# Math 2270-1

# Notes of 08/21/2019

## A Word of Caution

Most linear algebra calculations are done by computer. We are working through examples to recognize and understand the underlying concepts and principles.

In modern science and engineering linear systems can easily involve thousand, or millions, of equations.

The examples in the notes, class work, and textbook involve a great deal of copying. If we were actually learning how to do the calculations by hand we'd pay much attention to making the calculations efficient.

## 1.2 Row Reduction and Echelon Forms

- The textbook~~s~~ uses the phrases "Echelon Form" and "Row Echelon Form" interchangeably. Many texts insist on including the work "Row".
- Remember that we solved a linear system by applying elementary row operations to introduce zeros below the diagonal.

- There are three kinds of elementary row operations that do not change the solution set of a linear system:

1. **Add a multiple of one row to another row.**
2. **Interchange two rows.**
3. **Multiply a row by a nonzero constant.**

- In this section we have a closer look.
- The concepts apply to any matrix, whether augmented or not.
- recall: matrix, square, rectangular, rows, columns, entries.
- A row is **nonzero** if it contains at least one non-zero entry.
- the **leading entry** of a row is the leftmost non-zero entry in that row.
- A rectangular matrix is in **echelon form** if it has the following three properties:

1. **All nonzero rows are above any rows of all zeros.**
2. **Each leading entry of a row is in a column to the right of the leading entry of the row above it.** 0...0 ■ ——
   0...0 0 •
3. **All entries in a column below a leading entry are zeros.**

- Property 3. is a consequence of property 2., it's included for emphasis.
- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**.

**4. The leading entry in each non-zero row is 1.**

**5. Each leading 1 is the only nonzero entry in its column.**

- A matrix in **(reduced) echelon form** is also called a **(reduced) echelon matrix**.
- **Examples.** Let $*$ denote an arbitrary (and possibly zero) entry, and $\bullet$ the leading and hence non-zero) entry in a row. 0 or 1 mean the entry equals 0 or 1. Determine if the given matrix is in (reduced) row echelon form. If not indicate what would need to be changed to make it so.

$\begin{bmatrix} \bullet \end{bmatrix}$

$\begin{bmatrix} 0 \end{bmatrix}$

- There are just 4 types of $2 \times 2$ matrices in row echelon form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \bullet \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \bullet & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \bullet & * \\ 0 & \bullet \end{bmatrix}.$$

$\cancel{\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}}$ $\cancel{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$

- **Exercise:** how many types of $3 \times 3$ echelon matrices are there?

- More examples. We show the echelon matrix with its corresponding reduced echelon form.

$$\begin{bmatrix} \bullet & * & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & * & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet & * \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & * \\ 0 & \mathbf{1} & 0 & 0 & * \\ 0 & 0 & \mathbf{1} & 0 & * \\ 0 & 0 & 0 & \mathbf{1} & * \end{bmatrix}$$

$$\begin{bmatrix} \bullet & * & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & * & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

$$\begin{bmatrix} \bullet & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} \mathbf{1} & * & 0 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & \mathbf{1} & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Two matrices are **row equivalent** if one can be formed into the other by a finite sequence of elementary row operations.
- It's plausible, but perhaps not obvious, that every matrix is row equivalent to a unique matrix in reduced row echelon form. The proof given in appendix A of the textbook uses concepts from chapter 4, however.

## Pivot Positions

- **pivot position** in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the reduced row echelon from of $A$. A **pivot row** of $A$ is a row that contains a pivot position and a **pivot column** of $A$ is a column that contains a pivot position.

Many fundamental concepts in the first four chapters of the textbook will be connected in one way or other with pivot positions in a matrix. (That's OK, but you should know that in many LA textbooks row echelon forms and pivot positions play a much less prominent role.)

# Computing the reduced echelon form

- The textbook describes a general algorithm to compute the reduced row echelon form by elementary row operations.

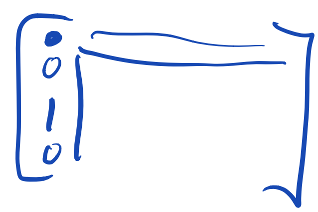
**1.** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is the top entry.

**2.** Select a nonzero entry in the pivot column as a **pivot**. If necessary interchange rows to move the pivot into the pivot position.

In principle it does not matter which nonzero entry is chosen as the pivot. However, the choice may be significant numerically if inexact arithmetic, as on a calculator, is used.

**3.** Subtract multiples of the pivot row from the rows underneath to introduce zeros in the pivot column below the pivot row.

**4.** Apply the first three steps to the submatrix below the pivot row and to the right of the pivot column. Repeat until there are no nonzero rows left to modify.



**5.** Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not equal to 1, make it 1 by scaling the row.

- Steps 1–4 are called **forward elimination** or the **forward phase** of the algorithm. Step 5 is called the **backward phase**.

- Let's work through the textbook example:

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \qquad r_2 \leftarrow r_2 - r_1$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \qquad r_3 \leftarrow r_3 - \frac{3}{2} r_2$$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{matrix} \div 3 \\ \div 2 \\ \phantom{0} \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{matrix} r_1 \leftarrow r_1 - 2r_3 \\ r_2 \leftarrow r_2 - r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \qquad r_1 \leftarrow r_1 + 3r_2$$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

## Solution of Linear Systems

- Example: Suppose the augmented matrix of a $3 \times 3$ matrix has been reduced to the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- the pivot columns correspond to **basic variables**, and the columns not containing any pivot elements to **free variables**. If the linear system is consistent then arbitrary values may be assigned to the free variables, which determines uniquely the basic variables.
- A linear system is consistent if the reduced row echelon formof its augmented matrix does not contain a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & b \end{bmatrix}$$

  with $b$ being nonzero.
- In other words, a linear system is consistent if an only if the last column of its augmented matrix is **not** a pivot column.
- The solution of a consistent linear system is unique if there are no free variables. There are infinitely many solutions of a consistent linear system if there are free variables, and the solutions can be expressed in terms of those free variables.

## Backward Substitution

- Some work can be saved by obtaining the solution from the (unreduced) row echelon form by working backward from the last pivot row.