Math 2270-1

Notes of 10/16/2019

4.7 Change of Basis

- There are infinitely many bases of a vector space. Which is the best or most convenient for a particular purpose depends on the context. But we want to be able to easily convert from one to another.
- The textbook describes the concept of change of basis in terms of the coordinate vectors corresponding to given bases of a vector space.
- All (finite-dimensional) vector spaces are isomorphic to $\mathbb{R}^n$.
- So let's suppose that we are considering only R^n and its subspaces. This will greatly simplify things.
- Consider three bases of $\mathbb{R}^n$:

$$\begin{aligned}\mathcal{I} &= \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}\\ \mathcal{B} &= \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}\\ \mathcal{C} &= \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}\end{aligned}$$

- $\mathbf{e}_i$ is the standard basis vector that is all zero except that the i-th entry is 1.

- Moreover, suppose $\mathbf{x}$ is a vector in $\mathbb{R}^n$, i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \mathbf{e}_i = \sum_{i=1}^{n} \beta_i \mathbf{b}_i = \sum_{i=1}^{n} \gamma_i \mathbf{c}_i.$$

- Of course, the β_i are the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{B}$ and the γ_i are the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{C}$. The entries $\mathbf{x}_i$ of $\mathbf{x}$ are the coordinates of $\mathbf{x}$ with respect to the standard basis.

$$[\mathbf{x}]_I = \mathbf{x} \qquad [\mathbf{x}]_{\mathcal{B}} = [\beta_i]_{i=1,\ldots,n} \qquad [\mathbf{x}[_{\mathcal{C}} = [\gamma_i]_{i=1,\ldots,n}$$

- We want to know how to convert from one set of coordinates to another!

- As usual, let's collect our sets of vectors into matrices:

$$\begin{aligned} I &= [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_n\,] \\ B &= [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \ldots \quad \mathbf{b}_n\,] \\ C &= [\,\mathbf{c}_1 \quad \mathbf{c}_2 \quad \ldots \quad \mathbf{c}_n\,] \end{aligned}$$

- Since the underlying vector sets are bases we know that I, B, and C are square and invertible.

- We know that

$$\mathbf{x} = B[\mathbf{x}]_{\mathcal{B}} = C[\mathbf{x}]_{\mathcal{C}}$$

In other words

$$B^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}} \quad \text{and} \quad C^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{C}}.$$

- Hence

$$[\mathbf{x}]_{\mathcal{B}} = B^{-1}C[\mathbf{x}]_{\mathcal{C}} \quad \text{and} \quad [\mathbf{x}]_{C} = C^{-1}B[\mathbf{x}]_{\mathcal{B}}.$$

- This formula lets us convert from any basis to any other basis.
- The textbook uses the notations

$$C^{-1}B = \underset{\mathcal{C} \longleftarrow \mathcal{B}}{P}$$

$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$

and

$$B^{-1}C = \underset{\mathcal{B} \longleftarrow \mathcal{C}}{P} = \left(\underset{\mathcal{C} \longleftarrow \mathcal{B}}{P}\right)^{-1}$$

- Example 2, textbook. Let

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix} \right\} \quad \text{with} \quad B = \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 14 \\ -11 \end{bmatrix} x_d$$

$$\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\} = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\} \quad \text{with} \quad C = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} -19 \\ -1 \end{bmatrix} = x$$

- Suppose we want to convert from the basis $\mathcal{B}$ to the basis $\mathcal{C}$:

$$\mathbf{x}_{\mathcal{C}} = C^{-1} B \mathbf{x}_{\mathcal{B}}.$$

$$\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$$

- We get

$$C^{-1} = \frac{1}{7} \begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 42 & 28 \\ -35 & -21 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C^{-1} B = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} 14 \\ -11 \end{bmatrix} = [x]_{\mathcal{C}} \div 7$$

- Example: check with

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -19 \\ -1 \end{bmatrix} = x$$

- Example, polynomials revisited. Consider again the space V of quadratic polynomials.
- The standard basis of that space is

$$\mathcal{C} = \{1, x, x^2\}$$

- Actually, in many applications it is convenient to use other bases to express polynomials.
- An example of another useful basis is the **Bernstein-basis**:

$$\mathcal{B} = \{x^2, 2x(1-x), (1-x)^2\}$$

- (These basis concepts can of course be generalized to polynomials of degree n.)
- So we can write

$$p(x) = ax^2 + bx + c = \alpha x^2 + 2\beta x(1-x) + \gamma(1-x)^2$$

- We want to express α, β, and γ in terms of a, b, c, and vice versa.
- expanding the term on the right gives

$$\begin{aligned} p(x) &= \alpha x^2 + 2\beta(x - x^2) + \gamma(1 - 2x + x^2) \\ &= (\alpha - 2\beta + \gamma)x^2 + (2\beta - 2\gamma)x + \gamma \end{aligned}$$

- This gives the change of basis formula

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$\mathcal{C} \xleftarrow{P} \mathcal{B}$

- Inverting the matrix in this formula gives the corresponding formula

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$$

Linear Transformations

- Suppose A is an $m \times n$ matrix. It defines a linear transformation

$$\mathbf{y} = A\mathbf{x} \qquad \begin{array}{l} x \to Ax \\ \mathbb{R}^n \to \mathbb{R}^m \end{array}$$

  from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose we want to express **the same linear transform** in terms of a basis

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$$

  of $\mathbb{R}^n$ and a basis

$$\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_m\}$$

  of $\mathbb{R}^m$.

- In other words, we want to find a matrix T such that

$$[\mathbf{y}]_{\mathcal{C}} = T[\mathbf{x}]_{\mathcal{B}}$$

- As usual, we collect our bases vectors into matrices:

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{bmatrix}$$

  and

$$C = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_m \end{bmatrix}.$$

- What is T in terms of A, B, and C?
- Think about the sizes of T, A, B, and C.

- The situation is illustrated in this "commuting diagram":

$$
\begin{array}{cccccc}
 & \mathbf{x} & \longrightarrow & A\mathbf{x} & \\
B\mathbf{x} & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m & \mathbf{x} \\
\Big\uparrow & \Big\uparrow & & \Big\downarrow & \Big\downarrow \\
\mathbf{x} & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m & C^{-1}\mathbf{x} \\
 & \mathbf{x} & \longrightarrow & T\mathbf{x} & \\
 & & & =? &
\end{array}
$$

$Tx = C^{-1} A B x$

$T = C^{-1} A B$

$[y]_C = T[x]_B$

- start in the lower left corner. Move to the lower right corner either by going directly to the right, or in three steps by going up, right, and then down. We want T to be such that in either way we get to the same vector.

- Clearly,

$$T = C^{-1}AB.$$

- By the same token,

$$A = CTB^{-1}.$$

$m \times m \quad m \times n \quad n \times n \quad = m \times n$

- check the dimensions.

In the special case that $m = n$ and $B = C$ we get that

$$T = B^{-1}AB.$$

- In this case A and B are said to be similar, and the formula is called a **similarity transform**.
- This will become significant when we talk about eigenvalues and vectors in chapter 5.

- In the mean time, start with difference equations, specifically the Fibonacci sequence.

$$F_n$$

$$F_0 = 1 \qquad F_1 = 1$$

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

$$F_{n+2} = F_n + F_{n+1} \qquad n = 0, 1, 2, \ldots$$