Math 2270-1 — Fall 2019 — Exam 4 Answers

Note that this answer set contains more information than you needed to provide on the exam.

-1- (Gram Schmidt Process.) Using the Gram-Schmidt Process, find an orthonormal basis of the space

$$W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\} \quad \text{where} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Discussion:

We subtract from $\mathbf{x}_2$ the projection of $\mathbf{x}_2$ onto $\mathbf{x}_1$ and then normalize. This gives

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \bullet \mathbf{x}_1}{\mathbf{x}_1 \bullet \mathbf{x}_1}\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Normalizing gives the required orthonormal basis:

$$= \left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

-2- (Inner Product Spaces.) Suppose W is the inner product space of polynomials of degree up to 20, with the inner product

$$< p, q >= \int_0^1 p(x)q(x)\mathrm{d}x.$$

Compute the norm of $p(x) = x^3$.

Discussion:

We get

$$\|p\| = \sqrt{< p, p >} = \sqrt{\int_0^1 p^2(x)\mathbf{x}} = \sqrt{\int_0^1 x^6 \mathrm{d}x} = \sqrt{\left[\frac{x^7}{7}\right]_0^1} = \frac{1}{\sqrt{7}}.$$

-3- (Inner Products.) Again, let W be the inner product space of polynomials of degree up to 20. Suppose you define

$$< p, q >= p(1)q(1) + p(2)q(2) + p(3)q(3) + p(4)q(4).$$

Is $< \cdot, \cdot >$ and inner product on W? Why, or why not?

Discussion:

It's not an inner product because $< p, p >= 0$ for the non-zero polynomial

$$p(x) = (x - 1)(x - 2)(x - 3)(x - 4).$$

-4- (Least Squares.) Suppose you want to solve the least squares problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min$$

where A is $m \times n$ with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$ is a given vector. Describe and derive the normal equations giving the solution of this problem.

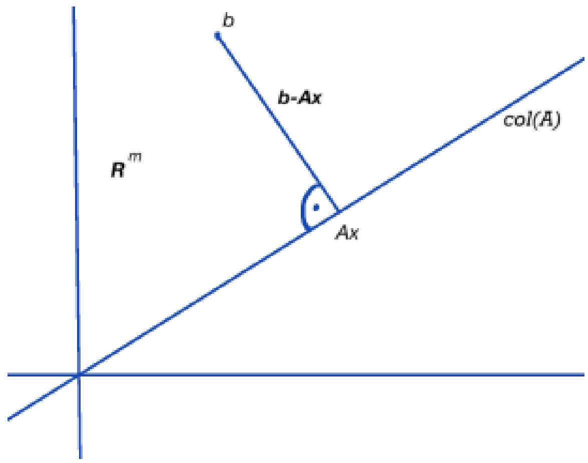

Figure 1. Normal Equations.

Discussion:

The geometric argument is illustrated in Figure 1. The residual $b - Ax$ must be orthogonal to the column space of A which gives

$$A^T(\mathbf{b} - A\mathbf{x}) = 0 \qquad \Longrightarrow \qquad \boxed{A^T A\mathbf{x} = A^T\mathbf{b}}.$$

Of course, using the QR factorization to solve the Least Squares problem is better than setting up and solving the normal equations.

-5- (True or False.) Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers. Unless stated otherwise, all linear spaces are inner product spaces and finite dimensional.

1. T F All orthogonal matrices are symmetric.

F False, for example the asymmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal.

2. T F The determinant of an orthogonal matrix is real.

T True, actually it's ± 1.

3. T F The zero vector in an inner product space is orthogonal to all vectors in that space.

T True, the inner product of zero with any other vector is zero.

4. T F Suppose W is a subspace of a finite dimensional inner product space and $W^\perp$ is its orthogonal complement. Then the intersection of W and $W^\perp$ is empty.

F False, the intersection contains the zero vector.

5. T F The orthogonal projection of the zero vector into a subspace of an inner product space is the zero vector.

T True, the zero vector is contained in every subspace and is closest to itself.

6. T F Every inner product space has an orthogonal basis.

T True, just take any basis and orthogonalize it.

7. T F Suppose W is a subspace of an inner product space V, and $W^\perp$ is its orthogonal complement. Then the dimension of V equals the sum of the dimensions of W and $W^\perp$.

T True, this follows from the uniqueness and existence of the orthogonal decomposition of a vector in V.

8. T F The projection of a vector $\mathbf{y}$ into a subspace W is unique.

T True, the projection is the unique point in W that is closest to $\mathbf{y}$.

9. T F The solution of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ is unique.

F False, $\mathbf{x}$ is unique only if the columns of A are linearly independent.

10. T F For any two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space it is true that

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

T True, this is the statement of the triangle inequality.