Math 2270-1

Notes of 10/18/2019

4.8 Finite Difference Equations

The Fibonacci Sequence

- We start with an example.
- consider **the set of sequences** S

$$y_0, y_1, y_2, \dots$$

satisfying the **finite difference equation**

$$y_{k+2} - y_{k+1} - y_k = 0, \qquad k = 0, 1, 2, \dots$$

$u_k = y_k + z_k$

$z_{k+2} - z_{k+1} - z_k = 0$

- This is a linear space. Determine a basis and the dimension of this space.

$y_0, y_1, y_2, y_3, \dots$

$F: \quad S \longrightarrow \mathbb{R}^2$

$F(y_0, y_1, y_2, \dots) = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

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Fibonacci: F(0) = F(1) = 1,
F(k) = F(k-1) + F(k-2)

 k      F(k)    F(k)/F(k-1)

 0         1
 1         1
 2         2  2.000000000000
 3         3  1.500000000000
 4         5  1.666666666667
 5         8  1.600000000000
 6        13  1.625000000000
 7        21  1.615384615385
 8        34  1.619047619048
 9        55  1.617647058824
10        89  1.618181818182
11       144  1.617977528090
12       233  1.618055555556
13       377  1.618025751073
14       610  1.618037135279
15       987  1.618032786885
16      1597  1.618034447822
17      2584  1.618033813400
18      4181  1.618034055728
19      6765  1.618033963167
20     10946  1.618033998522
21     17711  1.618033985017
22     28657  1.618033990176
23     46368  1.618033988205
24     75025  1.618033988958
25    121393  1.618033988670
26    196418  1.618033988780
27    317811  1.618033988738
28    514229  1.618033988754
29    832040  1.618033988748
30   1346269  1.618033988751
```

r

$Y_{k+1} \approx r Y_k$

$Y_{k+1} = r Y_k$

$Y_0 = 1$

- What is the limit of that ratio?

$y_k = f(k) \qquad f = ?$

Try: $\quad y_k = r^k$

$\cancel{(r+1)(r-1) = r^2 - 1}$

$r^{k+2} - r^{k+1} - r^k = 0$

$r^k(r^2 - r - 1) = 0$

$r^2 - r - 1 = 0 \quad \Big| + \frac{5}{4} \qquad (r-a)^2 = r^2 - 2ar + a^2$

$r^2 - r + \frac{1}{4} = \frac{5}{4}$

$\left(r - \frac{1}{2}\right)^2 = \frac{5}{4}$

$r_\pm = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$= \frac{1 \pm \sqrt{5}}{2} \qquad \begin{array}{l} r_+ = 1.6\ldots \\ r_- = -0.6\ldots \end{array}$

Basis $\quad \left\{ y_k = r_+^k, \quad y_k = r_-^k \right\}$

$y_k = \alpha r_+^k + \beta r_-^k$

- Let's consider the general **linear** finite **difference equation**

$$\sum_{j=0}^{n} a_j y_{n+k-j} = a_0 y_{n+k} + a_1 y_{k+n-1} + \ldots + a_n y_k = z_k$$

where
$$k = 0, 1, 2, \ldots \tag{1}$$
and $a_0 a_n \neq 0$.

- The difference equation is **homogeneous** if $z_k = 0$ for all $k = 0, 1, 2, \ldots$ and **inhomogeneous otherwise.**

Recall our major concept that **the general solution of a linear problem equals any particular solution plus the general solution of the homogeneous version.**

- Usually, finding the general solution of the homogeneous problem is easy, and finding a particular solution of the inhomogeneous problem is hard.

- The key to the general solution of the homogeneous version of (1) is the **characteristic polynomial**

$$p(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n = \sum_{j=0}^{n} a_j x^{n-j}.$$

- Suppose r is a root of p, i.e., $p(r) = 0$.

- Then the sequence

$$y_j = r^j, \qquad j = 0, 1, 2, \dots \qquad j = k$$

is a solution of

$$\sum_{j=0}^{n} a_j y_{n+k-j} = 0$$

$$\sum_{j=0}^{n} a_j r^{n+k-j} = r^k \underbrace{\sum_{j=0}^{n} a_j r^{n-j}}_{p(r) = 0} = 0$$

- **Example:** Compute all solutions of the finite difference equation

$$y_{k+2} - 5y_{k+1} + 6y_k = 0, \qquad k = 0, 1, 2, \ldots$$

$y_0 = 2$

$y_1 = 3$

$y_{k+2} = 5y_{k+1} - 6y_k$

$y_2 = 3$

$p(r) = r^2 - 5r + 6 = (r-2)(r-3)$

$y_k = 2^k$

$y_k = 3^k$

$y_k = \alpha 2^k + \beta 3^k$

$y_0 = 2 \quad y_1 = 3$

$y_0 = \alpha 2^0 + \beta 3^0 = \alpha + \beta = 2$

$y_1 =$

$2\alpha + 3\beta = 3$

$2\alpha + 2\beta = 4$

$\beta = -1$

$\alpha = 3$

$y_k = 3 \cdot 2^k - 3^k$

$y_2 = 3 \cdot 4 - 3^2 = 3$

Repeated Roots

- Consider the equation

$$y_{k+2} - 2y_{k+1} + y_k = 0.$$

$$p(r) = r^2 - 2r + 1 = (r-1)^2$$

$$y_k = 1^k = 1$$

$$\underline{y_k = k}$$

$$y_{k+2} - 2y_{k+1} + y_k = k+2 - 2(k+1) + k = 0$$

- In general, if $p(r) = p'(r) = 0$ then

$$y_k = r^k \quad \text{and} \quad y_k = kr^k$$

are solutions of

$$\sum_{j=0}^{n} a_j y_{n+k-j} = 0$$

$$= \sum_{j=0}^{n} a_j (n+k-j) r^{n+k-j}$$

$$= n \sum_{j=0}^{n} a_j r^{n+k-j} + \sum_{j=0}^{n} a_j (k-j) r^{n+k-j}$$

$$= n r^n \underbrace{\sum_{j=0}^{n} a_j r^{k-j}}_{p(r)=0} + r^{n+1} \underbrace{\sum_{j=0}^{n} a_j (k-j) r^{k-j-1}}_{p'(r)=0}$$

$$= 0$$

An Inhomogeneous Equation

- Suppose

$$p(1) = 0 \quad \text{and} \quad \sum_{j=0}^{n} a_j y_{k+n-j} = z = \text{constant}.$$

Try: $y_k = \gamma k$ $\qquad \gamma = ?$

$$\sum_{j=0}^{n} a_j \gamma (k+n-j) = \gamma \underbrace{\sum_{j=0}^{n} a_j}_{p(1)=0} k + \gamma \underbrace{\sum_{j=0}^{n} a_j (n-j)}_{p'(1)}$$

$$\gamma p'(1) = z$$

$$\gamma = \frac{z}{p'(1)}$$

Ex.: what if $p'(1) = 0$

Complex Roots

Ex.: $p(x) = x^2 + 1$

$$y_{k+2} + y_k = 0$$

$$x_{k+2} = -x_k$$

$\{y_k\} = 1, 0, -1, 0, 1, 0, -1, \ldots$

$\{y_k\} = 0, 1, 0, -1, 0, 1, \ldots$