Math 2270-1

Notes of 8/20/19

1.1 Linear Systems

- Linear Algebra is the mathematics of matrices and vectors.
- A major part of linear algebra consists of the mathematics of **linear systems**.
- That's our starting point.
- Much of what we'll do today is transfer our familiarity of linear systems of two or three equations in as many variables to the general case of having n equations.
- An example for a linear system:

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ 2x_1 - 4x_2 &= -6 \end{aligned} \tag{1}$$

$x_1 = 1$
$x_2 = 2$

- In the past we most likely would have written this something like:

$$\begin{aligned} 3x + 2y &= 7 \\ 2x - 4y &= -6 \end{aligned}$$

- Since now we contemplate a general number of variables using a different letter for each variable is no longer feasible or desirable.

- In general, a **linear equation** in n **variables** x_1, x_2, …, x_n can be written as

$$b = \quad a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

where a_1, …, a_n and b are real (or complex) numbers. The a_i, $i = 1, \ldots, n$ are the **coefficients**.

- The textbook does not give a name to b. It's sometimes called, somewhat awkwardly, the **right hand side**.

- **A system of linear equations** or, more simply, a **linear system** is a collection of one or more linear equations in the same variables, say

$$x_1, x_2, \ldots, x_n$$

The study of linear systems will be a major part of our course.

- A **solution** of a linear system is a set

$$\{s_1, s_2, \ldots, s_n\}$$

such that setting

$$x_i = s_i, \qquad i = 1, \ldots, n$$

makes all the equations true.

- For example, $x_1 = 1$ and $x_2 = 2$ is a solution of (1).

- A linear system may have none, one, or infinitely many solutions.

One linear equation in one variable

$$ax = b$$

Case I: $a \neq 0$ $\quad x = \frac{b}{a}$ unique soln

Case II: $a = 0$

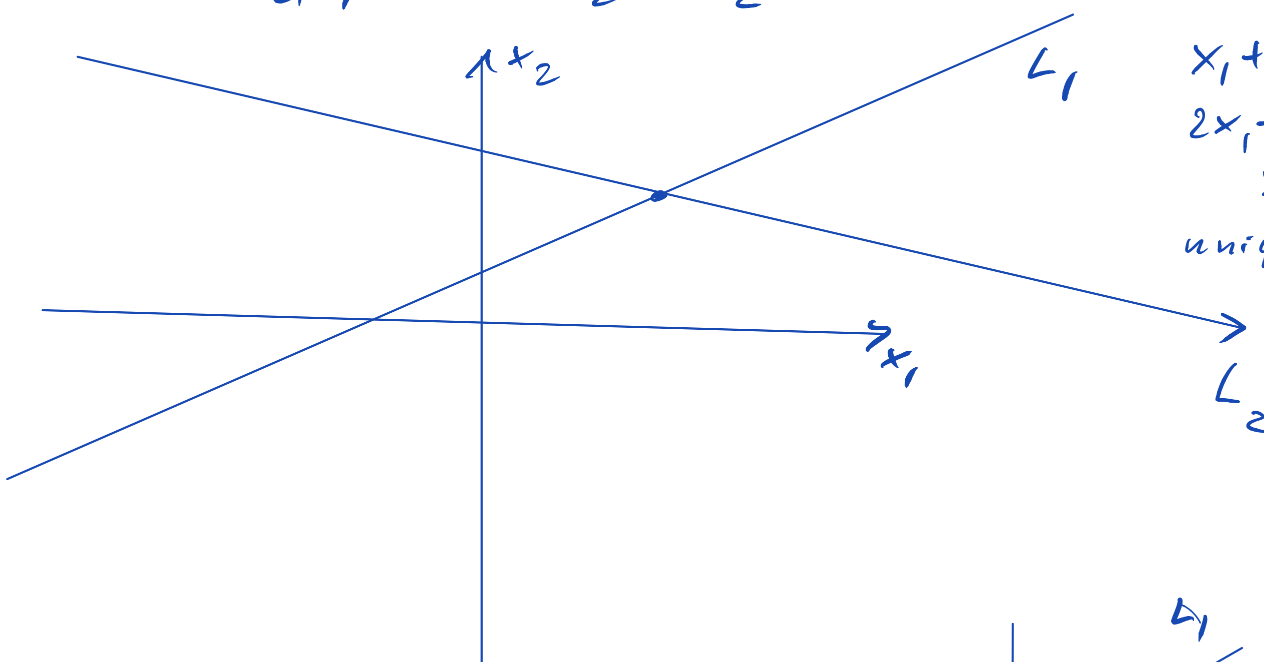
II a $\quad b = 0 \quad 0x = 0 \quad x$ arbitrary

II b $\quad b \neq 0 \quad 0x = 0 = b \neq 0$

no soln

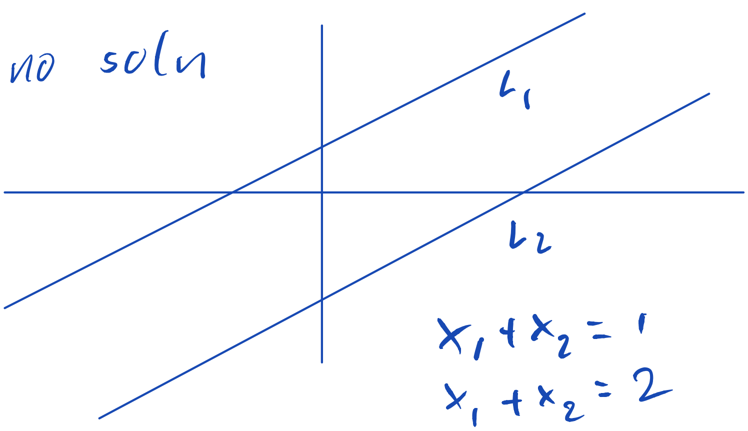
Two linear equations in two variables

$$a_{11}x_1 + a_{12}x_2 = b_1 \qquad ax + by = c$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

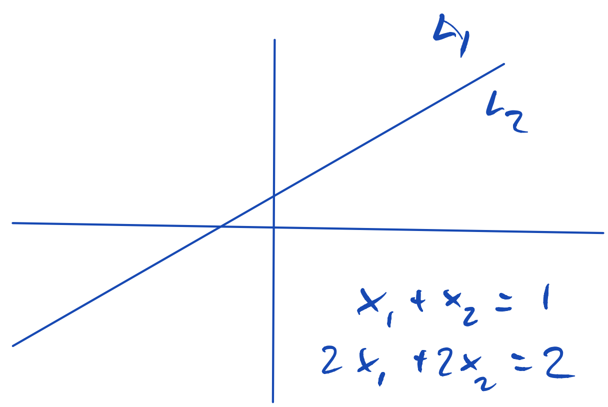


$x_1 + x_2 = 3$
$2x_1 + x_2 = 4$
$x_1 = 1 \quad x_2 = 2$
unique soln

no soln



$x_1 + x_2 = 1$
$x_1 + x_2 = 2$



$x_1 + x_2 = 1$
$2x_1 + 2x_2 = 2$

- The set S of all solutions of a linear system is the **solution set** of that linear system.
- Two linear systems are **equivalent** if they have the same solution sets.
- S may be empty, have one element, or have infinitely many elements.
- We'll see soon that these are all possible cases. A linear system cannot have exactly two, or two hundred, solutions, for example.

Matrix Form

- It is convenient to collect the information about a linear system in a rectangular table called a **matrix**.
- The next example is from the textbook.
- The linear system

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ 5x_1 & & & - & 5x_3 & = & 10 \end{array} \qquad (2)$$

- Associated with the system (2) are the **coefficient matrix**

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

and the **augmented matrix**

$$W = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

- There is a large body of terminology associated with matrices. The following is only a small beginning:
  - **row**
  - **column**
  - **entry**: The number in the i-th row and the j-column of a matrix A is usually denoted by a_{ij} and is called the i, j-entry.
  - The **diagonal** of a matrix A is the sequence of numbers a_{ii}.
  - We say that a matrix with m rows and n columns is an $m \times n$ ("m by n") matrix.
- The pair (m, n) is sometimes called the **dimension** or **size** of the matrix.
  - A matrix that has as many columns as rows is a **square matrix**.
- For example, for the above coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

  we have:

- The number of rows is: 3
- The number of columns is: 3
- the 2, 3 entry is: -8
- A is square

Solving the System

$$\begin{array}{ccccccc} 1 & & 0 & & -1 & & \\ x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ 5x_1 & & & - & 5x_3 & = & 10 \end{array} \tag{3}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \begin{matrix} \\ \div 2 \\ \div 5 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 1 & 0 & -1 & 2 \end{bmatrix} \quad r_3 \leftarrow r_3 - r_1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 2 & -2 & 2 \end{bmatrix} \quad r_3 \leftarrow r_3 - 2r_2$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 6 & -6 \end{bmatrix}$$

$x_1 - 2x_2 + x_3 = x_1 - 0 - 1 = 0 \quad x_1 = 1$

$x_2 - 4x_3 = 4 = x_2 + 4 \quad x_2 = 0$

$6x_3 = -6 \quad x_3 = -1$

Elementary Row Operations

- the basic idea of solving the linear system is to replace it with a sequence of equivalent linear systems obtained **elementary row operations** that do not change the solution set of the linear system.
- This idea underlies the most common algorithms for solving a linear systems, and also is the basis for much of the mathematics we will study in this class.
- There are just three types elementary row operations:

1. **Add a multiple of one row to another row.**
2. **Interchange two rows.**
3. **Multiply a row by a non-zero constant.**

Existence and Uniqueness

- There are two fundamental Questions about a linear system:

1. **Existence:** Is the system **consistent**, i.e., does it have a solution?
2. **Uniqueness:** If a solution exists, is it unique, or is there another one?

- We already saw that (2) has a unique solution.
- Example: Consider the system

$$
\begin{array}{rrrrrrrl}
 & & x_2 & - & 4x_3 & = & 8 & \quad -10 \\
2x_1 & - & 3x_2 & + & 2x_3 & = & 1 & \quad 2 \\
4x_1 & - & 8x_2 & + & 12x_3 & = & 1 & \quad 24
\end{array}
$$

$$
\begin{array}{cl}
\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right] & \begin{array}{l} 2 \\ -10 \\ 24 \quad r_3 \leftarrow r_3 - 2r_1 \end{array}
\end{array}
$$

$$
\begin{array}{cl}
\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{array}\right] & \begin{array}{l} 2 \\ -10 \\ 20 \quad r_3 \leftarrow r_3 + 2r_2 \end{array}
\end{array}
$$

$$
\begin{array}{cl}
\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right] & \begin{array}{l} 2 \\ -10 \\ 0 \end{array}
\end{array}
$$

no soln

$$\begin{bmatrix} 2 & -3 & 2 & 2 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ infinitely many solns

x_3 arbitrary

$$x_2 = -10 + 4x_3$$

$$2x_1 = 2 + 3x_2 - 2x_3$$

$$x_1 = \frac{1}{2}(2 + 3x_2 - 2x_3)$$

- Modify this calculation to analyze the linear system

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & -10 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 2 \\ 4x_1 & - & 8x_2 & + & 12x_3 & = & 24 \end{array}$$