Write your name here:

Math 2270-1 — Fall 2019 — Final Exam

1	2	3	4	5	6	7	8	9	10	11	12	Total

Instructions

1. **This exam is closed books and notes. Do not use a calculator or another electronic device. Do not use scratch paper.**
2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it's obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.
4. To avoid distraction and disruption **I am unable to answer questions during the exam.** If you believe there is something wrong with a problem state so, and if you are right you will receive generous credit. I will also be unable to discuss individual problems and grading issues with you after you are done while the exam is still in progress.
5. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to **check your answers.**
6. When you are done hand in your exam, pick up an answer sheet, and leave the room. Do not return to your seat.
7. All questions have equal weight.
8. Clearly indicate (for example, by circling or boxing) your final answers.

Note that this answer set contains more information than you needed to provide on the exam.

-1- (LU factorization.) Compute the LU factorization of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}.$$

Discussion:

Recall that U is the upper triangular matrix we obtain by Gaussian Elimination, and L is the matrix with 1s along the diagonal, and the multipliers below the diagonal. Thus, subtracting twice the first row from the second gives

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

We check by multiplication that $A = LU$:

$$\begin{array}{ccl} & \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} & = U \\ L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} & = LU = A \end{array}$$

-2- (Determinants.) For what value of t does

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{bmatrix} = 0?$$

Discussion:

Expanding the determinant about the third column gives

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{bmatrix} = 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + t \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -9 + 36 - 3t = 27 - 3t = 0$$

which gives

$$t = 9.$$

-3- (Reduced Row Echelon Form.) Suppose the reduced row echelon form of a matrix A is

$$R = \begin{bmatrix} 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Compute the rank of A, the dimensions of its column and null spaces, and give a basis of the null space.

Discussion:

Variables 1 and 3 are pivot variables, variables 2, 4, and 5 are free variables. Thus the rank is 2, the dimension of the row space is 2, and the dimension of the null space is $5-2=3$ The basis $\{s_1, s_2, s_3\}$ of the null space can be obtained by setting the free variables, and using the first two equations to compute the pivot variables:

$$\begin{matrix} x_2 = 1 \\ x_4 = 0 \\ x_5 = 0 \end{matrix} \longrightarrow s_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{matrix} x_2 = 0 \\ x_4 = 1 \\ x_5 = 0 \end{matrix} \longrightarrow s_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{matrix} x_2 = 0 \\ x_4 = 0 \\ x_5 = 1 \end{matrix} \longrightarrow s_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

-4- (Inner Product Spaces and the Pythagorean Theorem.) Let V be an inner product space with an inner product $< \cdot, \cdot >$. List the defining properties of $< \cdot, \cdot >$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V. Define what we mean by $\|\mathbf{u}\|$. Also define what we mean when we say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Finally, prove the Pythagorean Theorem:

$$\|u + \mathbf{v}\|^2 = \|u\|^2 + \|\mathbf{v}\|^2 \quad \iff \quad \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

Discussion:

The inner product $< \cdot, \cdot >$ is a function that maps two vectors $\mathbf{u}$ and $\mathbf{v}$ to a number $< \mathbf{u}, \mathbf{v} >$ such that the following properties hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and scalars c:

1. $< \mathbf{u}, \mathbf{v} > = < \mathbf{v}, \mathbf{u} >$,
2. $< \mathbf{u} + \mathbf{v}, \mathbf{w} > = < \mathbf{u}, \mathbf{w} > + < \mathbf{v}, \mathbf{w} >$,
3. $< c\mathbf{u}, \mathbf{v} > = c < \mathbf{u}, \mathbf{v} >$,
4. $< \mathbf{u}, \mathbf{u} > \geq 0 \quad \text{and} \quad < \mathbf{u}, \mathbf{u} > = 0 \quad \Longrightarrow \quad \mathbf{u} = 0.$

The norm $\|\mathbf{u}\|$ of $\mathbf{u}$ is defined by

$$\|\mathbf{u}\| = \sqrt{< \mathbf{u}, \mathbf{u} >}.$$

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their inner product is zero:

$$< \mathbf{u}, \mathbf{v} > = 0.$$

We can now easily see the Pythagorean Theorem from

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= < \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} > \\ &= < \mathbf{u}, \mathbf{u} > + 2 < \mathbf{u}, \mathbf{v} > + < \mathbf{v}, \mathbf{v} > \\ &= \|\mathbf{u}\|^2 + 2 < \mathbf{u}, \mathbf{v} > + \|\mathbf{v}\|^2 \end{aligned}$$

$$\Longrightarrow$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad \iff \quad < \mathbf{u}, \mathbf{v} > = 0$$

This is a beautiful illustration of the fact that sometimes generalizing a statement simplifies the argument showing that it is true.

-5- (Invertibility and Diagonalizability.) Give examples—as simple as possible—for matrices that are

a. invertible and diagonalizable: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. invertible and non-diagonalizable: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

c. singular and diagonalizable: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d. singular and non-diagonalizable: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

These examples are pretty simple.

-6- (Positive Definiteness.) Let

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 6 & 3 & 1 \\ 1 & 3 & 10 & 5 \\ 1 & 1 & 5 & 12 \end{bmatrix}$$

Show that A is positive definite.

Discussion:

A matrix is positive definite if it is symmetric and all of its eigenvalues are positive. Clearly A is symmetric. By the Gershgorin Theorem, all of its eigenvalues are positive. Hence A is positive definite.

-7- (Linear Transformation.) Suppose we write a cubic polynomial as

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

as usual. Find the 4×4 matrix M that maps the coefficient vector of p with respect to the basis $\{1, x, x^2, x^3\}$ onto the vector of the coefficients of the Taylor expansion about $x = 1$. In other words, find M such that

$$M \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1)/2 \\ p'''(1)/6 \end{bmatrix}$$

Note: I assume you are familiar with Taylor series, but for reference, the Taylor expansion of a function f about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots.$$

Discussion:

We get the columns of M by applying the transformation to the basis polynomials 1, x, x^2, and x^3. We get the Taylor coefficients

$$\begin{aligned}
p(x) = 1 &\implies p(1) = 1, p'(1) = p''(1)/2 = p'''(1)/6 = 0 \\
p(x) = x &\implies p(1) = 1, p'(1) = 1, p''(1)/2 = p'''(1)/6 = 0 \\
p(x) = x^2 &\implies p(1) = 1, p'(1) = 2, p''(1)/2 = 1, p'''(1)/6 = 0 \\
p(x) = x^3 &\implies p(1) = 1, p'(1) = 3, p''(1)/2 = 3, p'''(1)/6 = 1
\end{aligned}$$

This gives

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

-8- (Linear Spaces.) Let S be the set of polynomials p of degree 2 that satisfy $p(0) = 0$. Show that S is a subspace of the space of quadratic polynomials, compute the dimension of S, and give a basis of S.

Discussion:

S is clearly closed under addition and multiplication with a scalar, and so S is a linear space. The space of all quadratic polynomials has dimension 3, but non-zero constant functions are not in S, and so S has dimension **at most 2.** The two functions x and x^2 are linearly independent and are in S. Therefore the dimension of S is **at least 2.** Therefore the dimension is **equal to 2**, and the two functions form a basis $\{x, x^2\}$ of S.

-9- (Orthogonal Matrices.) Give an example of an orthogonal matrix with complex (non-real) eigenvalues. Your example should be as simple as possible.

Discussion:

The eigenvalues of the orthogonal matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\pm i$.

A more general example is the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

-10- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. T F A linear system may have exactly 2 solutions.

$\boxed{\textbf{F}}$ A linear system can only have none, one, or infinitely many solutions.

2. T F A linear system with fewer equations than unknowns always has at least one solution.

$\boxed{\textbf{F}}$ For example, the linear system of 2 equations in three unknowns

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

has no solution.

3. T F Every vector in a linearly dependent set can be written as a linear combination of the other vectors.

$\boxed{\textbf{F}}$ The first vector in the linearly dependent set

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

is not a linear combination of the last two vectors.

4. T F If a set of more than one vectors is linearly dependent then at least one of those vectors can be written as a linear combination of the others.

$\boxed{\mathbf{T}}$ Take a non-trivial dependence relation (linear combination equals zero) of vectors and solve for one of the vectors in that relation.

5. T F The matrix transformation $\mathbf{y} = A\mathbf{x}$ is one-to-one if and only if the columns of A are linearly independent.

$\boxed{\mathbf{T}}$ The phrase "one-to-one" means that every vector in the column space of A can be written uniquely as a linear combination of the columns, and this is equivalent to saying the columns are linearly independent.

6. T F Any vector in the span of a linearly dependent set can be written in more than one way as a linear combination of the given vectors.

$\boxed{\mathbf{T}}$ Just write it in one way, and add a non-trivial linear combination of vectors that equals zero, to get another way.

7. T F The solution set of $A\mathbf{x} = \mathbf{0}$ is a linear space

$\boxed{\mathbf{T}}$ That's the definition of the null space, or kernel, of A.

8. T F If $\mathbf{u}$ and $\mathbf{v}$ are solutions of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ then $\mathbf{u} - \mathbf{v}$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

$\boxed{\mathbf{T}}$ We get $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$ and hence $A\mathbf{u} - A\mathbf{v} = A(\mathbf{u} - \mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

9. T F The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ may be inconsistent.

$\boxed{\mathbf{F}}$ The zero vector is a solution of $A\mathbf{x} = \mathbf{0}$.

10. T F The general solution of any consistent linear problem is any particular solution, plus the general solution of the associated homogeneous problem.

$\boxed{\mathbf{T}}$ Applied to general linear problems, this is one of the central principles of mathematics.

-11- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers. This question deals with matrix multiplication. Throughout assume that A and B are matrices.

1. T F The matrix product $C = AB$ can be formed only if A has as many rows as B.

$\boxed{\mathbf{F}}$ A must have as many columns as B has rows.

2. T F If A and B are two non-square matrices it is impossible for AB and BA both to be defined.

$\boxed{\mathbf{F}}$ If A is $m \times n$ and B is $n \times m$ then AB is $m \times m$ and BA is $n \times n$, but they are both defined.

3. T F The j-th column of AB equals the product of A and the j-th column of B.

$\boxed{\mathbf{T}}$ This is one of the six views of matrix multiplication that we considered in class.

4. T F The i-th row of AB equals the product of A and the i-th row of B.

$\boxed{\mathbf{F}}$ The i-th row of AB equals the product of the i-th row of A, and B.

5. T F The product AB of a non-zero $m \times 1$ matrix A and a non-zero $1 \times n$ matrix B is an $m \times n$ matrix of rank 1.

$\boxed{\textbf{T}}$ Every row of AB is a multiple of A, and every column of AB is a multiple of B. Moreover, AB is non-zero, so its rank is greater than zero.

6. T F Every column of AB is in the column space of A.

$\boxed{\textbf{T}}$ A column of AB is A multiplied with the corresponding column of B. Thus it is a linear combination of the columns of A.

7. T F Every row of AB is in the row space of A.

$\boxed{\textbf{F}}$ Every row of AB is in the row space of B, not A.

8. T F Assuming A is $m \times p$ and B is $p \times n$ then $C = AB$ is $m \times n$ and

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \qquad i = 1, \ldots m, \quad j = 1, \ldots, n.$$

$\boxed{\textbf{T}}$ that's how we define matrix multiplication.

9. T F If A and B are both $n \times n$ then $AB = BA$.

$\boxed{\textbf{F}}$ Matrix multiplication does not commute.

10. Assuming the $m \times p$ matrix A is the standard matrix of a function $f : \mathbb{R}^p \longrightarrow \mathbb{R}^m$, and the $p \times n$ matrix B is the standard matrix of a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, the product AB is the standard matrix of the function $g \circ f$.

$\boxed{\textbf{T}}$ That's precisely why we define matrix multiplication the way we do.

-12- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. T F The determinant of an $n \times n$ matrix A equals the product of its eigenvalues.

$\boxed{\textbf{T}}$ This follows from evaluating

$$\det(A - \lambda I) = (-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i)$$

at $\lambda = 0$.

2. T F A symmetric matrix A is positive definite if and only if all its eigenvalues are positive.

$\boxed{\textbf{T}}$ If there is a nonpositive eigenvalue λ with corresponding eigenvector $\mathbf{x}$ then for that vector $\mathbf{x}^T A \mathbf{x}$ is nonpositive. If all eigenvalues are positive use an orthogonal basis of eigenvectors to see that $\mathbf{x}^T A \mathbf{x} > 0$ for all non-zero $\mathbf{x}$.

3. T F The eigenvalues of a triangular matrix are its diagonal entries.

$\boxed{\textbf{T}}$ This follows directly from

$$\det(A - \lambda I) = \prod_{i=1}^{n} (\lambda - a_{ii}).$$

4. T F Multiplying a rank 1 matrix A with a non-singular matrix B gives a rank 1 matrix $C = AB$

$\boxed{\textbf{T}}$ All the columns of AB are in the one-dimensional column space of A.

5. T F Suppose A is an $m \times n$ matrix. Then AA^T and A^TA are both symmetric and positive semidefinite.

$\boxed{\textbf{T}}$ These matrices are symmetric and

$$\mathbf{x}^T A^T A \mathbf{x} = \mathbf{y}^T \mathbf{y} \geq 0$$

where $\mathbf{y} = A\mathbf{x}$. The argument for AA^T is similar.

6. T F A square matrix A is invertible if and only if A^T is invertible.

$\boxed{\textbf{T}}$ This is one of our criteria for invertibility.

7. T F The processes of inverting and transposing a matrix commute.

$\boxed{\textbf{T}}$ We can verify directly that $(A^T)^{-1} = (A^{-1})^T$.

8. T F If A and B are invertible $n \times n$ matrices then $(AB)^{-1} = A^{-1}B^{-1}$.

$\boxed{\textbf{F}}$ $(AB)^{-1} = B^{-1}A^{-1}$.

9. T F Suppose A is an $n \times n$ matrix and k is a scalar. Then $\det(kA) = k \det A$.

$\boxed{\textbf{F}}$ Actually, $\det(kA) = k^n \det A$.

10. T F Interchanging two columns of a matrix does not change its determinant.

$\boxed{\textbf{F}}$ Interchanging two columns (or rows) multiplies the determinant with -1.

-13- **(True or False.)** Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. T F Suppose A, B, and P are $n \times n$ matrices and

$$B = P^{-1}AP.$$

The A and B have the same eigenvalues.

$\boxed{\textbf{T}}$ This is the essence of a similarity transform.

2. T F The eigenvectors of a triangular matrix are the standard basis vectors.

$\boxed{\textbf{F}}$ The eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The second basis vector, $e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector.

3. T F Suppose $\mathbf{a}$ is orthogonal to $\mathbf{b}$ and $\mathbf{b}$ is orthogonal to $\mathbf{c}$. Then $\mathbf{a}$ is orthogonal to $\mathbf{c}$.

$\boxed{\textbf{F}}$ For example, we could have $\mathbf{a} = \mathbf{c} \neq \mathbf{0}$.

4. T F The normal equations for the Least Squares problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min$$

are always consistent.

T This is another way of saying that the projection of **b** into the column space of A exists.

5. T F The zero vector in $\mathbb{R}^n$ is orthogonal to all vectors in $\mathbb{R}^n$.

T The inner product of **0** with any vector is zero.

6. T F A singular matrix may be orthogonal.

F Every orthogonal matrix has an inverse, its transpose.

7. T F The matrix Σ in the singular value decomposition of an invertible square matrix is positive definite.

T The eigenvalues of Σ are the singular values of A which are in general non-negative, and are positive for an invertible matrix.

8. T F The singular values of a negative definite matrix are negative.

F Singular values are non-negative by definition.

9. T F Suppose f is a scalar valued function of several variables. Then it assumes a minimum at a point if the gradient at that point is zero and the matrix of second derivatives at that point is positive definite.

T Positive definiteness is the natural analog of positive numbers.

10. T F Linear Algebra is cool.

T That's certainly my opinion, but of course you are entitled to your own. I counted both possible answers as correct.

I enjoyed meeting you, and it's been great fun teaching this class. I hope you found it a worthwhile experience. Best wishes to you, and perhaps I'll meet you again in some future class!

Peter Alfeld