Math 2270-1

Notes of 10/30/19

6.1 Inner Product, Length, Orthogonality

- Today's topic is familiar from Math 2210 where we discussed the dot product, the norm of a vector, and orthogonality of vectors.
- In our context, the terminology is slightly different, and we consider the space $\mathbb{R}^n$ for general n, instead of mostly, or just, $\mathbb{R}^2$ and $\mathbb{R}^3$.
- The **inner product**, previously called the **dot product**, of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, is defined to be

$$\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i.$$

- Examples:

- It's straightforward to verify the following algebraic properties of the inner product:
- **Theorem 1, p. 333**. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and c be a scalar. Then

  a. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$

  b. $(\mathbf{u} + \mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{w}$

  c. $(c\mathbf{u}) \bullet \mathbf{v} = c(\mathbf{u} \bullet \mathbf{v})$

  d. $\mathbf{u} \bullet \mathbf{u} \geq 0, \quad \text{and} \quad \mathbf{u} \bullet \mathbf{u} = 0 \Longrightarrow \mathbf{u} = \mathbf{0}$.

- The **length** or **norm**[−1−] of a vector $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}.$$

- Examples.

[−1−] also called **Standard Norm**, **Euclidean Norm**, or **2-norm**.

- Identifying points and vectors as usual, the distance between two vectors (points) $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u}-\mathbf{v}\|$.

- If $\mathbf{u}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$ then $\|\mathbf{u}\|$ agrees with our ordinary concept of the length of a vector.

- The concept of orthogonality in $\mathbb{R}^2$ and $\mathbb{R}^3$ generalized to orthogonality in $\mathbb{R}^n$.
- Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if

$$\mathbf{u} \bullet \mathbf{v} = 0.$$

- In 2210 we learned that

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) \tag{1}$$

  where θ is the angle formed by $\mathbf{u}$ and $\mathbf{v}$.

- This works also in $\mathbb{R}^n$. You can take (1) as the **definition** of θ.

the zero vector is orthogonal to all vectors in $\mathbb{R}^n$.

Orthogonal Complements

- Suppose W is a subspace of $\mathbb{R}^n$. Then the set

  $$W^\perp = \{\mathbf{z} : \mathbf{z}\text{is orthogonal to all vectors in } W\}$$

  is a linear space, called the **orthogonal complement** of W.
- $W^\perp$ is read as "W-perpendicular" or, more commonly, just "W-perp".
- Example: line and plane in $\mathbb{R}^3$.

- **Theorem 3**, p. 337: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T

$$(\text{Row}A)^\perp = \text{Nul} \quad \text{and} \quad (\text{Col}A)^\perp = \text{Nul}A^T.$$