Math 2270-1

Notes of 11/1/19

6.2 Orthogonal Sets

- Recall: two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if
  $\Sigma u_i v_i = u^T v = \mathbf{u} \bullet \mathbf{v} = 0.$

- A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ from $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors from that set is orthogonal, i.e.,

$$i \neq j \quad \Longrightarrow \quad \mathbf{u}_i \bullet \mathbf{u}_j = 0.$$

- Examples:
  - The standard basis of $\mathbb{R}^n$. $\{e_1, e_2, \ldots, e_n\}$ $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ i-th
  - The set $\{\mathbf{u}, \mathbf{0}\}$.
  - Example 1, textbook, the set

$$S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

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- **Theorem 4**, p. 340, textbook. If

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$$

is an orthogonal set of **nonzero** vectors in $\mathbb{R}^n$, then S is linearly independent. (Hence S is a basis of span(S).)

$$z = \sum_{i=1}^{p} \alpha_i u_i = 0 \quad \Rightarrow \quad \alpha_k = 0, \quad k = 1, \ldots, p$$

$$u_k \cdot z = u_k \cdot \sum_{i=1}^{p} \alpha_i u_i$$

$$= \sum_{i=1}^{p} \alpha_i u_k \cdot u_i$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(ka) \cdot b = k(a \cdot b)$$

$$= \alpha_k \underbrace{u_k \cdot u_k}_{\neq 0} \quad \Rightarrow \quad \alpha_k = 0$$

$$k = 1, \ldots, p$$

$$\alpha_k = \frac{u_k \cdot z}{u_k \cdot u_k}$$

- Naturally, an **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- For example, the set in Example 1 is an orthogonal basis of $\mathbb{R}^3$.
- Orthogonal Bases are nice! You can compute coefficients without solving a linear system.
- Suppose

  $$\mathrm{B} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$$

  is a basis of a subspace W of $\mathbb{R}^n$,

  $$B = [\,\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\,],$$

  and $\mathbf{y}$ is a vector in W. Then, in general, computing the coordinate vector

  $$[\mathbf{y}]_{\mathrm{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

  of $\mathbf{y}$ requires the solution of the linear system

  $$B[\mathbf{y}]_{\mathrm{B}} = \mathbf{y}.$$

- However, if B is an orthogonal basis we can compute the components of $[\mathbf{y}]_{\mathrm{B}}$ directly:

  $$c_j = \frac{\mathbf{y} \bullet \mathbf{u}_j}{\mathbf{u}_j \bullet \mathbf{u}_j}.$$

- Example 2, p. 341. Express the vector

$$\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

as a linear combination of the vectors in the set

$$S = \left\{ \begin{array}{ccccc} u_1 & & u_2 & & u_3 \\ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} & , & \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} & , & \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \end{array} \right\} \qquad \frac{17}{4} + \frac{49}{4} = \frac{33}{2}$$

in Example 1.

$$y = \sum_{i=1}^{3} \frac{u_i \cdot y}{u_i \cdot u_i} u_i$$

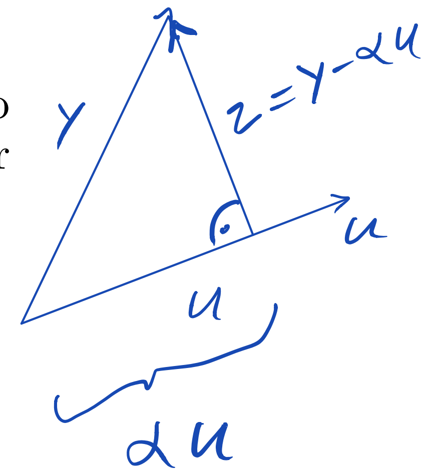
$$= \frac{11}{11}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + \frac{-12}{6}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \frac{-33}{33/2}\begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

$$1 \qquad\qquad -2 \qquad\qquad -2$$

$$= \begin{bmatrix} 3+2+1 \\ 1-4+4 \\ 1-2-7 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

Orthogonal Projections onto a Line

- Again, this is a review and generalization from Math 2210.
- Given a non-zero vector $\mathbf{u}$ in $\mathbb{R}^n$ we wish to write $\mathbf{y}$ in $\mathbb{R}^n$ as a multiple of $\mathbf{u}$ and a vector orthogonal to $\mathbf{u}$.



- That is we wish to write

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \alpha\mathbf{u} + \mathbf{z}$$

where

$$\mathbf{z} \bullet \mathbf{u} = 0.$$

- We want formulas for α and $\mathbf{z}$. They are easy to obtain.

$$(y - \alpha u) \cdot u = 0$$

$$y \cdot u = \alpha\, u \cdot u$$

$$\alpha = \frac{y \cdot u}{u \cdot u}$$

$$z = y - \frac{y \cdot u}{u \cdot u}\, u$$

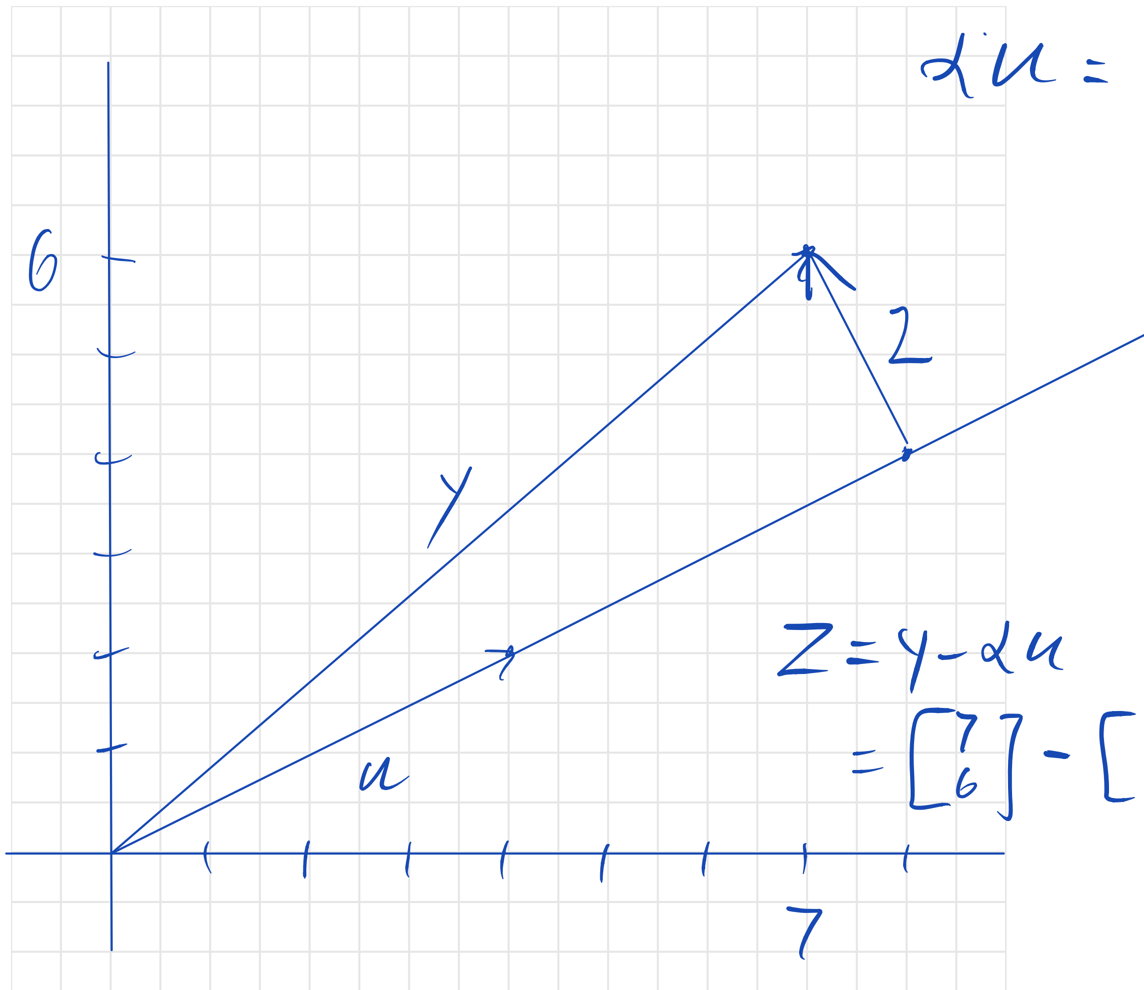
- Example 3, pg. 342, textbook. Let

$$\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

  Write $\mathbf{y}$ as a linear combination of a vector in Span$\{\mathbf{u}\}$ and a vector that is orthogonal to $\mathbf{u}$.

$$\alpha = \frac{y \cdot u}{u \cdot u} = \frac{40}{20} = 2$$

$$\alpha u = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$



$$z = y - \alpha u = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Figure 1. Example 2.

- An orthogonal set is called an **orthonormal set** is all of its vectors are unit vectors.
- Example: The standard basis, and any (nonempty) subset of it.
- **Theorem 6**, p. 345. An $m \times n$ matrix U has orthonormal columns if and only if

$$\underset{n \times m}{U^T}\,\underset{m \times n}{U} = \overset{n \times n}{I}$$

(where I is the $n \times n$ identity matrix.).

- **Theorem 7**, p. 345. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and y be vectors in $\mathbb{R}^n$. Then:

  a. $\|U\mathbf{x}\|^2 = \|\mathbf{x}\|^2$

  b. $(U\mathbf{x}) \bullet (U\mathbf{y}) = \mathbf{x} \bullet \mathbf{y}$

  c. $(U\mathbf{x}) \bullet (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \bullet y = 0$

$$\|Ux\|^2 = (Ux)^T Ux = x^T \underbrace{U^T U}_{I} x = x^T x = \|x\|^2$$

$$(Ux) \cdot (Uy) = (Ux)^T (Uy)$$

$$= x^T U^T U y$$

$$= x^T y = x \cdot y$$

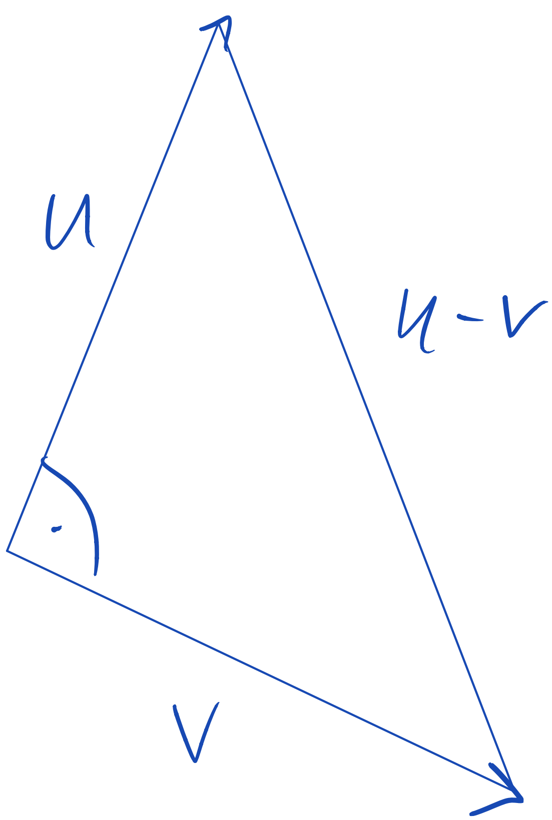
The Pythagorean Theorem

- Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

- More precisely, we should say that

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad \Longleftrightarrow \quad \mathbf{u} \bullet \mathbf{v} = 0.$$



$$\|u - v\|^2 = (u-v)^T(u-v)$$

$$= u^T u - 2u^T v + v^T v$$

$$= \|u\|^2 - 2u^T v + \|v\|^2$$

$u, v \in \mathbb{R}^n$

$(u,v) \in \mathbb{R}$ inner product

if

$$(u,v) = (v,u)$$

$$(u,u) \geq 0$$

$$(u,u) = 0 \Rightarrow u = 0$$

$$(u+v, w) = (u, w) + (v, w)$$

$$(ku, v) = k(u, v)$$

Ex.: $(u, v) = u \cdot v$

Ex.: $$(u, v) = \sum_{i=1}^{n} w_i u_i v_i$$

$$w_i > 0$$

Ex.: $$(u, v) = u^T A v$$

$$A = A^T$$

A non-singular

$$u^T A u \geq 0$$

$V = P_n$ $$(p, q) = \int_0^1 p(x) q(x)\, dx$$