Math 2280-1 Notes of August 23, 2022

1.1 Differential Equations and Mathematical Models

- A differential equation is an equation that involves a function and some of its derivatives.
- DEs are important because they describe natural processes
- **Examples:**

  $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + t^2$

  $\frac{\mathrm{d}^2y}{\mathrm{d}t} = \frac{C}{y^2}$

Short List of Possible Questions

- Given a physical situation, find a differential equation that describes it.
- Given a differential equation, find a solution, either exactly, or approximately.
- Interpret a solution.
- Find families of solutions and describe their relationships and properties.
- Examine the dependence of a solution on certain parameters.

Families of Solutions

- Example 2, page 2.

$$y(x) = Ce^{x^2}$$

$$y' = 2x\underbrace{C_1 e^{x^2}}_{y} = 2xy = y'$$

Separation of Variables

$$\frac{dy}{dx} = 2xy \quad \Big| \div y \quad \cdot dx$$

$$\frac{dy}{y} = 2x\,dx \quad \Big| \int$$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln|y| = x^2 + \hat{C}_1 \quad \Big| \; e^{(\,)}$$

$$|y| = e^{x^2+\hat{C}_1} = \underbrace{e^{\hat{C}}}_{C} e^{x^2} = C e^{x^2}$$

$$\boxed{y(x) = C_1 e^{x^2}}$$

$$y' = 2x C_1 e^{x^2} = 2xy$$

Example 3, Newton's Law of Cooling

The rate of change of the temperature of an object is proportional to the difference of the ambient temperature and the temperature of the object.

$T(t)$ temp at t of the object

A ambient temp.

$$T' = k(T-A)$$

$$\frac{dT}{dt} = k(T-A)$$

$$\frac{dT}{T-A} = k\,dt \qquad k<0$$

$$\ln(T-A) = kt + \hat{c} \qquad \Big|\ e^{(\ )}$$

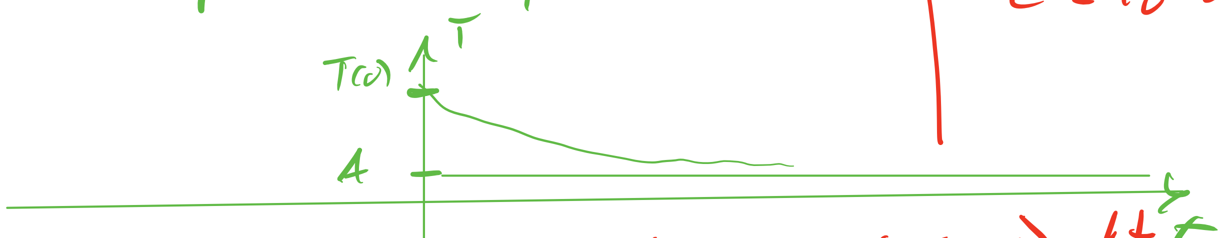
$$T-A = e^{\hat{c}}e^{kt} = ce^{kt}$$

$$T = A + ce^{kt}$$

$$T(0) = T_0$$

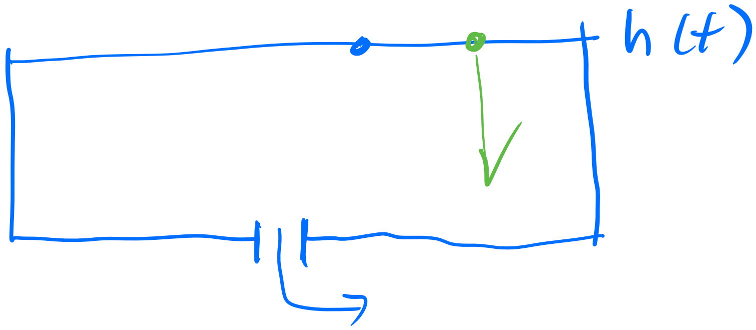
$$T(0) = A + c = T_0$$

$$c = T_0 - A$$



$$T(t) = A + (T_0 - A)e^{kt}$$

- **Toricelli's law**: Water drains at the speed of an object falling from a height that equals the depth of the water. This implies that the rate at which the water drains is proportional to the square root of the depth.



A : area of the bottom

a : area of the hole

$$V = Ah$$

$$V' = Ah'$$

$$= \overset{a}{\cancel{A}} \cdot \left(-g\left(\frac{2h}{g}\right)^{1/2}\right)$$

Madeline: volume change = area of hole × speed of fluid

$$= -\overset{a}{\cancel{A}}\sqrt{2hg}$$

$$= k\sqrt{h} \qquad k = -\overset{a}{\cancel{A}}\sqrt{2g}$$

$$\frac{f^3}{s} = f^2\sqrt{f\,\frac{f}{s^2}}$$

Falling objects

$$h(t) = -\frac{g}{2}t^2 + v_0 t + h_0$$

$$= -\frac{g}{2}t^2 + h = 0$$

$$t = \left(\frac{2h}{g}\right)^{1/2}$$

$$h'(t) = -gt$$

$$= -g\left(\frac{2h}{g}\right)^{1/2}$$

$$\begin{cases} V' = -a\sqrt{2hg} \\ h' = \dfrac{V'}{A} = -\dfrac{a}{A}\sqrt{2hg} \end{cases}$$

Sorry for the confusion :(

both rates are proportional to $\sqrt{h}$

- The **growth of a population** may be proportional to the size of the population. This gives rise to **exponential growth**.
- Note that we talk about the derivative of an integer valued function which strictly speaking does not make sense.

$p\%$ per year

$P(t)$ population at time t

$$P(t+1) = \underbrace{\left(1+\frac{p}{100}\right)}_{r} P(t)$$

$$P(t) = P_0 \left(1+\frac{p}{100}\right)^t$$

$$P(t+1) = P_0\left(1+\frac{p}{100}\right)^{t+1} = \left(1+\frac{p}{100}\right) P(t)$$

$$P(t) = P_0 r^t = P_0 \left(e^{\ln r}\right)^t$$

$$= P_0 e^{kt}$$

$$k = \ln r$$

$$P' = P_0 e^{kt} \cdot k$$

$$P' = kP$$

Some Terminology

Illustrate with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$$

$$\frac{dy}{y^2} = dx$$

$$-\frac{1}{y} = x + c'$$

$$y = \frac{-1}{x + c'}$$

- solving a differential equation

$$-\frac{1}{x+c_1}$$

- general solution

- particular solution

$$-\frac{1}{\pi+c_1}$$

- initial condition

$y(0)=1$ $\quad y(0) = -\frac{1}{0+c_1}$ $\quad c_1=-1$

- initial value problem

$y'=f(x,y)$ $\quad y(a)=y_0$

- parameter

$P'=kP$

- order of a differential equation

$y'=y$ (1st order) $\quad y''=y+y'$ (2nd order)

- ordinary differential equation

1 var

- partial differential equation

>1 var $\quad u_{xx}+u_{yy}=0$

Exercise:

- Example 10: Solve the initial value problem

$$y' = y^2, \quad y(1) = 2.$$

Exercise:

Suppose P denotes a population.

Discuss qualitative differences between the solutions of

$$P' = kP \quad \text{and} \quad P' = kP^2$$