Math 2270-1

Notes of 8/30/19

- No class on Monday
- hw 3 will close on Tuesday, as usual.
- When sending me queries about ww problems please use the "email instructor" button and the bottom of the problem page. I get more information that way, including a record of your answers at that time, and a link that takes me directly to your page exactly as you see it. This means, for example, that you don't need to take a screen shot.
- My name is frequently misspelled. It is Peter **Alfeld**.

1.7 Linear Independence

- Today we discuss one of the most central concepts in linear algebra, that of **linear independence**.
- Suppose we have a set of vectors

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n.$$

- Recall the concept of a linear combination of the vectors in S. Any vector $\mathbf{y}$ that can be written as

$$\mathbf{y} = \sum_{j=1}^{p} x_j \mathbf{v}_i = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p$$

is a **linear combination** of the $\mathbf{v}_j$.

- Clearly $\mathbf{y}$ is zero if all the coefficients x_j are zero.

but there may be other ways to get zero as a linear combination.

- S is said to be **linearly independent** if the only way to get zero as a linear combination of the vectors in S is by choosing all the coefficients equal to zero.

- Aside: The textbook calls S an indexed set. That's non-standard. Linear independence of a set S is independent of the ordering or indexing of the vectors in S.

$10 v_1 - 10 v_2 = 0$
$\Rightarrow \{v_1, v_2\}$
l. d.

Definition (p. 57, textbook) A set of of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights (or coefficients) c_1, c_2, …, c_p, not all equal to zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

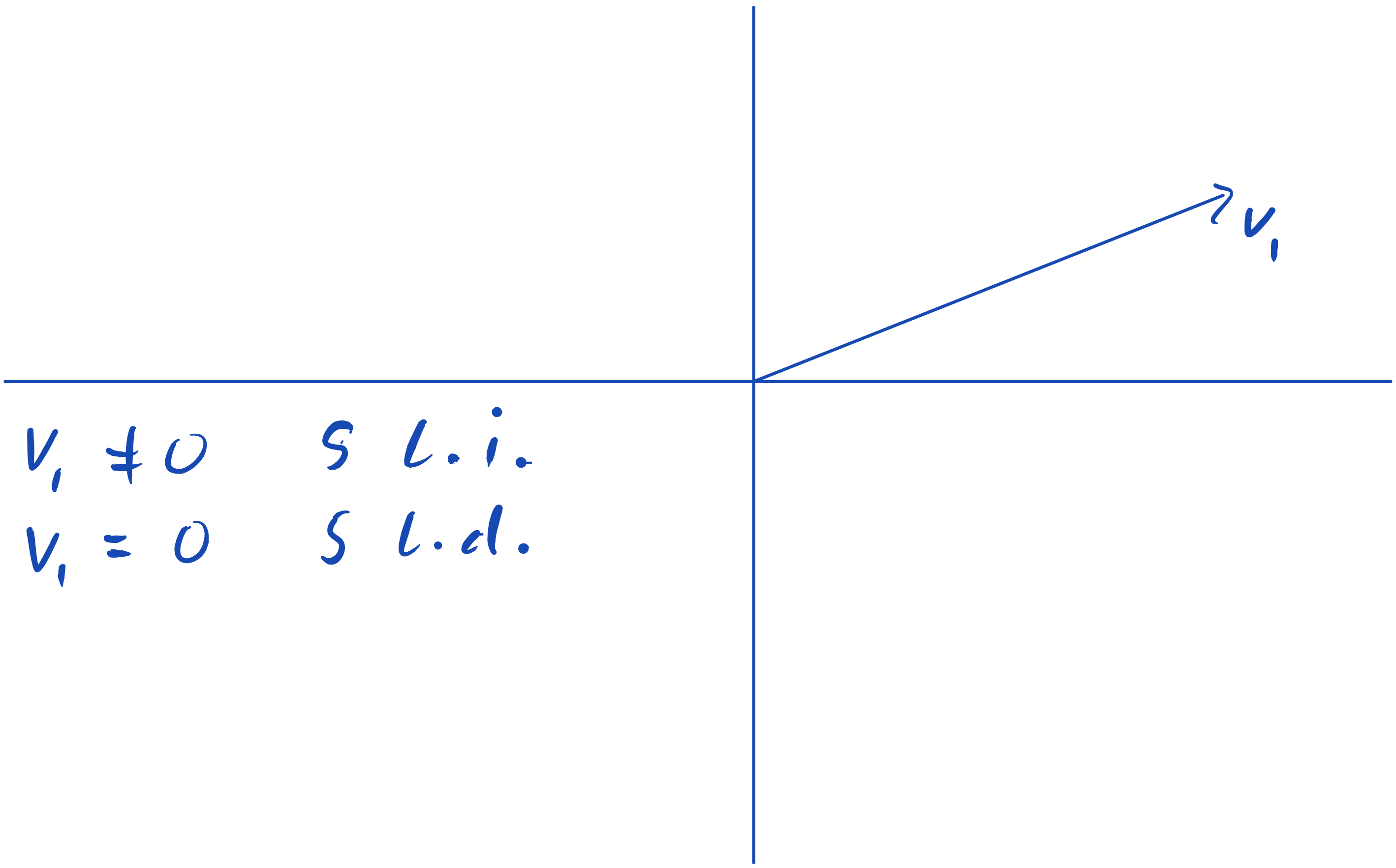
- (This is a quote directly from the textbook. I don't think it's necessary to change the names of the coefficients from x to c.)

- In other words, the set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if

$$\sum_{j=1}^{p} x_j \mathbf{v}_j = \mathbf{0} \quad \Longrightarrow \quad x_1 = x_2 = \ldots = x_p = 0.$$

$S = \{v_1\}$ $\mathbb{R}^2$

- Let's look at examples.
- Example 1. $p = 1$, $n = 2$

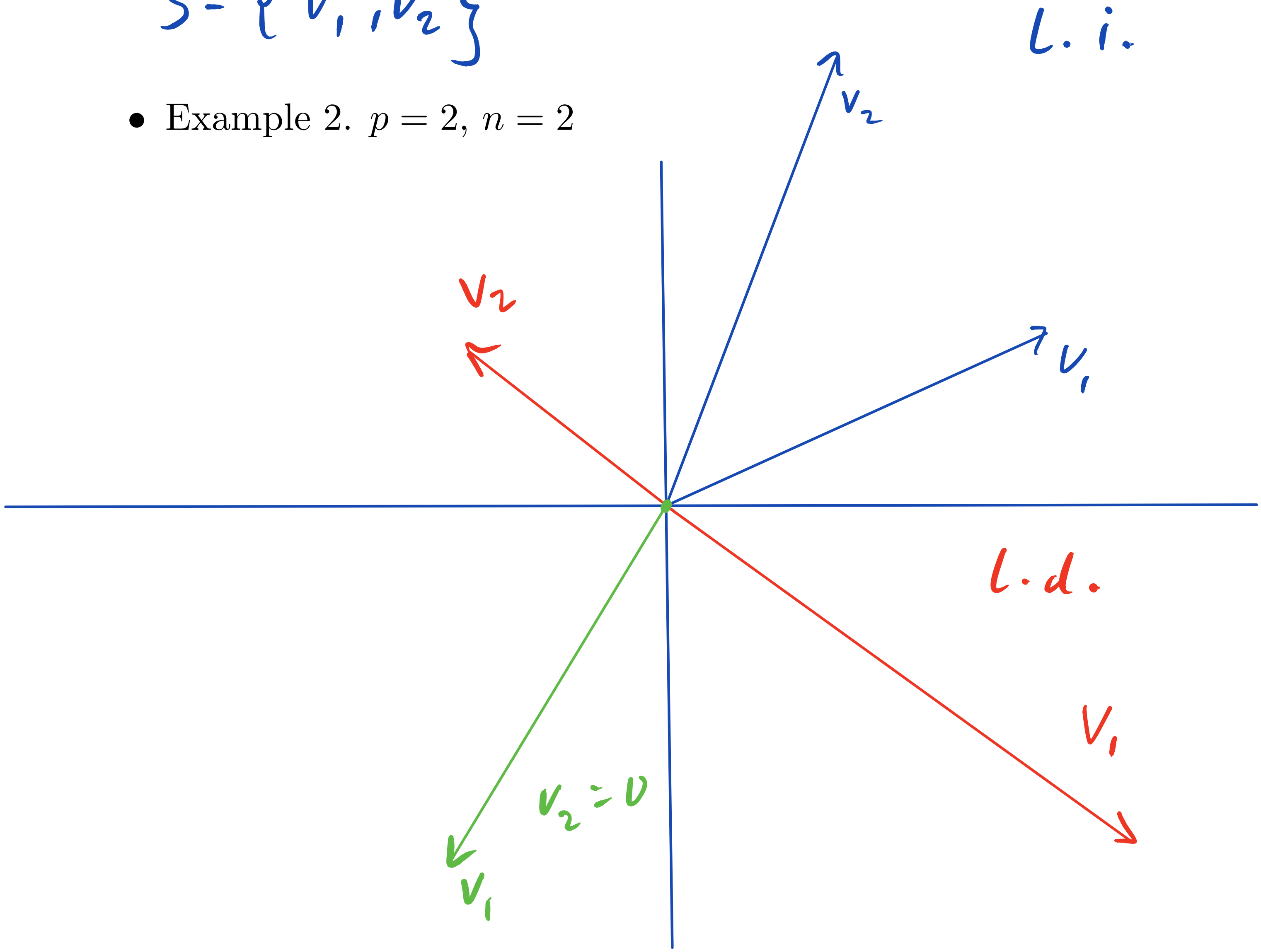


$v_1 \neq 0$ S l.i.

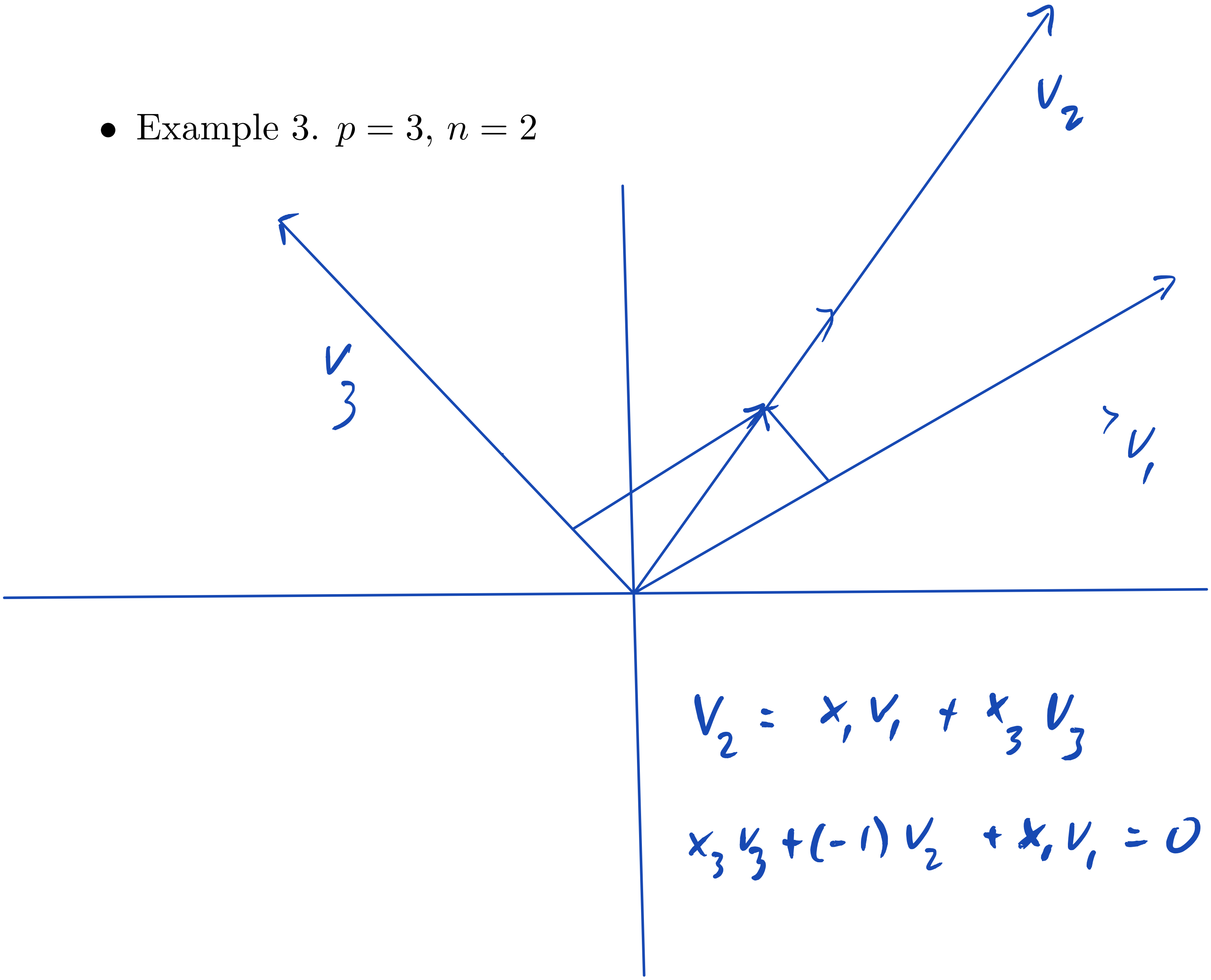
$v_1 = 0$ S l.d.

$S = \{ v_1, v_2 \}$

- Example 2. $p = 2$, $n = 2$



- Example 3. $p = 3$, $n = 2$



$$v_2 = x_1 v_1 + x_3 v_3$$

$$x_3 v_3 + (-1) v_2 + x_1 v_1 = 0$$

Matrix Form

- Suppose we collect the vectors $\mathbf{v}_j$, $j = 1, \ldots, p$ into the $n \times p$ matrix

$$A = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \mathbf{v}_p \,]$$

- Then the set S is linearly independent if and only if the homogeneous linear system

$$A\mathbf{x} = 0$$

has only the trivial solution.

- Example, (Example 1, p. 57 textbook). Determine if the set of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

is linearly dependent or independent.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \quad \begin{array}{l} r_2 = r_2 - 2r_1 \\ r_3 = r_3 - 3r_1 \end{array}$$

$$\to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 arbitrary

$$-3x_2 - 3x_3 = 0$$
$$x_2 = -x_3$$

$$x_1 + 4x_2 + 2x_3 = 0$$
$$x_1 = -4x_2 - 2x_3$$
$$= 4x_3 - 2x_3$$
$$= 2x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

- An equation of the form

$$\sum_{j=1}^{p} c_j \mathbf{v}_j = \mathbf{0}$$

  where some of the **coefficients** (or **weights**) c_j are non-zero is called an **linear dependence relation** among the v_j.

- Example continued: Find all linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in the preceding example.

Notes

- Suppose we have a linear dependence relation

$$\sum_{j=1}^{p} c_j \mathbf{v}_j = \mathbf{0} \tag{1}$$

  and $c_j \neq 0$ for some particular j. Then we can solve (1) for $\mathbf{v}_j$ and express $\mathbf{v}_j$ as a linear combination of the other $\mathbf{v}$'s.

- Apply this idea to the last example.

- The textbook (p. 59) has this Theorem 7:

  A set

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$$

of two or more vectors is linearly dependent if and only if at least one of the vectors in s is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$ then some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the vectors $\mathbf{v}_1$, …, $\mathbf{v}_{j-1}$.

- To see the last statement just pick any linear dependence relation and solve for the vector with the highest coefficient.

it is not true that in a linearly dependent set all vectors can be expressed as a linear combination of the others.

- Example:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Theorem 8 (page 60, textbook) Any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$ is linearly dependent if $p > n$.

$p \qquad\qquad p > n$

$$n \left[\qquad\qquad\qquad \right] \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$$

- Theorem 9. (page 60, textbook) Any set

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$$

  that contains the zero vector is linearly dependent.

True Statements

- To review, here is a list of conversationally stated true statements about linear independence. After you absorb the concept all of these should be obvious.

1. A set with exactly one vector is linearly dependent if and only if that vector is the zero vector.
2. A set with exactly two vectors is linearly dependent if and only if one of those vectors is a multiple of the other.
3. No vector in a linearly independent set can be written as a linear combination of the other vectors.
4. Every linearly dependent set with more than one vector contains at least one vector that can be written as a linear combination of the others.
5. A set is linearly independent if and only if the homogeneous linear system with the matrix that has the given vectors as its columns only has the trivial solution.
6. The linear dependence or independence of a set of vectors is independent of the ordering of the vectors.
7. A set of more vectors than the vectors have entries is linearly dependent.
8. A set of fewer vectors than the vectors have entries may also be linearly dependent.