Write your name here:

Math 2270-6 — Spring 2019 — Exam 4

1	2	3	4	5	6	Total

Instructions

1. **This exam is closed books and notes. Do not use a calculator or another electronic device. Do not use scratch paper.**
2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it's obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.
4. To avoid distraction and disruption **I am unable to answer questions during the exam.** If you believe there is something wrong with a problem state so, and if you are right you will receive generous credit. I will also be unable to discuss individual problems and grading issues with you after you are done while the exam is still in progress.
5. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to **check your answers.**
6. When you are done hand in your exam, pick up an answer sheet, and leave the room. Do not return to your seat.
7. All questions have equal weight.
8. Clearly indicate (for example, by circling or boxing) your final answers.

Note that this answer set contains more information than you needed to provide on the exam.

-1- (Gram Schmidt Process.) Using the Gram-Schmidt Process, find an orthonormal basis of the space

$$W = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\} \quad \text{where} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Discussion:

We subtract from $\mathbf{x}_2$ the projection of $\mathbf{x}_2$ onto $\mathbf{x}_1$ and then normalize. This gives

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \bullet \mathbf{x}_1}{\mathbf{x}_1 \bullet \mathbf{x}_1}\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Normalizing gives the required orthonormal basis:

$$= \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

-2- (Inner Product Spaces.) Suppose W is the inner product space of polynomials of degree up to 20, with the inner product

$$< p, q >= \int_0^1 p(x)q(x)\mathrm{d}x.$$

Compute the norm of $p(x) = x^3$.

Discussion:

We get

$$\|p\| = \sqrt{< p, p >} = \sqrt{\int_0^1 p^2(x)\mathbf{x}} = \sqrt{\int_0^1 x^6 \mathrm{d}x} = \sqrt{\left[\frac{x^7}{7}\right]_0^1} = \frac{1}{\sqrt{7}}.$$

-3- (Inner Products.) Again, let W be the inner product space of polynomials of degree up to 20. Suppose you define

$$< p, q >= p(1)q(1) + p(2)q(2) + p(3)q(3) + p(4)q(4).$$

Is $< \cdot, \cdot >$ and inner product on W? Why, or why not?

Discussion:

It's not an inner product because $< p, p >= 0$ for the non-zero polynomial

$$p(x) = (x-1)(x-2)(x-3)(x-4).$$

-4- (Least Squares.) Suppose you want to solve the least squares problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min$$

where A is $m \times n$ with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$ is a given vector. Describe and derive the normal equations giving the solution of this problem.

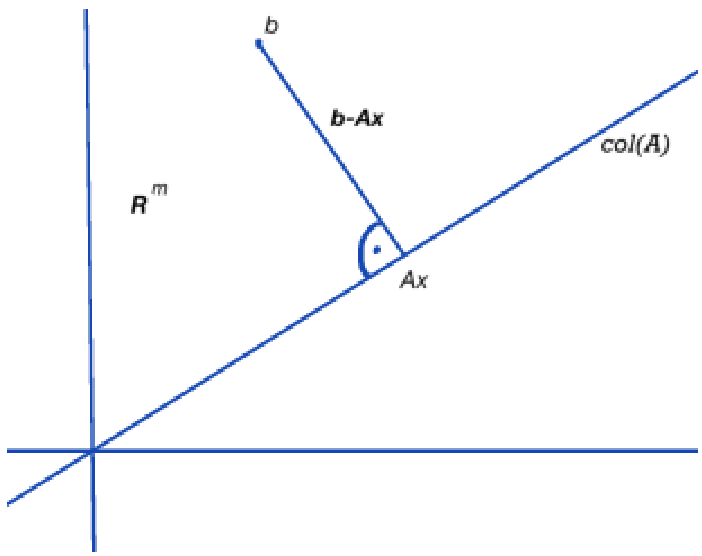

Figure 1. Normal Equations.

Discussion:

I apologize for the crude picture but the geometrix argument is illustrated in Figure 1. The residual $b - Ax$ must be orthogonal to the column space of A which gives

$$A^T(\mathbf{b} - A\mathbf{x}) = 0 \qquad \Longrightarrow \qquad \boxed{A^TA\mathbf{x} = A^T\mathbf{b}}.$$

Of course, using the QR factorization to solve the Least Squares problem is better than setting up and solving the normal equations.

-5- (True or False.) Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers. Unless stated otherwise, all linear spaces are inner product spaces and finite dimensional.

1. T F All orthogonal matrices are symmetric.

F False, for example the asymmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal.

2. T F The determinant of an orthogonal matrix is real.

T True, actually it's ± 1.

3. T F The zero vector in an inner product space is orthogonal to all vectors in that space.

T True, the inner product of zero with any other vector is zero.

4. T F Suppose W is a subspace of a finite dimensional inner product space and $W^\perp$ is its orthogonal complement. Then the intersection of W and $W^\perp$ is empty.

F False, the intersection contains the zero vector.

5. T F The orthogonal projection of the zero vector into a subspace of an inner product space is the zero vector.

T True, the zero vector is contained in every subspace and is closest to itself.

6. T F Every inner product space has an orthogonal basis.

T True, just take any basis and orthogonalize it.

7. T F Suppose W is a subspace of an inner product space V, and $W^\perp$ is its orthogonal complement. Then the dimension of V equals the sum of the dimensions of W and $W^\perp$.

T True, this follows from the uniqueness and existence of the orthogonal decomposition of a vector in V.

8. T F The projection of a vector $\mathbf{y}$ into a subspace W is unique.

T True, the projection is the unique point in W that is closest to $\mathbf{y}$.

9. T F The solution of the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ is unique.

F False, $\mathbf{x}$ is unique only if the columns of A are linearly independent.

10. T F For any two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space it is true that

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

T True, this is the statement of the triangle inequality.