Math 2270-1 — Fall 2019 — Final Exam Amswers

-1- **(LU factorization.)** Compute the LU factorization of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}.$$

Discussion:

Recall that U is the upper triangular matrix we obtain by Gaussian Elimination, and L is the matrix with 1s along the diagonal, and the multipliers below the diagonal. Thus, subtracting twice the first row from the second gives

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

We check by multiplication that $A = LU$:

$$\begin{array}{rccl} & & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} & = U \\ L = & \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} & = LU = A \end{array}$$

-2- **(Determinants.)** For what value of t does

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{bmatrix} = 0?$$

Discussion:

Expanding the determinant about the third column gives

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{bmatrix} = 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + t \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -9 + 36 - 3t = 27 - 3t = 0$$

which gives

$$t = 9.$$

-3- **(Reduced Row Echelon Form.)** Suppose the reduced row echelon form of a matrix A is

$$R = \begin{bmatrix} 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Compute the rank of A, the dimensions of its column and null spaces, and give a basis of the null space.

Discussion:

Variables 1 and 3 are pivot variables, variables 2, 4, and 5 are free variables. Thus the rank is 2, the dimension of the row space is 2, and the dimension of the null space is $5 - 2 = 3$ The basis $\{s_1, s_2, s_3\}$ of the null space can be obtained by setting the free variables, and using the first two equations to compute the pivot variables:

$$\begin{array}{l} x_2 = 1 \\ x_4 = 0 \\ x_5 = 0 \end{array} \longrightarrow s_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{array}{l} x_2 = 0 \\ x_4 = 1 \\ x_5 = 0 \end{array} \longrightarrow s_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{array}{l} x_2 = 0 \\ x_4 = 0 \\ x_5 = 1 \end{array} \longrightarrow s_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

-4- (Inner Product Spaces and the Pythagorean Theorem.) Let V be an inner product space with an inner product $< \cdot, \cdot >$. List the defining properties of $< \cdot, \cdot >$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V. Define what we mean by $\|\mathbf{u}\|$. Also define what we mean when we say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Finally, prove the Pythagorean Theorem:

$$\|u + \mathbf{v}\|^2 = \|u\|^2 + \|\mathbf{v}\|^2 \quad \Longleftrightarrow \quad \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

Discussion:

The inner product $< \cdot, \cdot >$ is a function that maps two vectors $\mathbf{u}$ and $\mathbf{v}$ to a number $< \mathbf{u}, \mathbf{v} >$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and scalars c:

1. $< \mathbf{u}, \mathbf{v} >=< \mathbf{v}, \mathbf{u} >$,
2. $< \mathbf{u} + \mathbf{v}, \mathbf{w} >=< \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >$,
3. $< c\mathbf{u}, \mathbf{v} >= c < \mathbf{u}, \mathbf{v} >$,
4. $< \mathbf{u}, \mathbf{u} >\geq 0 \quad \text{and} \quad < \mathbf{u}, \mathbf{u} >= 0 \quad \Longrightarrow \quad \mathbf{u} = 0.$

The norm $\|\mathbf{u}\|$ of $\mathbf{u}$ is defined by

$$\|\mathbf{u}\| = \sqrt{< \mathbf{u}, \mathbf{u} >}.$$

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their inner product is zero:

$$< \mathbf{u}, \mathbf{v} >= 0.$$

We can now easily see the Pythagorean Theorem from

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &=< \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} > \\ &=< \mathbf{u}, \mathbf{u} > +2 < \mathbf{u}, \mathbf{v} > + < \mathbf{v}, \mathbf{v} > \\ &= \|\mathbf{u}\|^2 + 2 < \mathbf{u}, \mathbf{v} > +\|\mathbf{v}\|^2\end{aligned}$$

$$\Longrightarrow$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \Longleftrightarrow \qquad < \mathbf{u}, \mathbf{v} >= 0$$

This is a beautiful illustration of the fact that sometimes generalizing a statement simplifies the argument showing that it is true.

-5- (Invertibility and Diagonalizability.) Give examples—as simple as possible—for matrices that are

a. invertible and diagonalizable: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. invertible and non-diagonalizable: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

c. singular and diagonalizable: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. singular and non-diagonalizable: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

These examples are pretty simple.

-6- (Positive Definiteness.) Let

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 6 & 3 & 1 \\ 1 & 3 & 10 & 5 \\ 1 & 1 & 5 & 12 \end{bmatrix}$$

Show that A is positive definite.

Discussion:

A matrix is positive definite if it is symmetric and all of its eigenvalues are positive. Clearly A is symmetric. By the Gershgorin Theorem, all of its eigenvalues are positive. Hence A is positive definite.

-7- **(Linear Transformation.)** Suppose we write a cubic polynomial as

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

as usual. Find the 4×4 matrix M that maps the coefficient vector of p with respect to the basis $\{1, x, x^2, x^3\}$ onto the vector of the coefficients of the Taylor expansion about $x = 1$. In other words, find M such that

$$M \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1)/2 \\ p'''(1)/6 \end{bmatrix}$$

Note: I assume you are familiar with Taylor series, but for reference, the Taylor expansion of a function f about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots.$$

Discussion:

We get the columns of M by applying the transformation to the basis polynomials 1, x, x^2, and x^3. We get the Taylor coefficients

$$\begin{aligned} p(x) = 1 &\implies p(1) = 1, p'(1) = p''(1)/2 = p'''(1)/6 = 0 \\ p(x) = x &\implies p(1) = 1, p'(1) = 1, p''(1)/2 = p'''(1)/6 = 0 \\ p(x) = x^2 &\implies p(1) = 1, p'(1) = 2, p''(1)/2 = 1, p'''(1)/6 = 0 \\ p(x) = x^3 &\implies p(1) = 1, p'(1) = 3, p''(1)/2 = 3, p'''(1)/6 = 1 \end{aligned}$$

This gives

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

-8- **(Linear Spaces.)** Let S be the set of polynomials p of degree 2 that satisfy $p(0) = 0$. Show that S is a subspace of the space of quadratic polynomials, compute the dimension of S, and give a basis of S.

Discussion:

S is clearly closed under addition and multiplication with a scalar, and so S is a linear space. The space of all quadratic polynomials has dimension 3, but non-zero constant functions are not in S, and so S has dimension **at most 2.** The two functions x and x^2 are linearly independent and are in S. Therefore the dimension of S is **at least 2.** Therefore the dimension is **equal to 2**, and the two functions form a basis $\{x, x^2\}$ of S.

-9- **(Orthogonal Matrices.)** Give an example of an orthogonal matrix with complex (non-real) eigenvalues. Your example should be as simple as possible.

Discussion:

The eigenvalues of the orthogonal matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\pm i$.

A more general example is the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

-10- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. A linear system may have exactly 2 solutions.

$\boxed{\textbf{F}}$ A linear system can only have none, one, or infinitely many solutions.

2. A linear system with fewer equations than unknowns always has at least one solution.

$\boxed{\textbf{F}}$ For example, the linear system of two equations in three unknowns

$$\begin{aligned} x+y+z&=1 \\ x+y+z&=2 \end{aligned}$$

has no solution.

3. Every vector in a linearly dependent set can be written as a linear combination of the other vectors.

$\boxed{\textbf{F}}$ The first vector in the linearly dependent set

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

is not a linear combination of the last two vectors.

4. If a set of more than one vectors is linearly dependent then at least one of those vectors can be written as a linear combination of the others.

$\boxed{\textbf{T}}$ Take a non-trivial dependence relation (linear combination equals zero) of vectors and solve for one of the vectors in that relation.

5. The matrix transformation $\mathbf{y} = A\mathbf{x}$ is one-to-one if and only if the columns of A are linearly independent.

$\boxed{\textbf{T}}$ The phrase "one-to-one" means that every vector in the column space of A can be written uniquely as a linear combination of the columns, and this is equivalent to saying the columns are linearly independent.

6. Any vector in the span of a linearly dependent set can be written in more than one way as a linear combination of the given vectors.

$\boxed{\textbf{T}}$ Just write it in one way, and add a non-trivial linear combination of vectors that equals zero, to get another way.

7. The solution set of $A\mathbf{x} = \mathbf{0}$ is a linear space

$\boxed{\textbf{T}}$ That's the definition of the null space, or kernel, of A.

8. If $\mathbf{u}$ and $\mathbf{v}$ are solutions of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ then $\mathbf{u} - \mathbf{v}$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$\boxed{\textbf{T}}$ We get $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$ and hence $A\mathbf{u} - A\mathbf{v} = A(\mathbf{u} - \mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

9. The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ may be inconsistent.

$\boxed{\textbf{F}}$ The zero vector is a solution of $A\mathbf{x} = \mathbf{0}$.

10. The general solution of any consistent linear problem is any particular solution, plus the general solution of the associated homogeneous problem.

$\boxed{\textbf{T}}$ Applied to general linear problems, this is one of the central principles of mathematics.

-11- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers. This question deals with matrix multiplication. Throughout assume that A and B are matrices.

1. The matrix product $C = AB$ can be formed only if A has as many rows as B.

F A must have as many columns as B has rows.

2. If A and B are two non-square matrices it is impossible for AB and BA both to be defined.

F If A is $m \times n$ and B is $n \times m$ then AB is $m \times m$ and BA is $n \times n$, but they are both defined.

3. The j-th column of AB equals the product of A and the j-th column of B.

T This is one of the six views of matrix multiplication that we considered in class.

4. The i-th row of AB equals the product of A and the i-th row of B.

F The i-th row of AB equals the product of the i-th row of A, and B.

5. The product AB of a non-zero $m \times 1$ matrix A and a non-zero $1 \times n$ matrix B is an $m \times n$ matrix of rank 1.

T Every row of AB is a multiple of A, and every column of AB is a multiple of B. Moreover, AB is non-zero, so its rank is greater than zero.

6. Every column of AB is in the column space of A.

T A column of AB is A multiplied with the corresponding column of B. Thus it is a linear combination of the columns of A.

7. Every row of AB is in the row space of A.

F Every row of AB is in the row space of B, not A.

8. Assuming A is $m \times p$ and B is $p \times n$ then $C = AB$ is $m \times n$ and

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \qquad i = 1, \dots m, \quad j = 1, \dots, n.$$

T that's how we define matrix multiplication.

9. If A and B are both $n \times n$ then $AB = BA$.

F Matrix multiplication does not commute.

10. Assuming the $m \times p$ matrix A is the standard matrix of a function $f : \mathbb{R}^p \longrightarrow \mathbb{R}^m$, and the $p \times n$ matrix B is the standard matrix of a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, the product AB is the standard matrix of the function $g \circ f$.

T That's precisely why we define matrix multiplication the way we do.

-12- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. The determinant of an $n \times n$ matrix A equals the product of its eigenvalues.

T This follows from evaluating

$$\det(A - \lambda I) = (-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i)$$

at $\lambda = 0$.

2. A symmetric matrix A is positive definite if and only if all its eigenvalues are positive.

T If there is a nonpositive eigenvalue λ with corresponding eigenvector $\mathbf{x}$ then for that vector $\mathbf{x}^T A\mathbf{x}$ is nonpositive. If all eigenvalues are positive use an orthogonal basis of eigenvectors to see that $\mathbf{x}^T A\mathbf{x} > 0$ for all non-zero $\mathbf{x}$.

3. The eigenvalues of a triangular matrix are its diagonal entries.

T This follows directly from

$$\det(A - \lambda I) = \prod_{i=1}^{n} (\lambda - a_{ii}).$$

4. Multiplying a rank 1 matrix A with a non-singular matrix B gives a rank 1 matrix $C = AB$

T All the columns of AB are in the one-dimensional column space of A.

5. Suppose A is an $m \times n$ matrix. Then AA^T and $A^T A$ are both symmetric and positive semidefinite.

T These matrices are symmetric and

$$\mathbf{x}^T A^T A\mathbf{x} = \mathbf{y}^T \mathbf{y} \geq 0$$

where $\mathbf{y} = A\mathbf{x}$. The argument for AA^T is similar.

6. A square matrix A is invertible if and only if A^T is invertible.

T This is one of our criteria for invertibility.

7. The processes of inverting and transposing a matrix commute.

T We can verify directly that $(A^T)^{-1} = (A^{-1})^T$.

8. If A and B are invertible $n \times n$ matrices then $(AB)^{-1} = A^{-1}B^{-1}$.

F $(AB)^{-1} = B^{-1}A^{-1}$.

9. Suppose A is an $n \times n$ matrix and k is a scalar. Then $\det(kA) = k \det A$.

F Actually, $\det(kA) = k^n \det A$.

10. Interchanging two columns of a matrix does not change its determinant.

F Interchanging two columns (or rows) multiplies the determinant with -1.

-13- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. Suppose A, B, and P are $n \times n$ matrices and

$$B = P^{-1}AP.$$

The A and B have the same eigenvalues.

T This is the essence of a similarity transform.

2. The eigenvectors of a triangular matrix are the standard basis vectors.

F The eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The second basis vector, $e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

3. Suppose $\mathbf{a}$ is orthogonal to $\mathbf{b}$ and $\mathbf{b}$ is orthogonal to $\mathbf{c}$. Then $\mathbf{a}$ is orthogonal to $\mathbf{c}$.

F For example, we could have $\mathbf{a} = \mathbf{c} \neq \mathbf{0}$.

4. The normal equations for the Least Squares problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min$$

are always consistent.

T This is another way of saying that the projection of $\mathbf{b}$ into the column space of A exists.

5. The zero vector in $\mathbb{R}^n$ is orthogonal to all vectors in $\mathbb{R}^n$.

T The inner product of $\mathbf{0}$ with any vector is zero.

6. A singular matrix may be orthogonal.

F Every orthogonal matrix has an inverse, its transpose.

7. The matrix Σ in the singular value decomposition of an invertible square matrix is positive definite.

T The eigenvalues of Σ are the singular values of A which are in general non-negative, and are positive for an invertible matrix.

8. The singular values of a negative definite matrix are negative.

F Singular values are non-negative by definition.

9. Suppose f is a scalar valued function of several variables. Then it assumes a minimum at a point if the gradient at that point is zero and the matrix of second derivatives at that point is positive definite.

T Positive definiteness is the natural analog of positive numbers.

10. Linear Algebra is cool.

T That's certainly my opinion, but of course you are entitled to your own. I counted both possible answers as correct.

I enjoyed meeting you, and it's been great fun teaching this class. I hope you found it a worthwhile experience.
Best wishes to you, and perhaps I'll meet you again in some future class!
Peter Alfeld