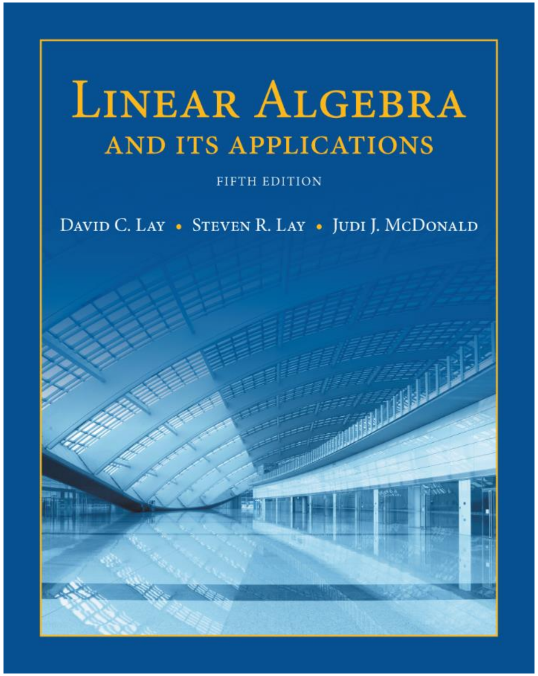
2 Matrix Algebra

2.4

PARTITIONED MATRICES



PARTITIONED MATRICES

- A key feature of our work with matrices has been the ability to regard matrix A as a list of column vectors rather than just a rectangular array of numbers.
- This point of view has been so useful that we wish to consider other **partitions** of A, indicated by horizontal and vertical dividing rules, as in Example 1 on the next slide.

PARTITIONED MATRICES

- **Example 1** The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

- Can also be written as the 2 × 3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

- Whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

ADDITION AND SCALAR MULTIPLICATION

- If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A + B$.
- In this case, each block of $A + B$ is the (matrix) sum of the corresponding blocks of A and B.
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

MULTIPLICATION OF PARTITIONED MATRICES

- Partitioned matrices can be multiplied by the usual row—column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B.
- **Example 3** Let

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- The 5 columns of A are partitioned into a set of 3 columns and then a set of 2 columns. The 5 rows of B are partitioned in the same way—into a set of 3 rows and then a set of 2 rows.

MULTIPLICATION OF PARTITIONED MATRICES

- We say that the partitions of A and B are conformable for block multiplication. It can be shown that the ordinary product AB can be written as

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

- It is important for each smaller product in the expression for AB to be written with the submatrix from A on the left, since matrix multiplication is not commutative.

MULTIPLICATION OF PARTITIONED MATRICES

- For instance,

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

- Hence the top block in AB is

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \end{bmatrix}$$

MULTIPLICATION OF PARTITIONED MATRICES

- **Theorem 10**: Column—Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then

$$AB = \begin{bmatrix} \mathrm{col}_1(A) & \mathrm{col}_2(A) & \cdots & \mathrm{col}_n(A) \end{bmatrix} \begin{bmatrix} \mathrm{row}_1(B) \\ \mathrm{row}_2(B) \\ \vdots \\ \mathrm{row}_n(B) \end{bmatrix} \tag{1}$$

$$= \mathrm{col}_1(A)\,\mathrm{row}_1(B) + \cdots + \mathrm{col}_n(A)\,\mathrm{row}_n(B)$$

- **Proof** For each row index i and column index j, the (i, j)-entry in $\mathrm{col}_k(A)$and b_{kj} from $\mathrm{row}_k(B)$ is the product of a_{ik} from $\mathrm{col}_k(A)$ and b_{kj} from $\mathrm{row}_k(B)$.

p

n

$A \quad m \qquad n$

$C = AB$

$C \quad m \times p$

$col_1(A)\, row_1(B)$

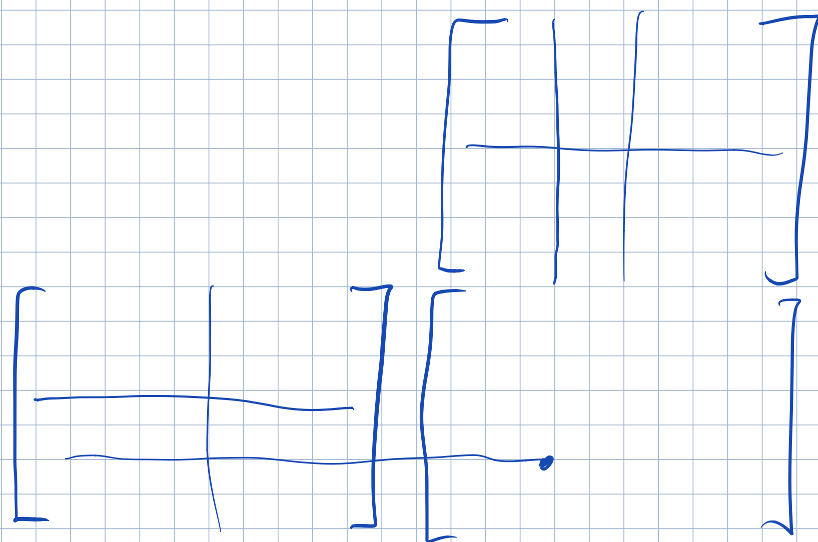
$m \times 1 \qquad 1 \times p$

$i \qquad j$

$a_{i1} \cdot b_{1j}$

$col_k(A) \cdot row_k(B) \qquad a_{ik}\, b_{kj}$

$$C = AB \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$



MULTIPLICATION OF PARTITIONED MATRICES

- Hence the (i, j)-entry in the sum shown in equation (1) is

$$\underset{(k=1)}{a_{i1}b_{1j}} + \underset{(k=2)}{a_{i2}b_{2j}} + \cdots + \underset{(k=n)}{a_{in}b_{nj}}$$

- This sum is also the (i, j)-entry in AB, by the row—column rule.

INVERSES OF PARTIONED MATRICES

- The next example illustrates calculations involving inverses and partitioned matrices.

$p+q=n$

suppose

$\begin{bmatrix} x \\ 0 \end{bmatrix}$ $A_{11}x = 0 \quad x \neq 0$

- **Example 5** A matrix of the form

$$A = \begin{array}{cc} p & p\times q \\ \left[\begin{matrix} A_{11} & A_{12} \\ 0 & A_{22} \end{matrix}\right] \\ q\times p & q \end{array}$$

$$\begin{array}{c} p \\ q \end{array} \begin{bmatrix} A_{11}x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

- Is said to be *block upper triangular*. Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible. Find a formula for A^{-1}.

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

INVERSES OF PARTIONED MATRICES

- **Solution** Denote A^{-1} by B and partition B so that

$$\begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \end{array} \begin{array}{cc} \begin{array}{cc} p & q \end{array} & \\ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} & \begin{array}{c} p \\ q \end{array} \end{array} = \begin{array}{cc} \begin{array}{cc} p & q \end{array} & \\ \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} & \begin{array}{c} p \\ q \end{array} \end{array} \quad (2)$$

- This matrix equation provides four equations that will lead to the unknown blocks $B_{11},\ldots,B_{22}$. Compute the product on the left side of equation (2), and equate each entry with the corresponding block in the identity matrix on the right.

INVERSES OF PARTIONED MATRICES

- That is, set

$B_{11} = A_{11}^{-1}$

$$A_{11}B_{11} + A_{12}B_{21} = Ip \qquad (3)$$

$A_{11}B_{11} = I_p$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \qquad (4)$$

$$A_{22}B_{21} = 0 \qquad (5)$$

$B_{21} = 0$

$$A_{22}B_{22} = I_q \qquad (6)$$

$B_{22} = A_{22}^{-1}$

$A_{11}B_{12} = -A_{12}A_{22}^{-1}$

$B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$

- By itself, equation (6) does not show that A_{22} is invertible. However, since A_{22} is square, the Invertible Matrix Theorem and (6) together show that A_{22} is invertible and $B_{22} = A_{22}^{-1}$.

INVERSES OF PARTIONED MATRICES

- Next, left-multiply both sides of (5) by A_{22}^{-1} and obtain

$$B_{21} = A_{22}^{-1}0 = 0$$

- So that (3) simplifies to

$$A_{11}B_{11} + 0 = I_p$$

- Since A_{11} is square, this shows that A_{11} is invertible and $B_{22} = A_{22}^{-1}$. Finally, use these results with (4) to find that

$$A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1} \quad \text{and} \quad B_{12} = A_{11}^{-1}A_{12}A_{22}^{-1}$$

INVERSES OF PARTIONED MATRICES

- Thus

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

- A **block diagonal matrix** is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.