Math 2270-1

Notes of 10/22/2019

Determinants Revisited

- Recall that the determinant of a 2×2 matrix is defined by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For $n > 2$ we have the recursive cofactor expansion

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} |A_{ij}|$$

  where A_{ij} is the $(n-1) \times (n-1)$ submatrix of A obtained by removing the i-th row and the j-th column from A.

- Observe that applying this definition gives the determinant as a sum of products of n entries of A. Each product has one factor from each row and each column.

- For example, for $n = 3$ we get

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned} \tag{1}$$

- The pattern continues. For our purposes the important thing is that the determinant is a sum of (signed) products of matrix entries, where each column and each row contributes one factor.

- However, as an aside, I mention that one can write down a formula for the determinant in terms of permutations. A permutation σ of the sequence

$$1, 2, \dots, n$$

is simply a rearrangement

$$\{\sigma_1, \sigma_2, \dots, \sigma_n\}$$

of the sequence. For example, there are six permutations of the set $\{1, 2, 3\}$ given in the table:

σ	signσ
$\{1,2,3\}$	$+1$
$\{1,3,2\}$	-1
$\{2,1,3\}$	-1
$\{2,3,1\}$	$+1$
$\{3,1,2\}$	$+1$
$\{3,2,1\}$	-1

- To obtain the **sign** of the permutation consider the number of **transpositions**, i.e., interchanges of neighboring elements, required to obtain the permutation. If that number is even the sign is plus 1, if the number is odd the sign is -1. For example, the sign of the last permutation is -1 because we can get the permutation with three transpositions:

$$\{1,2,3\} \longrightarrow \{1,3,2\} \longrightarrow \{3,1,2\} \longrightarrow 3,2,1.$$

- With these notions one can define the determinant in the more symmetric way as

$$\det A = \sum_\sigma \operatorname{sign}(\sigma) \prod_{i=1}^n a_{i\sigma_i} \tag{2}$$

where the sum goes over all $n!$ permutations of the set $\{1,2,\ldots,n\}$ and the large symbol π indicates the product of n factors, one from each row i, and the column σ_i.

- Note that this formula matches the expressions we got for $n=2$ and $n=3$.

5.2 The Characteristic Equation

- throughout A is a square, $n \times n$, matrix.
- Recall: a **non-zero** vector $\mathbf{x}$ is an **eigenvector** of A with corresponding **eigenvalue** λ if

$$A\mathbf{x} = \lambda\mathbf{x}.$$

- λ is an eigenvalue of A if and only if there is a non-zero solution $\mathbf{x}$ of the homogeneous system

$$(A - \lambda I)\mathbf{x} = 0.$$

- This is the case if and only if

$$\det(A - \lambda I) = 0.$$

- That equation is the **characteristic equation** of A.
- What kind of function is $A - \lambda I$?
- We obtain the determinant as a sum of products of entries of $A - \lambda I$, one from each row and column. The diagonal entries are $a_{ii} - \lambda$, the other entries are scalar.
- Thus we obtain a **polynomial** in λ.
- Each term in the sum has at most n factors that are from the diagonal, so the degree of that polynomial cannot exceed n

- Moreover, **there is exactly one term that has all n diagonal entries.** It is

  $$(a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda) = (-\lambda)^n + r(\lambda)$$

  where $r(\lambda)$ is a polynomial of degree $n-1$.
- You don't need the formula (2) to see this, it follows for example by using the cofactor expansion recursively, and expanding each cofactor about the first row.
- The upshot of this is that the determinant of $A - \lambda I$ is a polynomial of degree n, with leading term $(-\lambda)^n$.
- The polynomial

  $$p(\lambda) = \det(A - \lambda I)$$

  is the **characteristic polynomial** of A

Significantly, the leading coefficient of the characteristic polynomial, i.e., $(-1)^n$, is nonzero.

The eigenvalues of A are the **roots** of the characteristic polynomial.

- What do we know about the roots of a polynomial of exact degree n?

1. There are precisely n of them.

2. They may be repeated.

3. They may be complex.

4. If there are complex roots they occur in conjugate complex pairs.

- Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Eigenvalues of a real matrix may be complex!?
- really?
- yes, consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

- The natural way to compute eigenvalues and eigenvectors by hand proceeds in two steps:

1. Compute the characteristic polynomial and find its roots.

2. For each eigenvalue λ find the nullspace of $A - \lambda I$.

- This works well only for small matrices with exactly known entries.

- The reason for this is that the coefficients of the characteristic polynomial can be computed only approximately in inexact arithmetic, and the roots of a polynomial are extremely sensitive with respect to small changes in the coefficients.

- There is also a theoretical obstacle: In general, the roots of a polynomial can be expressed explicitly in terms of radicals only of the degree of the polynomial does not exceed four.

- Evariste Galois, 1811–132.

- For a discussion of computational methods for eigenvalue/vector calculations take Math 5600 or (preferably) 5610.

- On the other hand, computing the roots of a polynomial by computing the eigenvalues of its companion matrix, using modern software, works very well!

The companion matrix

- Actually, for every polynomial p of degree n with leading term $(-1)^n$ there exists a matrix A whose characteristic polynomial is p. Check:

$$\det\left(\begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} - \lambda I\right) = (-1)^n \left[\lambda^n - \sum_{j=0}^{n-1} \alpha_j \lambda^j\right].$$

- To see this expand the determinant of

$$A - \lambda I = \begin{bmatrix} \alpha_{n-1} - \lambda & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ 1 & -\lambda & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 \\ 0 & 0 & \cdots & 1 & -\lambda \end{bmatrix}$$

by cofactors about the first row.

Similarity

- **Definition:** Two matrices A and B are similar it there is a non-singular matrix P such that
$$B = P^{-1}AP.$$

- Similar matrices have the same eigenvalues, and their eigenvectors are related in a straightforward way. To see this suppose that
$$A\mathbf{x} = \lambda x$$
and note that
$$B(P^{-1}\mathbf{x}) = P^{-1}APP^{-1}\mathbf{x} = P^{-1}\lambda\mathbf{x} = \lambda(P^{-1}\mathbf{x}).$$

- In other words, the eigenvectors of B are those of A, multiplied with P^{-1}.

- Another way to see that similar matrices have the same eigenvalues is to observe that their characteristic polynomials are the same. Using the multiplicative property of determinants and the fact that the determinant of the inverse is the reciprocal of the determinant of the original matrix we see
$$\begin{aligned}|B - \lambda I| &= |P^{-1}AP - \lambda P^{-1}IP|\\ &= |P^{-1}(A - \lambda I)P|\\ &= |P^{-1}||A - \lambda I||P|\\ &= |A - \lambda I|\end{aligned}$$

- We will have to look at this more closely tomorrow.