

Math 2270-1

Notes of 8/27/2019

The Matrix Equation $A\mathbf{x} = \mathbf{b}$

- If A is an $m \times n$ matrix and \mathbf{x} is in \mathbb{R}^n , then the product of A and \mathbf{x} is the linear combination of the columns of A , with the weights being given by the entries of \mathbf{x} .
- More explicitly:

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n.$$



Note that the product $A\mathbf{x}$ is defined only if \mathbf{x} has as many rows as A has columns!

- Examples:

- Clearly (Theorem 3, p. 36, textbook): If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, and if \mathbf{b} is in \mathbb{R}^m , then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$M = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n \quad \mathbf{b}]$$

- Note that A is the **coefficient matrix** of the linear system described by M . A is an $m \times n$ matrix, and M is an $m \times (n + 1)$ matrix.
- Sometimes M is also written as

$$M = [A \quad \mathbf{b}].$$

- Example: Write the linear system

$$\begin{array}{rrcrcl} x_1 & + & 2x_2 & - & x_3 & = & 4 \\ & & - & 5x_2 & + & 3x_3 & = & 1 \end{array}$$

as a vector equation and as a matrix equation (and solve it).

- An important question for any matrix A is if the linear system $A\mathbf{x} = \mathbf{b}$ has a solution **for all right hand sides \mathbf{b}** .
- Example: Is the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

consistent for all right hand sides \mathbf{b} ?

- Example 3, textbook. Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Is the linear system $A\mathbf{x} = \mathbf{b}$ consistent for all right hand sides \mathbf{b} ?

- Theorem 4 in the textbook tells us four equivalent conditions for consistency for all right hand sides.
- **Theorem 4:** Let A be an $m \times n$ matrix. Then the following statements are equivalent. (This means that for a particular A they are all true or all false.)
 - a. For each \mathbf{b} in \mathbb{R}^m the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
 - b. Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
 - c. The columns of A span \mathbb{R}^m .
 - d. A has a pivot position in every row.
- Note that Theorem 4: makes a statement about a matrix, not a specific linear system or augmented matrix.
- We defined the product of a matrix A and a vector as a linear combination of the columns of A .
- However, recall the definition of the **dot product**:

$$\mathbf{u} \bullet \mathbf{v} = \sum_{i=1}^n u_i v_i.$$

We can thus think of the i -th entry of $A\mathbf{x} = \mathbf{b}$ as the dot product of the i -th row of A with \mathbf{x} .

- Example:

The Identity Matrix

- Compute

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}.$$

- The matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

is called the $(n \times n)$ identity matrix. It acts like the number 1, i.e., for all \mathbf{x} in \mathbb{R}^n ,

$$I\mathbf{x} = \mathbf{x}.$$

Algebraic Properties of the Matrix-Vector Product

- Suppose A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , and c is a scalar. Then it can be verified straight from the definition that
 - a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and
 - b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Linearity

- Let F be a function whose domain is \mathbb{R}^n and whose range is (a subset of \mathbb{R}^m), i.e.,

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

where \mathbf{y} is in \mathbb{R}^m and \mathbf{x} is in \mathbb{R}^n .

- We encountered functions like this in Math 2210.
- The function \mathbf{f} is said to be **linear** if
$$\mathbf{f}(\mathbf{u} + \mathbf{v}) = \mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v}) \quad \text{and} \quad \mathbf{f}(c\mathbf{u}) = c\mathbf{f}(\mathbf{u}),$$
for all vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n and scalars c .



the properties a. and b. above say that the function

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

is linear!