Math 2270-6

Notes of 10/28/19

The Gershgorin Theorem

- There should be a prize for the mathematical theorem that maximizes the ratio of usefulness and notoriety.
- My vote for the most useful and least known mathematical fact would go to the Gershgorin Theorem.
- For comprehensive information on the Gershgorin Theorem see the book: "Geršgorin and His Circles" by R.S. Varga, Springer Series in Computational Mathematics, Springer, 2010, ISBN 978-3-642-05928-5.
- The basic idea underlying the Gershgorin Theorem is this: the eigenvalues of a diagonal matrix are the diagonal entries. If a matrix is close to being diagonal then its eigenvalues should be close to the diagonal entries.

$$\begin{bmatrix} \times & & 0 \\ & \ddots & \\ 0 & & \times \end{bmatrix}$$

- The Gershgorin Theorem makes this notion precise. It says:

Gershgorin Theorem. Suppose A is an $n \times n$ matrix, and λ is one of its eigenvalues. Then, for some $i \in \{1, 2, \dots, n\}$

$$|a_{ii} - \lambda| \leq \sum_{j \neq i} |a_{ij}|.$$

- In other words, every eigenvalue lies in some circle whose center is a diagonal entry of A, and whose radius equals the sum of the absolute values of the off-diagonal entries in that row.
- Those circles are referred to as the **Gershgorin Circles**.
- Before seeing why this is true and extending this idea, let's look at some examples.
- **Example 0.** Suppose A is diagonal. Then the radii of the Gershgorin circles are 0, and the eigenvalues are the diagonal entries.

  Example 1. Suppose

$$A = \begin{bmatrix} -10 & 4 & 5 \\ 3 & 10 & 5 \\ 4 & 8 & 20 \end{bmatrix}.$$

- The eigenvalues of A (computed with matlab) are

$$\lambda_1 = 10.97, \quad \lambda_2 = 7.00 \quad \lambda_3 = 23.97$$

- The Gershgorin Circles and the eigenvalues are shown in Figure 1.
- Note that since the origin is not contained in any of the circles 0 is not an eigenvalue, which implies that the matrix is non-singular.

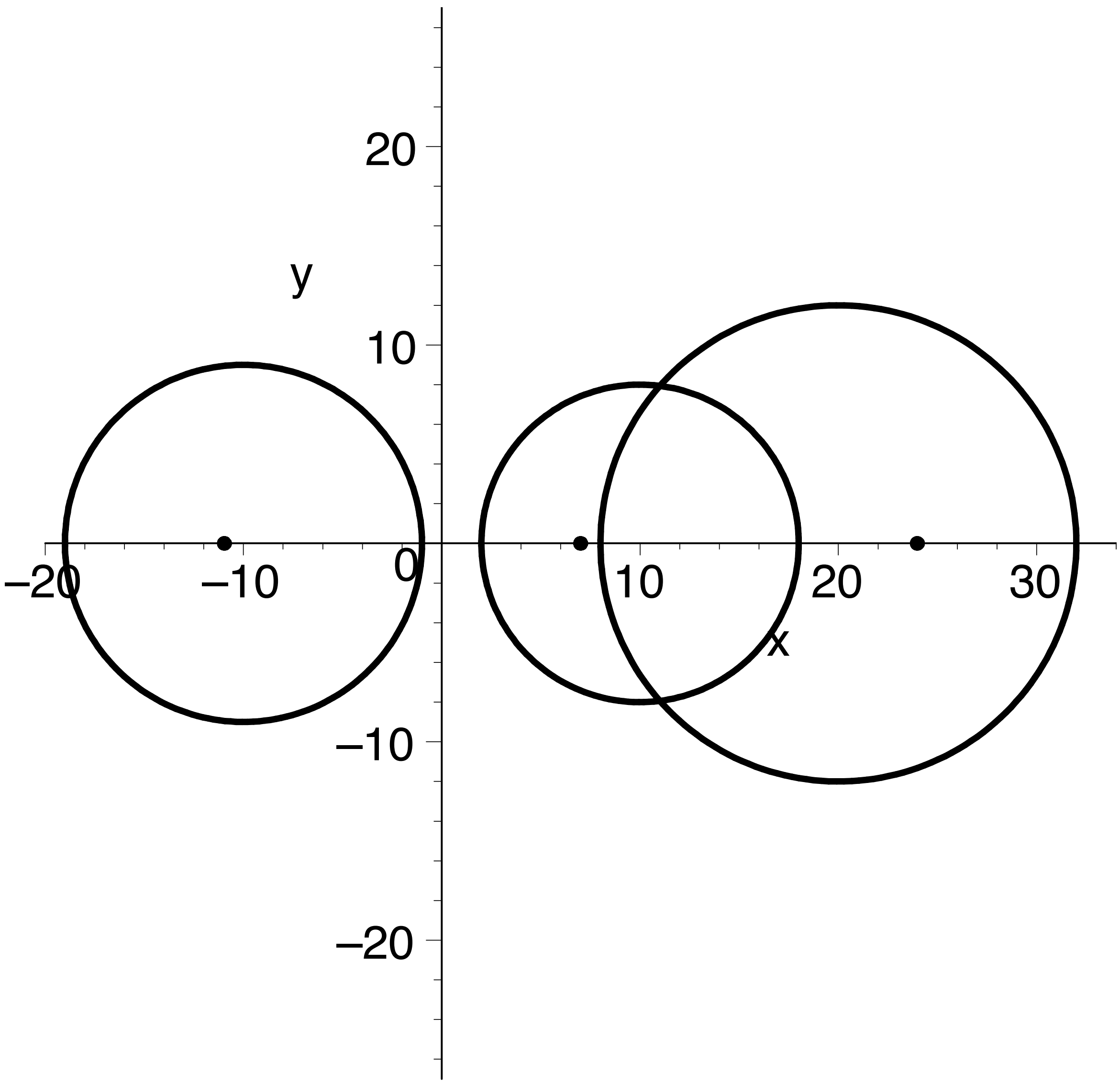


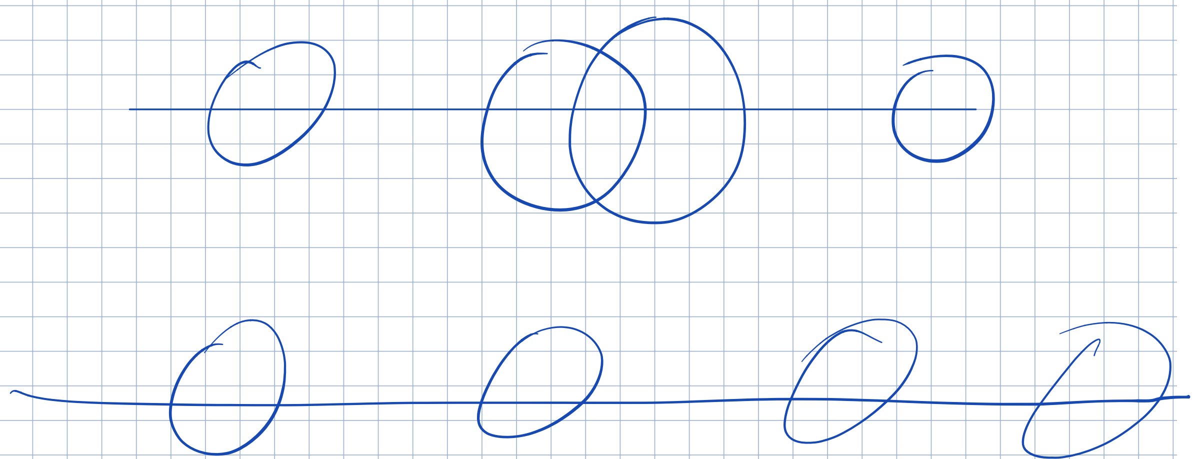
Figure 1. Gershgorin Circles of $\begin{bmatrix} -10 & 4 & 5 \\ 3 & 10 & 5 \\ 4 & 8 & 20 \end{bmatrix}$.

- This observation generalizes. The matrix A is **strictly diagonally dominant** if for all $i = 1, 2, \ldots, n$

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

- It follows immediately from Gershgorin's Theorem:

A strictly diagonally dominant matrix is non-singular



- It's easy to see that Gershgorin's Theorem is true.

$$Ax = \lambda x$$

$$1 = \max_{j=1,\dots,n} |x_j| = x_i \qquad (\text{def. of } i)$$

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i \qquad |a+b| \le |a|+|b|$$

$$\lambda x_i - a_{ii} x_i = \sum_{\substack{j=1 \\ j\ne i}}^{n} a_{ij} x_j$$

$$|\lambda - a_{ii}||x_i| = \left| \sum_{\substack{j=1 \\ j\ne i}}^{n} a_{ij} x_j \right| \qquad |x_i| = 1, \quad |x_j| \le 1$$

$$|\lambda - a_{ii}| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}| |x_j| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}|$$

- The eigenvalues of A are also the eigenvalues of A^T. Applying Gershgorin's Theorem to the transpose of A gives the result that the eigenvalues lie in the union of all circles with the same centers, but the radii being the sum of the absolute values of the off-diagonal entries in the corresponding column.

- Revisit Example 1:

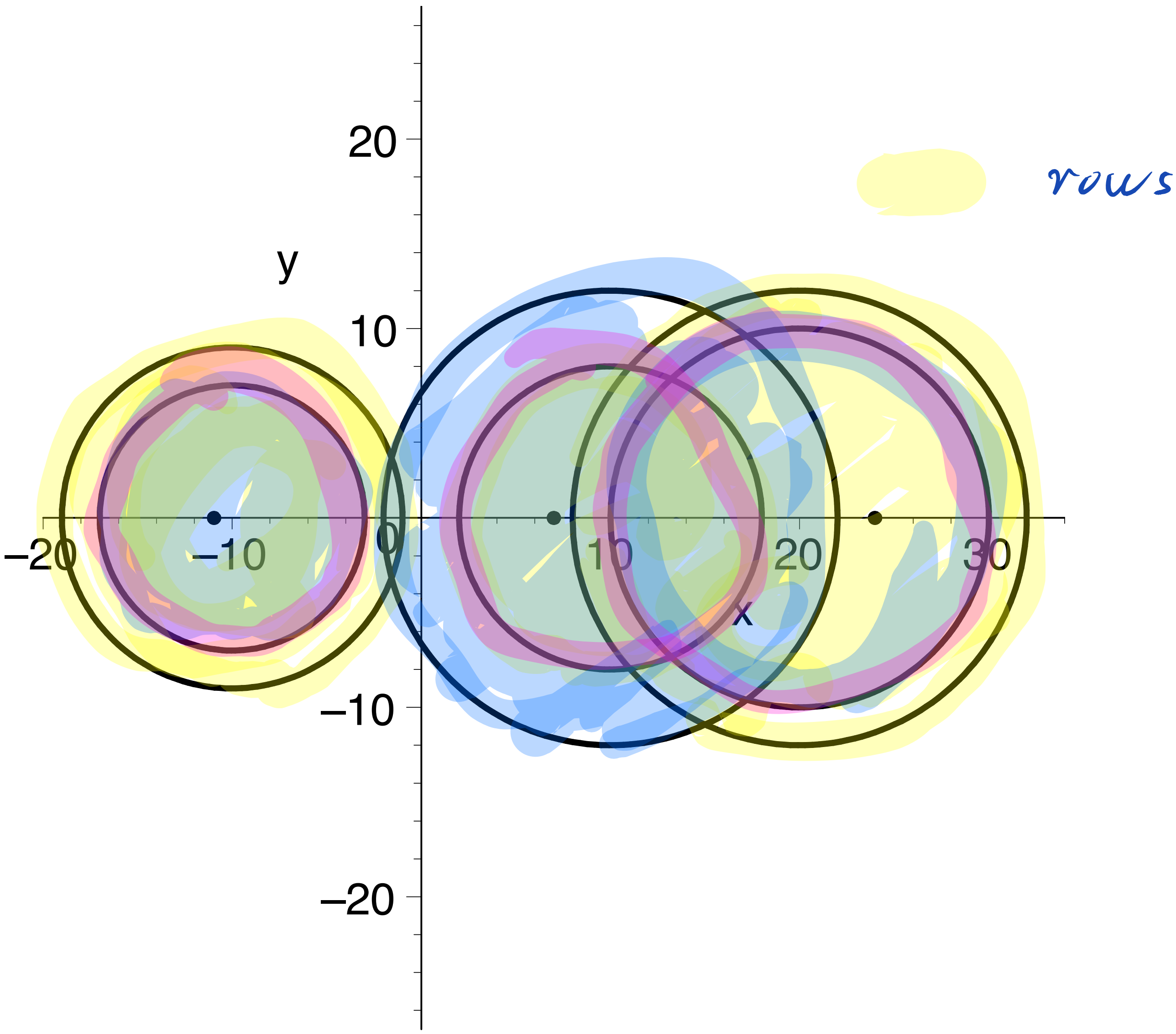
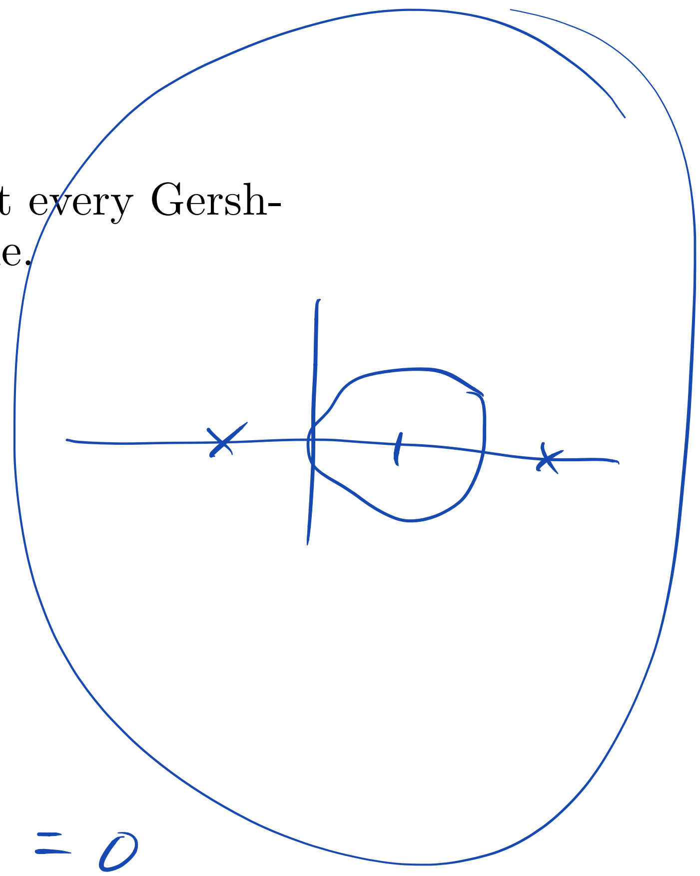


Figure 2. $\begin{bmatrix} -10 & 4 & 5 \\ 3 & 10 & 5 \\ 4 & 8 & 20 \end{bmatrix}$ revisited.

It's not true in general that every Gershgorin Circle contains an eigenvalue.



- Example 2: Suppose

$$A = \begin{bmatrix} 1 & 1 \\ 16 & 1 \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)^2 - 16 = 0$$

$$1 - \lambda = \pm 4$$

$$\lambda = 1 \pm 4$$

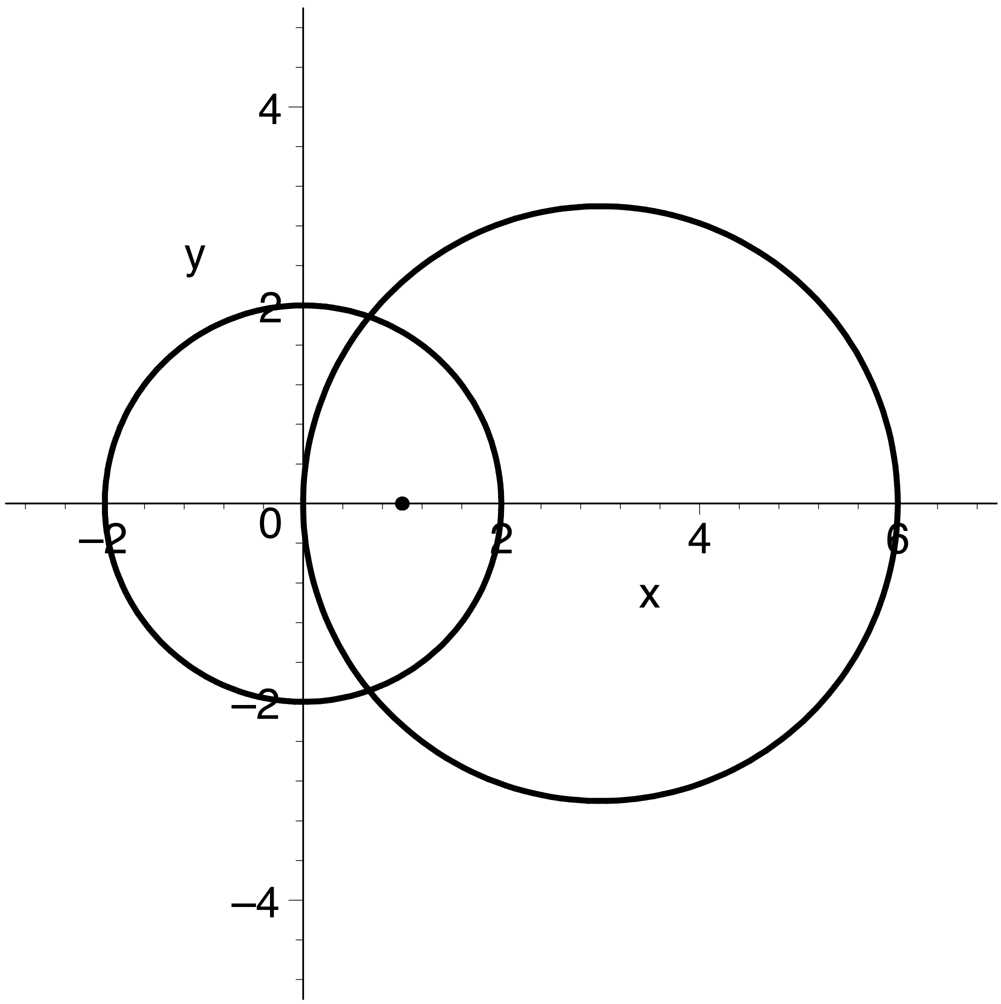


Figure 3. Example 3.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$|\lambda - 0| \le 2$
$|\lambda - 1| \le 2$
$|\lambda - 3| \le 3$

- **Example 3:** Suppose

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

- Gershgorin's Theorem by itself does not imply that 1 is an eigenvalue.
- However, it is! We can tell by observing that $A - I$ is singular, or finding an eigenvector!

On the other hand, it is true that any union of k Gershgorin Circles that does not overlap with any of the remaining Gershgorin Circles contains precisely k eigenvalues, counting multiplicity.

- To see this let

$$A = B + D$$

  where D is diagonal and the diagonal of B is zero.

- For example, for the matrix in Example 1 we get

$$A = \begin{bmatrix} -10 & 4 & 5 \\ 3 & 10 & 5 \\ 4 & 8 & 20 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 4 & 5 \\ 3 & 0 & 5 \\ 4 & 8 & 0 \end{bmatrix}}_{=B} + \underbrace{\begin{bmatrix} -10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 20 \end{bmatrix}}_{=D}.$$

- Next, let

$$A(t) = tB + D.$$

  Clearly,

$$A(0) = D \quad \text{and} \quad A(1) = A.$$

  The Gershgorin circles of D are actually points (which are the eigenvalues of D). As t changes from 0 to 1, the Gershgorin Circles of $A(t)$ start out as those of D and grow until they become those of $A(1)$. Since the centers don't change, once they overlap for some $t = t_0$ they will overlap for all $t > t_0$. In particular, the Gershgorin Circles forming the set S do not overlap with any other circles for all non-negative $t \leq 1$.

- Now, the eigenvalues of a matrix depend continuously on the matrix. Thus for no $t \in [0, 1]$ can an eigenvalue wander into or out of S.

- We are done with chapter 5, eigenvalues and eigenvectors.
- Next: return to topics we have seen in Calc III:
  - dot products
  - distance
  - orthogonality
  - projections