Math 2270-1

Notes of 10/15/19

Quick Review of Key Concepts

- **Linear Algebra** is the mathematics of linear functions on finite dimensional spaces. (The subject of linear operators on infinite-dimensional spaces, like function spaces, is called **Functional Analysis**.)
- Every linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by its standard matrix.
- Linear spaces are sets of vectors that are closed under addition and scalar multiplication, and are such that those rules hold.
- A basis is a linearly independent spanning sets. All bases have the same number of vectors, and that number is the dimension,
- Two linear spaces are isomorphic if there exists a linear function from one to the other that is one-to-one and onto.
- Two spaces are isomorphic if and only if they have the same dimension.
- Every isomorphism is invertible, and its inverse is also an isomorphism.
- The coordinate vector with respect to a given basis is the vector of coefficients with respect to that basis.

- The mapping from a vector space to the coordinate vector is an isomorphism.
- Hence, and in particular, all n-dimensional spaces are isomorphic to $\mathbb{R}^n$.
- All n-dimensional spaces are isomorphic.
- Two linear spaces are isomorphic if and only if they have the same dimension.
- Given the bases of two vector spaces of the same dimension the unique linear function that maps each basis vector in one basis to its counterpart in the other is an isomorphism.

4.6 Rank

- The key fact: Row and Column spaces of any matrix A have the same dimension. We call that number the **rank** of A.
- The rank of a matrix in row echelon form is the number of pivots.

- So we have to ask how do row operations affect the Row and Column Spaces, and their dimensions?
- Here are some facts about row and column spaces
- Theorem 13. If A and B are row equivalent they have the same row space.
- To see this assume that B is obtained from A by applying elementary row operations. Thus the rows of B are linear combinations of the rows of A. This means the row space of B is a subspace of the row space of A. Row operations are invertible, so by the same argument the row space of A is a subspace of the row space of B. The spaces must be the same.
- Theorem 6, page 214: The pivot columns of A form a basis for the column space of A.
- To see this suppose B is the row echelon form of A. Clearly, the pivot columns of B are linearly independent since none of them is a linear combination of the preceding columns. They also span the column space of B. Moreover, B is obtained from A by row operations which do not change the solution set of the linear system $A\mathbf{x} = \mathbf{0}$. Thus the columns of B have the same dependence relations as the columns of A and a set of columns of A is linearly independent if and only if the corresponding set of columns of B is linearly independent.

- The dimension of the column space of A equals the number of pivot columns of A, and that of its row space equals the number of pivot rows of B (and A). Those two numbers are equals, and so the dimensions of row and column spaces are equal.
- Moreover, consider the null space of A. Its dimension equals the number of free variables, which, for an $m \times n$ matrix A, equals n minus the rank of A.
- To summarize, we have these facts:

$$\begin{aligned}\mathrm{rank}A &= \mathrm{dimCol}A \\ &= \mathrm{dimRow}A \\ &= \text{the number of pivots of } A\end{aligned}$$

and

$$\mathrm{rank}A + \mathrm{dimNul}A = n.$$

- Example 2, textbook. Compute a basis of the column space, the row space, and the null space, as well as the rank, of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Use the fact that A is row equivalent to

$$B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

That's a weird row equivalent form because you would ordinarily expect the first rows to be identical, or multiples of each other.

Invertible Matrices

- Recall the invertible matrix theorem as we know it so far:

Invertibility Theorem Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \longrightarrow A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$

i. The linear transformation maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $CA = I$. (Of course, the left and right inverses C and D are actually equal.)

l. A^T is an invertible matrix.

- We can now add the following items:

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\mathrm{Col}A = \mathbb{R}^n$

o. $\dim\mathrm{Col}A = n$

p. $\mathrm{rank}A = n$

q. $\mathrm{Nul}A = \{\mathbf{0}\}$

r. $\dim\mathrm{Nul}A = 0$.

- Of course we also have an item that the textbook does not list

s. $|A| \neq 0$

- Instead of the column space in items **m**, **n**, **o**, we could also use the rows:

m'. The rows of A form a basis of $\mathbb{R}^n$.

n'. $\mathrm{Row}A = \mathbb{R}^n$

o'. $\dim\mathrm{Row}A = n$