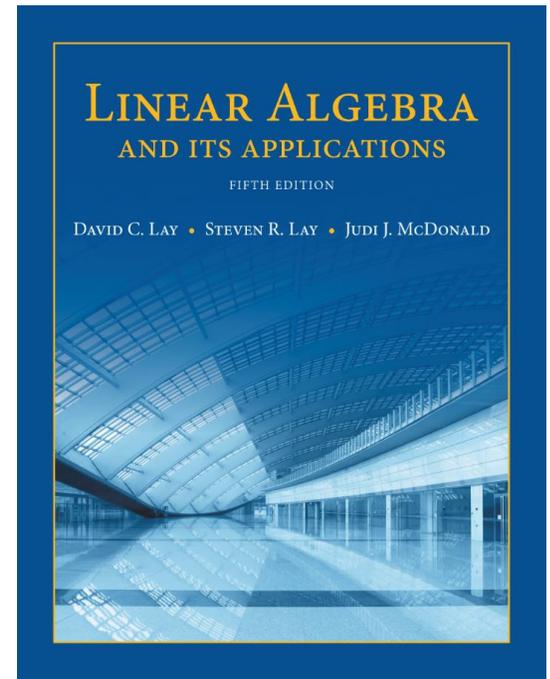


2

Matrix Algebra

2.2

THE INVERSE OF A MATRIX



MATRIX OPERATIONS

- An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix.

- In this case, C is an **inverse** of A .
- In fact, C is uniquely determined by A , because if B were another inverse of A , then

$$B = BI = B(AC) = (BA)C = IC = C.$$

- This unique inverse is denoted by A^{-1} , so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

$$A A^{-1} = I \quad A^{-1} = [a_1^{-1} \ a_2^{-1} \ \dots \ a_n^{-1}]$$

$$A [a_1^{-1} \ a_2^{-1} \ \dots \ a_n^{-1}] = [e_1 \ \dots \ e_n]$$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th place}$$

$$A a_i = e_i$$

$$[A \ I] \rightarrow [I \ A^{-1}]$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = ? \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \checkmark$$

MATRIX OPERATIONS

- **Theorem 4:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \begin{matrix} \\ \\ \\ \frac{1}{ad-bc} \end{matrix}$$

If $ad - bc = 0$, then A is not invertible.

- The quantity $ad - bc$ is called the **determinant** of A , and we write $\det A = ad - bc$
- This theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$

MATRIX OPERATIONS

- **Theorem 5:** If A is an invertible $n \times n$ matrix, then for each \mathbf{b} in \mathbb{R}^n , the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. $\overline{I}\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} = \mathbf{x}$
- **Proof:** Take any \mathbf{b} in \mathbb{R}^n .
- A solution exists because if $A^{-1}\mathbf{b}$ is substituted for \mathbf{x} , then $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$.
- So $A^{-1}\mathbf{b}$ is a solution.
- To prove that the solution is unique, show that if \mathbf{u} is any solution, then \mathbf{u} must be $A^{-1}\mathbf{b}$.
- If $A\mathbf{u} = \mathbf{b}$, we can multiply both sides by A^{-1} and obtain $A^{-1}A\mathbf{u} = A^{-1}\mathbf{b}$, $I\mathbf{u} = A^{-1}\mathbf{b}$, and $\mathbf{u} = A^{-1}\mathbf{b}$.

MATRIX OPERATIONS

■ Theorem 6:

- a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

- b. If A and B are $n \times n$ invertible matrices, then so is AB , and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

- c. If A is an invertible matrix, then so is A^T , and the inverse of A^T is the transpose of A^{-1} . That is,

$$(A^T)^{-1} = (A^{-1})^T$$

MATRIX OPERATIONS

- **Proof:** To verify statement (a), find a matrix C such that

$$A^{-1}C = I \quad \text{and} \quad CA^{-1} = I$$

- These equations are satisfied with A in place of C . Hence A^{-1} is invertible, and A is its inverse.
- Next, to prove statement (b), compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

- A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$.
- For statement (c), use Theorem 3(d), read from right to left, $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.
- Similarly, $A^T (A^{-1})^T = I^T = I$.

ELEMENTARY MATRICES

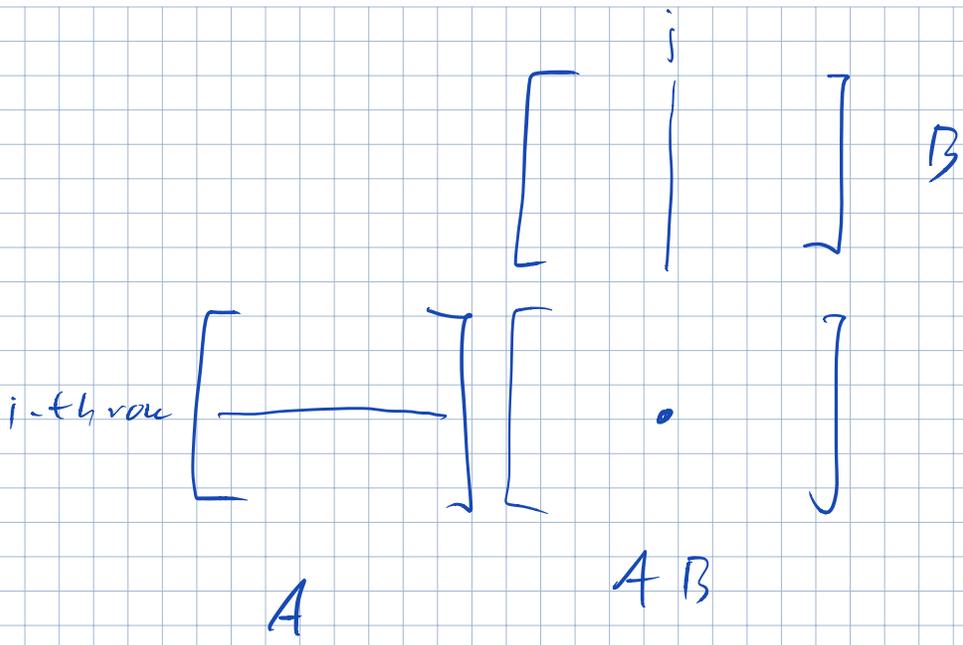
- Hence A^T is invertible, and its inverse is $(A^{-1})^T$.
- The generalization of Theorem 6(b) is as follows:
The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.
- An invertible matrix A is row equivalent to an identity matrix, and we can find A^{-1} by *watching the row reduction of A to I* .
- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

ELEMENTARY MATRICES

■ **Example 5:** Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A , E_2A , and E_3A , and describe how these products can be obtained by elementary row operations on A .



ELEMENTARY MATRICES

- **Solution:** Verify that

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ g - 4a & h - 4b & i - 4c \end{bmatrix}, E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

- Addition of -4 times row 1 of A to row 3 produces E_1A .

ELEMENTARY MATRICES

- An interchange of rows 1 and 2 of A produces E_2A , and multiplication of row 3 of A by 5 produces E_3A .
- Left-multiplication (that is, multiplication on the left) by E_1 in Example 1 has the same effect on any $3 \times n$ matrix.
- Since $E_1 \cdot I = E_1$, we see that E_1 *itself* is produced by this same row operation on the identity.

ELEMENTARY MATRICES

- Example 5 illustrates the following general fact about elementary matrices.
- If an elementary row operation is performed on an $m \times n$ matrix A , the resulting matrix can be written as EA , where the $m \times m$ matrix E is created by performing the same row operation on I_m .
- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I .

ELEMENTARY MATRICES

- **Theorem 7:** An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n , and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1} . $[A \ I] \rightarrow [I \ A^{-1}]$
- **Proof:** Suppose that A is invertible.
- Then, since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} (Theorem 5), A has a pivot position in every row.
- Because A is square, the n pivot positions must be on the diagonal, which implies that the reduced echelon form of A is I_n . That is, $A \sim I_n$.

ELEMENTARY MATRICES

- Now suppose, conversely, that $A \sim I_n$.
- Then, since each step of the row reduction of A corresponds to left-multiplication by an elementary matrix, there exist elementary matrices E_1, \dots, E_p such that $A \sim E_1 A \sim E_2 (E_1 A) \sim \dots \sim E_p (E_{p-1} \dots E_1 A) = I_n$
- That is,

$$E_p \dots E_1 A = I_n \quad (1)$$

- Since the product $E_p \dots E_1$ of invertible matrices is invertible, (1) leads to

$$(E_p \dots E_1)^{-1} (E_p \dots E_1) A = (E_p \dots E_1)^{-1} I_n$$

$$A = (E_p \dots E_1)^{-1}$$

ALGORITHM FOR FINDING A^{-1}

- Thus A is invertible, as it is the inverse of an invertible matrix (Theorem 6). Also,

$$A^{-1} = \left[(E_p \dots E_1)^{-1} \right]^{-1} = E_p \dots E_1$$

- Then $A^{-1} = E_p \dots E_1 \cdot I_n$, which says that A^{-1} results from applying E_1, \dots, E_p successively to I_n .
- This is the same sequence in (1) that reduced A to I_n .
- Row reduce the augmented matrix $[A \ I]$. If A is row equivalent to I , then $[A \ I]$ is row equivalent to $[I \ A^{-1}]$. Otherwise, A does not have an inverse.

ALGORITHM FOR FINDING A^{-1}

- **Example 2:** Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}, \text{ if it exists.}$$

- **Solution:**

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

- Theorem 7 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ANOTHER VIEW OF MATRIX INVERSION

- It is not necessary to check that $A^{-1}A = I$ since A is invertible.
- Denote the columns of I_n by $\mathbf{e}_1, \dots, \mathbf{e}_n$.
- Then row reduction of $[A \ I]$ to $[I \ A^{-1}]$ can be viewed as the simultaneous solution of the n systems

$$A\mathbf{x} = \mathbf{e}_1, A\mathbf{x} = \mathbf{e}_2, \dots, A\mathbf{x} = \mathbf{e}_n \quad (2)$$

where the “augmented columns” of these systems have all been placed next to A to form

$$\begin{bmatrix} A & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix}.$$

ANOTHER VIEW OF MATRIX INVERSION

- The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of A^{-1} are precisely the solutions of the systems in (2).