Math 2270-1

Notes of 9/27/2019

- Recall: a vector space is a set of objects that can be added and multiplied with scalars, such that those 10 axioms hold.
- A subspace of a vectors space V is a non-empty subset H of V that is closed under addition and scalar multiplication.
- Loose end from Wednesday:

  Span
- the span of a set of vectors in a space V is a subspace of V.

4.2 Null and Column Spaces

- Today's topic: Vector Spaces associated with a matrix or a linear transformation.
- Suppose, as usual that A is an $m \times n$ matrix. We denote the columns of A by $\mathbf{a}_1$, ..., $\mathbf{a}_n$, i.e.,
$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n\,].$$
- We associate with A the linear (matrix) transformation
$$T(\mathbf{x}) = A\mathbf{x}$$
- There are four spaces associated with A.

The Column Space of A

- The **column space** of A is the set of all linear combinations of columns of A. In other words,

$$\begin{aligned}\mathrm{Col}A &= \mathrm{span}\left\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\right\} \\ &= \left\{\mathbf{b} : \mathbf{b} = A\mathbf{x} \quad \text{for some } \mathbf{x} \in \mathbb{R}^n\right\}.\end{aligned}$$

- $\mathrm{Col}A$ being the span of a set of vectors clearly is a linear space.
- It's also obvious that the set $\mathrm{Col}A$ contains the zero vector, and is closed under addition and scalar multiplication.
- $\mathrm{Col}A$ is a subspace of $\mathbb{R}^m$.
- $\mathrm{Col}A$ equals $\mathbb{R}^m$ if and only if the matrix transform T is onto.
- $\mathrm{Col}A$ is also called the **range** of T.

The Null Space of A

- The **null space** of A is the set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\mathrm{Nul}A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}.$$

- The set $\mathrm{Nul}A$ contains the origin and is closed under addition and scalar multiplication. It is a vector space!
- $\mathrm{Nul}A$ is a subspace of $\mathbb{R}^n$.
- $\mathrm{Nul}A$ is the zero space,

$$\mathrm{Nul}A = \{\mathbf{0}\},$$

  if and only if the columns of A are linearly independent.
- $\mathrm{Nul}A$ is the zero space if and only if the matrix transformation T is one-to-one.
- $\mathrm{Nul}A$ is also called the **kernel** of T.

The Row Space of A

- The Row Space of A is the set of all linear combinations of rows of A.
- You can interpret it as the column space of A^T.
- The row space is a subset of $\mathbb{R}^n$.

The Left Null Space of A

- The left Null Space of A is the null space of A^T.
- It is a subspace of $\mathbb{R}^m$.

the textbook does not mention the row space or the left null space. I list them only for completeness.

- Consider Example 5 in the textbook. (modified to simplify the arithmetic)

$$A = \begin{bmatrix} 2 & 4 & -2 & 2 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

- Discuss the table on page 206 of the textbook

Kernel and Range of a Linear Transformation

- Suppose we have a linear transformation

  $$T : V \longrightarrow W$$

  from a vector space V to a vector space W.

- Recall that T is linear if

  $$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

  for all $\mathbf{u}$ and $\mathbf{v}$ in V, and

  $$T(c\mathbf{v}) = cT(\mathbf{v})$$

  for all scalars c and $\mathbf{v}$ in V.

- We define the **kernel** of the T to be the set of all vectors $\mathbf{v}$ in V such that

  $$T(\mathbf{v}) = 0.$$

- As before, the **range of** T is the set of all $\mathbf{w}$ in W that can be written as $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V.

- If T is defined by a matrix transform,

  $$T(\mathbf{x}) = A\mathbf{x},$$

  then the range of T is the column space of A, and the kernel of T is the null space of A.

- Example 8. Suppose V is the set of all continuously differentiable functions defined on some interval $I = [a, b]$. Let W be the space of all continuous functions defined on I. Then the differentiation operator $Df = f'$ is a linear operator from V to W. What is its kernel?

- What if D denoted the second derivative (and V is the space of twice continuously differentiable functions)?

- Example 9: Suppose

$$Dy = y'' + \omega^2 y$$

for some fixed ω? Compute the kernel of D

- What about the kernel of

$$Dy = y'' - \omega^2 y?$$