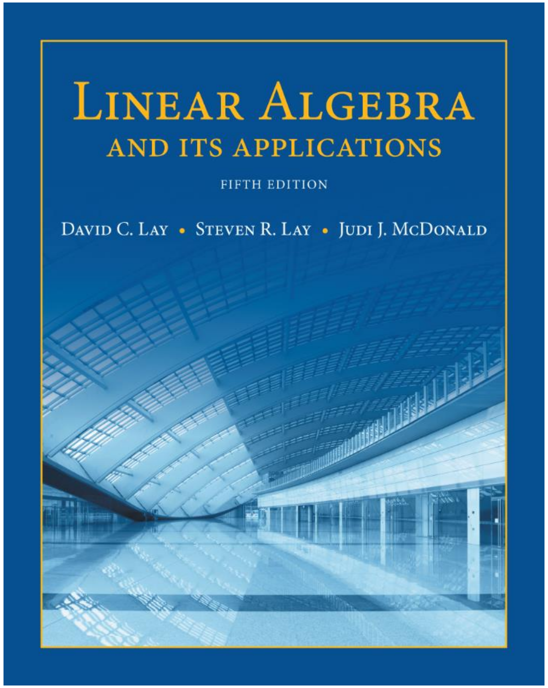
2 Matrix Algebra

2.1

MATRIX OPERATIONS



MATRIX OPERATIONS

- If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A. See the Fig. 1 below.
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} & & \text{Column} & & \\ & & j & & \end{array} \\
\text{Row } i & \left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] = A \\
 & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}
$$

Matrix notation.

MATRIX OPERATIONS

- The columns are denoted by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and the matrix A is written as

$$A = [a_1 \quad a_2 \quad \ldots \quad a_n]$$

- The number a_{ij} is the ith entry (from the top) of the jth column vector $\mathbf{a}_j$.
- The **diagonal entries** in an $m \times n$ matrix $A = \left[a_{ij} \right]$ are $a_{11}, a_{22}, a_{33}, \ldots$, and they form the **main diagonal** of A.
- A **diagonal matrix** is a square $\overset{n \times n}{\cancel{n \times m}}$ matrix whose nondiagonal entries are zero.
- An example is the $n \times n$ identity matrix, I_n.

O matrix

SUMS AND SCALAR MULTIPLES

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

- The two matrices are **equal** if they have the same size (*i.e.*, the same number of rows and the same number of columns) and if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal.

- If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.

SUMS AND SCALAR MULTIPLES

- Since vector addition of the columns is done entrywise, each entry in $A+B$ is the sum of the corresponding entries in A and B.

- The sum $A+B$ is defined only when A and B are the same size.

- **Example 1:** Let $A=\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B=\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix},$

  and $C=\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. Find $A+B$ and $A+C$.

SUMS AND SCALAR MULTIPLES

- **Solution:** $A+B=\begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$ but $A+C$ is not defined because A and C have different sizes.

- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

- **Theorem 1:** Let A, B, and C be matrices of the same size, and let r and s be scalars.
  a. $A+B=B+A$

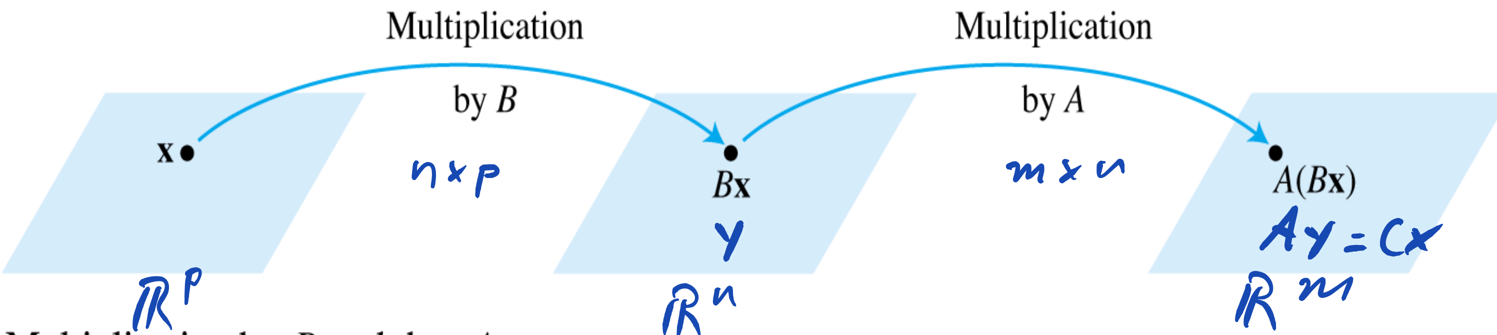
SUMS AND SCALAR MULTIPLES

b. $(A+B)+C=A+(B+C)$
c. $A+0=A$
d. $r(A+B)=rA+rB$
e. $(r+s)A=rA+sA$
f. $r(sA)=(rs)A$

- Each quantity in Theorem 1 is verified by showing that the matrix on the left side has the same size as the matrix on the right and that corresponding columns are equal.

MATRIX MULTIPLICATION

- When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$. See the Fig. 2 below.



Multiplication by B and then A.

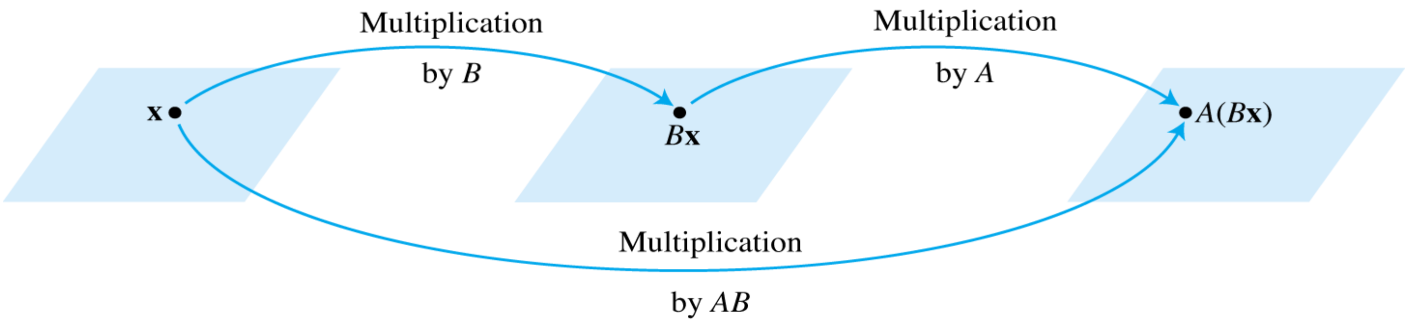
$C = AB$

$m \times p$

- Thus $A\,(B\mathbf{x})$ is produced from x by a *composition of mappings*—the linear transformations.

MATRIX MULTIPLICATION

- Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that $A(B\text{x})=(\text{AB})\text{x}$. See Fig. 3 below



Multiplication by AB.

- If A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in $\mathbb{R}^p$, denote the columns of B by $\mathbf{b}_1, \ldots, \mathbf{b}p$ and the entries in $\mathbf{x}$ by $\mathbf{x}_1, \ldots, \mathbf{x}_p$.

MATRIX MULTIPLICATION

- Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_p\mathbf{b}_p$$

- By the linearity of multiplication by A,

$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1) + \ldots + A(x_p\mathbf{b}_p)$$
$$= x_1 A\mathbf{b}_1 + \ldots + x_p A\mathbf{b}_p$$

- The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$, using the entries in $\mathbf{x}$ as weights.
- In matrix notation, this linear combination is written as

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}\mathbf{x}$$

MATRIX MULTIPLICATION

- Thus multiplication by $\underbrace{\begin{bmatrix} A\mathrm{b}_1 & A\mathrm{b}_2 & \cdots & A\mathrm{b}_p \end{bmatrix}}_{C}$ transforms $\mathbf{x}$ into $A(B\mathbf{x})$.

- **Definition:** If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$.

- That is,

$$\underset{m\times n}{A}\underset{n\times p}{B} = A\begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \cdots & \mathrm{b}_p \end{bmatrix} = \begin{bmatrix} A\mathrm{b}_1 & A\mathrm{b}_2 & \cdots & A\mathrm{b}_p \end{bmatrix}$$

- *Multiplication of matrices corresponds to composition of linear transformations.*

MATRIX MULTIPLICATION

- **Example 3:** Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

- **Solution:** Write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, and compute:

$$C = AB = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = B$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} = AB$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 8 \\ 7 & 10 \end{bmatrix} = BA$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix} = BA$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 2 & 2 \end{bmatrix} AB$$

MATRIX MULTIPLICATION

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 0 \\ 13 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

- Then

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix} \\ \begin{array}{ccc} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{array} \end{array}$$

MATRIX MULTIPLICATION

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Row—column rule for computing AB

- If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = +a_{i2}b_{2j} + \dots a_{i1}b_{1j} + a_{in}b_{nj}$$

$$c_{ij} = (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

PROPERTIES OF MATRIX MULTIPLICATION

- **Theorem 2:** Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.
  a. $A(BC) = (AB)C$ (associative law of multiplication)
  b. $A(B+C) = AB + AC$ (left distributive law)
  c. $(B+C)A = BA + CA$ (right distributive law)
  d. $r(AB) = (rA)B = A(rB)$ for any scalar r
  e. $I_m A = A = AI_n$ (identity for matrix multiplication)

PROPERTIES OF MATRIX MULTIPLICATION

- **Proof:** Property (a) follows from the fact that matrix multiplication corresponds to composition of linear transformations (which are functions), and it is known that the composition of functions is associative. Let

$$C = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_p \end{bmatrix}$$

- By the definition of matrix multiplication,

$$BC = \begin{bmatrix} B\mathbf{c}_1 & \cdots & B\mathbf{c}_p \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} A(B\mathbf{c}_1) & \cdots & A(B\mathbf{c}_p) \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

- The definition of AB makes $A(B\mathrm{x}) = (AB)\mathrm{x}$ for all $\mathbf{x}$, so
$$A(BC) = \begin{bmatrix} (AB)\mathrm{c}_1 & \cdots & (AB)\mathrm{c}_p \end{bmatrix} = (AB)C$$
- The left-to-right order in products is critical because AB and BA are usually not the same.
- Because the columns of AB are linear combinations of the columns of A, whereas the columns of BA are constructed from the columns of B.
- The position of the factors in the product AB is emphasized by saying that A is *right-multiplied* by B or that B is *left-multiplied* by A.

PROPERTIES OF MATRIX MULTIPLICATION

- If $AB = BA$, we say that A and B **commute** with one another.

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **Warnings:**
  1. In general, $AB \neq BA$.
  2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
  3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

POWERS OF A MATRIX

- If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}$$

- If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.

- If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself. $A^0 = I$

- Thus A^0 is interpreted as the identity matrix.

THE TRANSPOSE OF A MATRIX

- Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

THE TRANSPOSE OF A MATRIX

- The transpose of a product of matrices equals the product of their transposes in the *reverse* order.