Math 2270-1 — Fall 2019 — Exam 3 Answers

-1- (Coordinate Vectors.) Let V be the vector space of quadratic polynomials. Compute the coordinate vector $[p]_{\mathrm{B}}$ of the polynomial

$$p(x) = (2x-1)(x-3)$$

with respect to the basis

$$\mathrm{B} = \left\{1, x, x^2\right\}$$

Discussion:

Expanding p gives

$$p(x) = 3 - 7x + 2x^2.$$

This gives

$$[p]_{\mathrm{B}} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}$$

-2- (Dimensions.) Recall that a (real) square matrix A is symmetric if

$$A = A^T.$$

Let V be the space of symmetric $n \times n$ matrices. What is the dimension of V if $n = 3$? What is it for general n?

Discussion:

In class we discussed the dimension of the space of triangular matrices. That's the same as the dimension of the space of symmetric matrices since the upper triangle is the reflection of the lower triangle. So the dimension of V is the number of entries along the diagonal, and in the part above the diagonal. That number is

$$\dim V = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

For the special case that $n = 3$ we get that $\dim V = 6$.

-3- (Isomorphism.) Construct an isomorphism from the vector space of quadratic polynomials to the vector space of upper triangular 2×2 matrices.

Discussion:

We discussed in class that we get an isomorphism by mapping a linear combination of vectors in a basis of the domain to the linear combination with the same coefficients of vectors in a basis of the image. So, for example, using the basis

$$\left\{1, x, x^2\right\}$$

for the space of quadratic polynomials, and the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

we get the isomorphism

$$I(a + bx + cx^2) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

-4- (Compute Eigenvalues and Eigenvectors.) Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$$

Discussion:

The characteristic equation of A is

$$p(\lambda) = |A - \lambda I| = (-2-\lambda)(7-\lambda) - 36 = \lambda^2 - 5\lambda - 50 = (\lambda - 10)(\lambda + 5) = 0$$

The eigenvalues are

$$\lambda = 10 \quad \text{and} \quad \lambda = -5.$$

To find an eigenvector $\mathbf{v}_1$ corresponding to $\lambda = -5$ we solve the system

$$(A + 5I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(or any non-zero multiple).

Similarly, to find the eigenvector $\mathbf{v}_2$ corresponding to $\lambda = 10$ we solve the system

$$(A - 10I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(or any non-zero multiple).

-5- (Gershgorin Theorem.) State and prove the Gershgorin Theorem.

Discussion:

The Gershgorin theorem says that if λ is an eigenvalue of an $n \times n$ matrix A then for some i in $\{1, 2, \ldots, n\}$

$$|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|.$$

In other words, every eigenvalue lies in a Gershgorin circle which is centered at the diagonal entry and whose radius is the sum of the absolute values of the off-diagonal entries.

To see this suppose $\mathbf{x}$ is an eigenvector of the $n \times n$ matrix A, with corresponding eigenvalue λ. Thus

$$A\mathbf{x} = \lambda \mathbf{x}.$$

Since an eigenvector is determined only up to a non-zero factor we may assume that $\mathbf{x}$ is normalized such that

$$\max_{j=1,\ldots,n} |x_j| = x_i = 1 \tag{1}$$

for some i in $\{1, 2, \ldots, n\}$. This fixes i. If there are several such indices i we pick any particular one of them.

The i-th component of the vector equation (1) is

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i = \lambda.$$

Subtracting $a_{ii} x_i = a_{ii}$ on both sides gives the equation

$$\lambda - a_{ii} = \sum_{j \neq i} a_{ij} x_j$$

Taking absolute values on both sides, applying the triangle inequality, and observing that $|x_j| \leq 1$ for all j shows that λ lies in the Gershgorin Circle centered at x_i:

$$|\lambda - a_{ii}| = \left| \sum_{j \neq i} a_{ij} x_j \right| \leq \sum_{j \neq i} |a_{ij} x_j| = \sum_{j \neq i} |a_{ij}||x_j| \leq \sum_{j \neq i} |a_{ij}|$$

-6- (Singularity versus Defectiveness.) Complete the Follwoing Table:

	singular	**invertible**
defective:	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
diagonalizable:	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(2)

-7- (True or False.) Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers.

1. T F All finite dimensional spaces are isomorphic to $\mathbb{R}^n$ for some n.

$\boxed{\textbf{T}}$ Actually, all vector spaces of the same dimension are isomorphic.

2. T F The set of singular $n \times n$ matrices forms a linear space.

$\boxed{\textbf{F}}$ The set of singular matrices is not closed under addition. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the 2×2 identity matrix is not singular.

3. T F The set of antisymmetric square matrices (those satisfying $A = -A^T$) form a linear space.

$\boxed{\textbf{T}}$ Antisymmetry is preserved by addition and scalar multiplication.

4. T F Two isomorphic vector spaces have the same dimension.

$\boxed{\textbf{T}}$ Since the isomorphism is one-to-one and onto the image of a basis of the domain is a basis of the image.

5. T F The set of all sequences

$$y_0, y_1, y_2, \ldots$$

satisfying the infinitely many equations

$$y_{n+3} = y_{n+2} - 2y_{n+1} + y_n, \qquad n = 0, 1, 2, \ldots$$

form a liner space.

$\boxed{\textbf{T}}$ This is a homogeneous linear difference equation:

$$y_{n+3} - y_{n+2} + 2y_{n+1} - y_n = 0.$$

Its solution set is closed under addition and scalar multiplication. Its dimension is 3.

6. T F The set of all triangular matrices forms a linear space.

$\boxed{\textbf{F}}$ The sum of a lower triangular matrix and an upper triangular matrix is (usually) not triangular.

7. T F Suppose A is an invertible $n \times n$ matrix and V is a subspace of R^n of dimension k. Then the set

$$W = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} = A\mathbf{v} \quad \text{for some } \mathbf{v} \in V\}$$

is a linear space with the same dimension as V.

$\boxed{\textbf{T}}$ A is an isomorphism.

8. T F Suppose V and W are subspaces of $\mathbb{R}^n$. Then the set of all vectors in $\mathbb{R}^n$ that can be written as $\mathbf{v}+\mathbf{w}$ where $\mathbf{v}$ is in V and $\mathbf{w}$ is in W, is a subspace of $\mathbb{R}^n$.

$\boxed{\textbf{T}}$ That set is closed under addition and scalar multiplication.

9. T F The union of two subspaces of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

$\boxed{\textbf{F}}$ For example, the union of two distinct lines is an X-shaped region that is clearly not closed under addition.

10. T F The intersection of two subspaces of $\mathbb{R}^n$ is a subspace o $\mathbb{R}^n$.

$\boxed{\textbf{T}}$ It's easy to check that the intersection is closed under addition and scalar multiplication.

-8- (True or False.) Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers.

1. T F Every singular matrix is defective.

$\boxed{\textbf{F}}$, the zero matrix is singular and non-defective.

2. T F Row operations preserve eigenvalues but not eigenvectors.

$\boxed{\textbf{F}}$, they preserve neither.

3. T F A square matrix may not have any eigenvalues.

$\boxed{\textbf{F}}$, an $n \times n$ matrix has precisely n eigenvalues, counting multiplicity.

4. T F Similar matrices have the same eigenvalues.

$\boxed{\textbf{T}}$, that's the essence of similarity transforms.

5. T F Similarity transforms preserve the null space of a matrix.

$\boxed{\textbf{F}}$, for example, the matrices $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ are similar since

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

However, they have different null spaces.

6. T F All square matrices are similar to a diagonal matrix.

$\boxed{\textbf{F}}$, matrices are similar to a diagonal matrix only if they are non-defective.

7. T F The eigenvalues of a real matrix may be complex.

$\boxed{\textbf{T}}$, for example the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalues $\pm i$.

8. T F The eigenvalues of a symmetric matrix may be complex.

$\boxed{\textbf{F}}$, the eigenvalues of a symmetric matrix are real.

9. T F Every defective matrix is singular.

$\boxed{\textbf{F}}$, for example the non-singular matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is defective.

10. T F A matrix is invertible if and only if all of its eigenvalues are non-zero

$\boxed{\textbf{T}}$, that's one of our criteria for invertibility.