Write your name here:

Math 2270-1 — Fall 2019 — Exam 1

1	2	3	4	5	6	7	Total

Instructions

1. **This exam is closed books and notes. Do not use a calculator or other electronic devices. Do not use scratch paper.**
2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it's obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.
4. To avoid distraction and disruption **I am unable to answer questions during the exam.** If you believe there is something wrong with a problem state so, and if you are right you will receive generous credit. I will also be unable to discuss individual problems and grading issues with you after you are done while the exam is still in progress.
5. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to **check your answers.**
6. When you are done hand in your exam, pick up an answer sheet, and leave the room. Do not return to your seat.
7. All questions have equal weight.
8. Clearly indicate (for example, by circling or boxing) your final answers.

Note that this answer set contains more information than you needed to provide on the exam.

-1- (Linear Systems.) Find all solutions of the linear system

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Discussion:

Clearly the second equation is just twice the first, and is therefore superfluous. Solving the first equation for x_1 gives

$$x_1 = 3 - 2x_2$$

where the free variable x_2 is arbitrary. You could also solve for x_2 which gives

$$x_2 = \frac{1}{2}(3 - x_1).$$

Either answer can be checked by substituting in the original system. For example, substituting the first solution in the first equation gives

$$3 - 2x_2 + 2x_2 = 3.$$

-2- (Row Echelon Form.) Compute the reduced row echelon form of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Indicate the pivot rows and columns.

Discussion:

Denoting the i-th row of the matrix by r_i we get

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \qquad r_2 = r_2 - r_1 \quad \text{and} \quad r_3 = r_3 - 2r_1 \\
&\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_3 = r_3 - r_2 \\
&\longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 = r_1 - r_2
\end{aligned}$$

Rows 1 and 2 are pivot rows, and columns 1 and 2 are pivot columns.

-3- (Linear Transformations.) Let T be the linear transformation such that if $\mathbf{y} = T(\mathbf{x})$ then $\mathbf{y}$ is the vector obtained from $\mathbf{x}$ by rotating $\mathbf{x}$ **clockwise** by the angle t. Compute the standard matrix of the transformation.

Discussion:

In general the standard matrix of a linear transform of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by

$$A = [\,T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n)\,]$$

where $\mathbf{e}_i$ is the i-th standard basis vector (which is all zero except that the i-th entry equals zero) in $\mathbb{R}^n$.

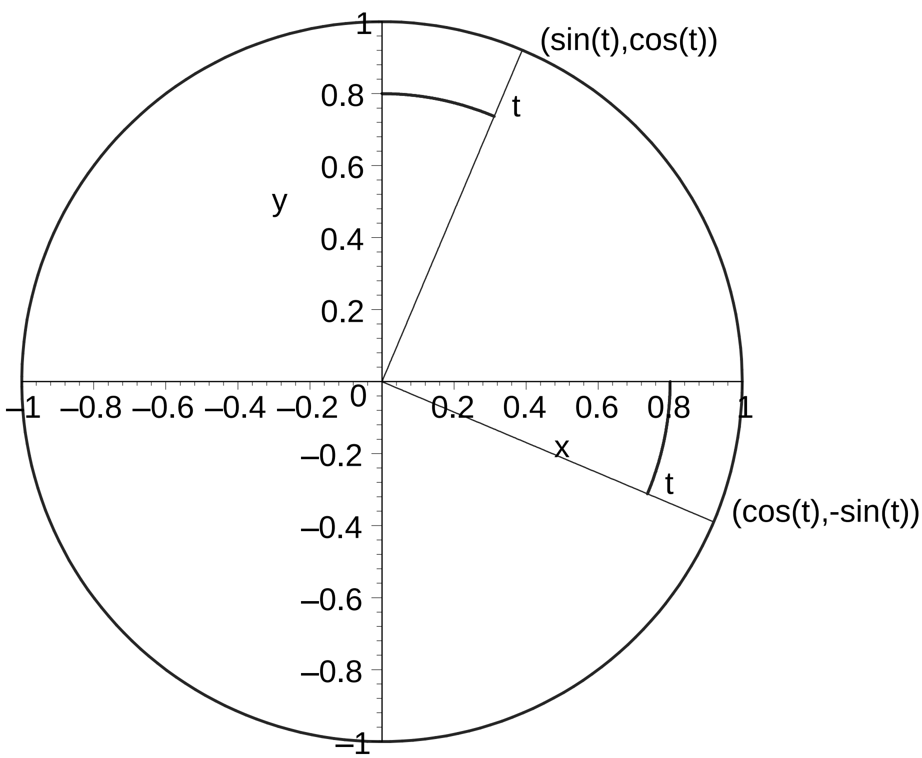

Figure 1. Clockwise Rotation.

As shown in Figure 1, in this case we get

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos t\\-\sin t\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\sin t\\\cos t\end{bmatrix}.$$

Hence

$$T(\mathbf{x}) = A\mathbf{x} \qquad \text{where} \qquad A = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}$$

-4- (Linear Transformations and Linear Systems.) Let

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

be the linear transformation satisfying

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.$$

Compute the standard matrix A of T.

Discussion:

We need to find the 2×2 matrix A that satisfies

$$T(\mathbf{x}) = A\mathbf{x}.$$

This is very simple if you notice that in both the given equations the first entry of the image is the sum, and the second the difference, of the preimage. Thus

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and indeed,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.$$

However, if you don't recognize this you can proceed as in problem 6 of hw LA3. Letting

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2a+b \\ 2c+d \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3a+5b \\ 3c+5d \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

This gives the linear system

$$\begin{array}{ccccccccc} 2a & + & b & & & & & = & 3 \\ & & & & 2c & + & d & = & 1 \\ 3a & + & 5b & & & & & = & 8 \\ & & & & 3c & + & 5d & = & -2 \end{array}$$

Technically, this is a 4×4 linear system, but of course it reduces to two 2×2 linear systems that share the same coefficient matrix:

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Solving these two linear systems gives

$$a = b = c = 1 \quad \text{and} \quad d = -1, \quad \text{i.e.,} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

as before.

-5- (Linear Independence.) Consider the set of three vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

Is this set linearly independent? Why or why not?

Discussion:

Again, there are many ways to show that the set is linearly dependent. The textbook approach would be to compute the row echelon form of the augmented matrix

$$A = \begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{bmatrix} \quad r_2 = r_2 - 2r_1 \quad \text{and} \quad r_3 = r_3 - 3r_1$$

$$\longrightarrow \begin{bmatrix} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_3 = r_3 - 2r_2$$

This shows that x_3 is free and arbitrary, and so the set of vectors is linearly dependent.

However, calling the three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, respectively, you may have seen directly that

$$\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$$

which is a linear dependence relation.

-6- (Row Echelon Forms.) List all possible row echelon forms of a 2×2 matrix. Use 0 to denote the entry zero, a bullet $\bullet$ to denote a pivot, and an asterisk $*$ or plus $+$ to denote an arbitrary entry.

Discussion:

As we discussed on August 21, there are four possible row echelon forms:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \bullet \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \bullet & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \bullet & * \\ 0 & \bullet \end{bmatrix}.$$

-7- (True or False.) Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers.

1. T F A linear system of 2 equations in 3 unknowns always has a solution.

F

consider, for example

$$x + y + z = 2 \quad \text{and} \quad x + y + z = 3$$

2. T F The linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

T

we **define** $A\mathbf{x}$ to be the linear combination of the columns of A with the entries of $\mathbf{x}$ being the coefficients.

3. T F A linear system of 3 equations in 2 unknowns cannot have a solution.

F

consider for example $x = y = 1$ and the linear system

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \\ 2x + 3y &= 5 \end{aligned}$$

4. T F In a linearly dependent set of vectors every vector can be written as a linear combination of the other vectors.

F

consider for example the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

The second is a linear combination (multiple) of the first, but the first cannot be written as a multiple of zero. A more complicated example not involving the zero vector is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}.$$

Here, no vector is the zero-vector, the first vector is not a linear combination of the other two, and the third vector equals twice the second.

5. T F In a linearly independent set of vectors no vector can be written as a linear combination of the other vectors.

T

If a vector could be written as a linear combination of the others we could subtract the right side from the left and obtain a linear dependence relation.

6. **T** **F** For the linear system $A\mathbf{x} = \mathbf{b}$ to have a unique solution A must have at least as many rows as it has columns.

$\boxed{\textbf{T}}$ If the system is consistent and we have fewer equations than unknowns then there must be free variable which can be set arbitrarily.

7. **T** **F** The span of the set of columns of a 3×2 matrix may be all of $\mathbb{R}^3$.

$\boxed{\textbf{F}}$ The span of a set of two vectors in $\mathbb{R}^3$ is a plane that cannot fill all of $\mathbb{R}^3$.

8. **T** **F** The standard matrix of a linear transformation from $\mathbb{R}^s$ to $\mathbb{R}^t$ is an $s \times t$ matrix.

$\boxed{\textbf{F}}$ it's $t \times s$.

9. **T** **F** The range (image) of a linear transformation contains the origin of its codomain.

$\boxed{\textbf{T}}$ If T is the linear transformation and A its standard matrix we get

$$T(\mathbf{0}) = A\mathbf{0} = \mathbf{0}.$$

So $\mathbf{0}$ is the image of $\mathbf{0}$.

10. **T** **F** A linear system can have a unique solution only if it is square.

$\boxed{\textbf{F}}$ start with a square linear system with a unique solution, and add any linear combination of the original equations to the system, to get a system with a unique solution, but more equations than unknowns.