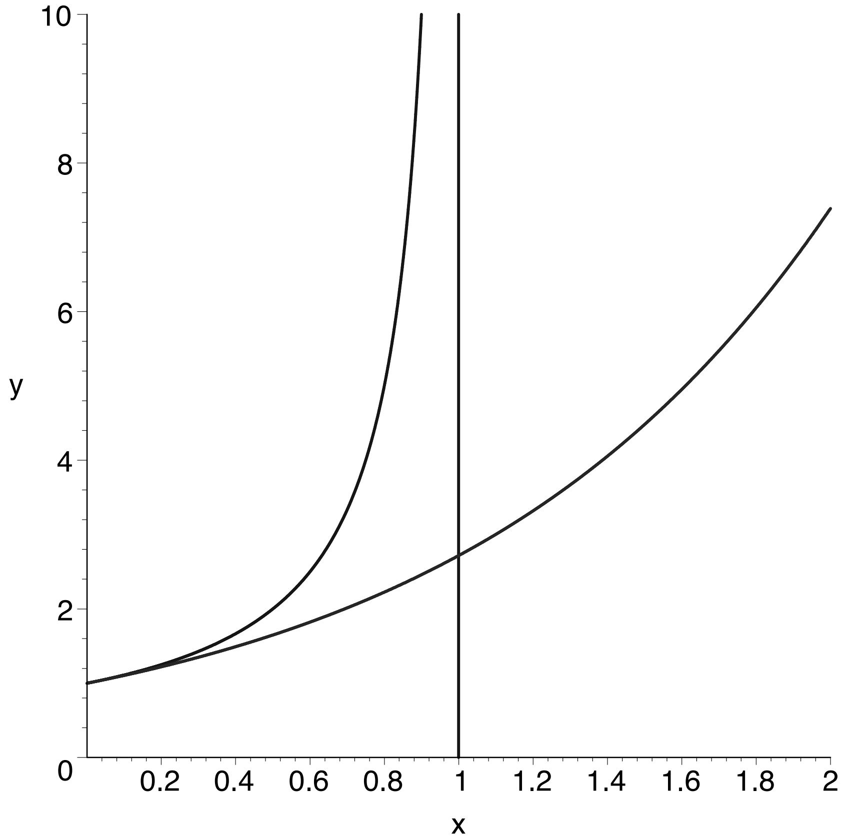


Figure 1. Two populations.

- Figure 1 shows the graphs of the two functions for the case $k = 1$ an d $P_0 = 1$.
- Of course, no growth lasts forever, we will revisit that topic later.

1.2 Integrals as General and Particular Solutions

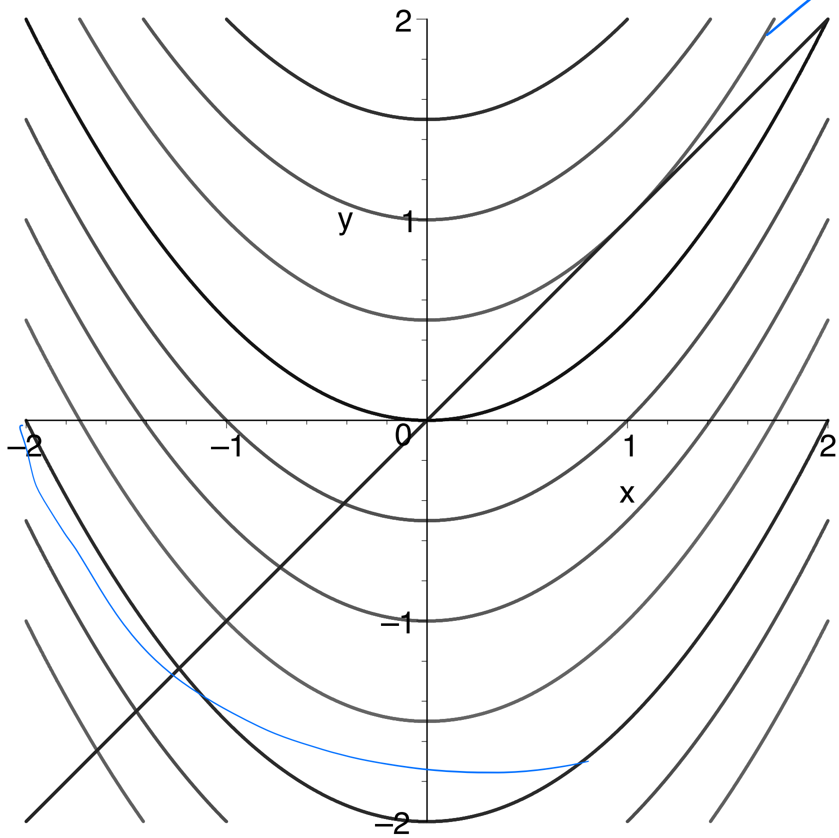
- We have seen in several examples that DEs have a general solution and particular solutions. The particular solution is determined by a parameter that we have usually denoted by T.
- A particularly simple case of a differential equation is of course one where the derivative only depends on the independent variable (and not on the dependent variable).
- We actually studied that case in depth in Calculus, without using the phrases differential equation, of general or particular solution.

differential equation:	y'	$=$	$f(x)$		$F = y$
general solution:	$y(x)$	$=$	$\int f(x)\mathrm{d}x$	indefinite integral	$= F(x) + C$, $F' = f$
particular solution:	$y(x)$	$=$	$\int_0^x f(t)\mathrm{d}t$	definite integral	
			$= y(x) - y(0)$		$y(0) = 0$

- Particular solutions of $y' = f(x)$ differ by constants!
- This is illustrated in Figure 2
- We already considered one example:

  height $=$ integral of velocity

  velocity $=$ integral of acceleration



y'

$y' = x$

Figure 2. Antiderivatives $y(x) = \frac{x^2}{2} + C$ of $f(x) = x$.

- While a function has infinitely many antiderivatives we may be interested in a particular one that is determined by an additional condition.

- **Examples:**

  $f(x) = x$ and $y(1) = 2$ $\quad y(x) = \frac{x^2}{2} + c_1 \quad y(1) = \frac{1}{2} + c_1 = 2 \quad c = \frac{3}{2}$

  $f(x) = \sin x$ and $y\left(\frac{\pi}{2}\right) = 5$ $\quad y(x) = -\cos x + c_1 \quad y(\frac{\pi}{2}) = c_1 = 5$

  $y' = f(x) = m$ and $y(0) = b$

$$y(x) = mx + c_1$$
$$= mx + b$$

- A car is traveling at 80mph when the driver sees an accident 300 feet ahead and slams on the brakes. What constant deceleration is required to avoid a pileup? What about traveling at 100mph? A mile is 5280 feet and an hour is 3600 seconds.

$s(t)$ distance from accident

$$s'' = a > 0 \qquad a = ?$$

$$s' = v = at + v_0 \qquad v_0 = -80\text{mph}$$

$$s = \frac{at^2}{2} + v_0 t + s_0 \qquad s_0 = 300f$$

$$v(t) = at + v_0 = 0 \qquad t_0 = -\frac{v_0}{a}$$

$$s(t_0) = \frac{a\frac{v_0^2}{a^2}}{2} - \frac{v_0^2}{a} + s_0 = 0$$

$$-\frac{v_0^2}{2a} + s_0 = 0$$

$$\frac{v_0^2}{2a} = s_0$$

$$\frac{2a}{v_0^2} = \frac{1}{s_0}$$

$$a = \frac{v_0^2}{2s_0}$$

$$\frac{f}{s^2} = \frac{\frac{f^2}{s^2}}{f} = \frac{f}{s^2}$$

$$a = \frac{\left(\frac{80 \cdot 5280}{3600}\right)^2}{2 \cdot 300} \approx 23 f/s^2$$

More Information

- Suppose the distance from the accident is $s(t)$, the velocity is $v(t)$, and the acceleration is $a(t) = a$ where a is the constant to be determined. Let's denote the initial velocity by v_0 and the initial distance (the distance at which you start braking at time $t = 0$) by s_0.

  We get

$$\begin{aligned} a(t) &= a \\ v(t) &= at + v_0 \\ s(t) &= a\frac{t^2}{2} + v_0 t + s_0 \end{aligned}$$

  We want to compute a for which $v(t)$ is zero exactly when $s(t)$ is zero. Let's denote by t_0 the time at which this happens. As usual in this kind of problems we first compute t_0 and then compute a.

  Setting

$$v(t) = at + v_0 = 0$$

  and solving for t gives

$$t = t_0 = -\frac{v_0}{a}.$$

  This is actually a positive quantity since the initial velocity (rate at which the distance from the accident is changing) is negative. t_0 is the time at which the car comes to a stop. We evaluate the distance from the accident at that time, set it equal to s_0 and solve for a. This gives

$$\begin{aligned} s(t_0) &= a\frac{t_0^2}{2} + v_0 t_0 + s_0 \\ &= a\frac{v_0^2}{2a^2} - v_0\frac{v_0}{a} + s_0 \\ &= -\frac{v_0^2}{2a} + s_0 \\ &= 0 \end{aligned}$$

Thus

$$a = \frac{v_0^2}{2s_0}.$$

Converting the velocities from mph to feet per second gives the following table of numerical values for various speeds and distances.

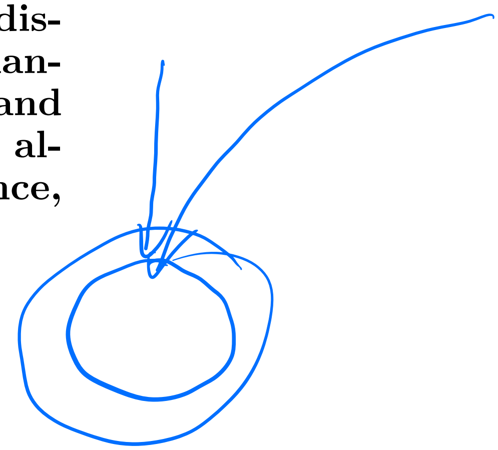
Each entry gives two values, the time t_0 giving the time in seconds until reaching the accident, and the necessary deceleration, measured in multiples of $g = 32$ feet per second squared.

dist(feet) speed(mph)	50	100	150	200	250	300
10	6.82 .07	13.64 .03	20.45 .02	27.27 .02	34.09 .01	40.91 .01
20	3.41 .27	6.82 .13	10.23 .09	13.64 .07	17.05 .05	20.45 .04
30	2.27 .60	4.55 .30	6.82 .20	9.09 .15	11.36 .12	13.64 .10
40	1.70 1.08	3.41 .54	5.11 .36	6.82 .27	8.52 .22	10.23 .18
50	1.36 1.68	2.73 .84	4.09 .56	5.45 .42	6.82 .34	8.18 .28
60	1.14 2.42	2.27 1.21	3.41 .81	4.55 .60	5.68 .48	6.82 .40
70	.97 3.29	1.95 1.65	2.92 1.10	3.90 .82	4.87 .66	5.84 .55
80	.85 4.30	1.70 2.15	2.56 1.43	3.41 1.08	4.26 .86	5.11 .72
90	.76 5.45	1.52 2.72	2.27 1.81	3.03 1.36	3.79 1.09	4.55 .91
100	.68 6.72	1.36 3.36	2.05 2.24	2.73 1.68	3.41 1.34	4.09 1.12

Table: Braking Times (seconds, top) and decelerations (g, bottom)

- A realistic typical stopping distance from 60mph might be about 150 feet. This corresponds to a deceleration of 0.81g, so you can't expect to be able to brake harder than that!

- **Disclaimer: These calculations are meant as a mathematical exercise, not as advice, promise, or guarantee, for your actual driving. Your stopping distance will vary and depends on your car, the manner of your braking, the condition of the road, and your reaction time. When operating a vehicle, always stay alert and focused, keep a safe distance, and travel at a safe speed.**



- **Exercise** (Modification and elaboration of Example 2 on page 12. That example has you falling at a **constant** velocity towards the moon which greatly simplifies the analysis but of course is not physical.) Your private spaceship is close to landing on a planet with gravity $-g$. You are currently (at time 0) at a height h_0 and falling under the influence of gravity towards the surface with a velocity v_0. Your retrorockets will provide a constant acceleration of a. At what height do you need to engage them so that you touch down at the surface at the precise moment when your velocity reaches zero?

- Note: this is a little tricky!

If you want to make this more realistic, but also more complicated, assume that gravity isn't constant but inversely proportional to the square of your distance from the center of the planet.

Direction Fields

We consider the DE $y' = f(x, y)$