Math 2270-1 12/11/19

A, C square

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \cdot \det C$$

A, B, C, D

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq \det A \det D - \det B \det C$$

in general

SVD $\quad A = U \Sigma V^T \qquad A \; m \times n \quad m \geq n$

$U \; m \times n \quad U = U^T$

$U^T A V = \Sigma \qquad V \; n \times n \quad V = V^T$

$$\operatorname{rank} A = \operatorname{rank} \Sigma = r \qquad \Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \\ & & & \sigma_n \\ & 0 & \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_n = 0$

Consider $\sigma_i = 0$ if $\frac{\sigma_i}{\sigma_1} \leq \tau = 10^{-16}$

$$\det A = \det U \det \Sigma \det V^T = \pm \prod_{i=1}^{n} \sigma_i$$

$$Q = Q^T \Rightarrow \det Q^T Q = \det I = 1$$

$$\det Q = \pm 1$$

Linear systems $Ax = b$

$$U\Sigma V^T x = b$$
$$\Sigma V^T x = U^T b$$
$$\Sigma z = c$$

$z = V^T x$

$x = Vz$

Least Squares

$Q = Q^T$ $\quad \|Qx\|^2 = (Qx)^T Qx = x^T Q^T Q x$
$= x^T x$
$= \|x\|^2$

$$\|Ax - b\| = \|U^T(Ax - b)\|$$
$$= \|U^T A V V^T x - U^T b\|$$
$$= \|\Sigma z - c\| = \min$$

$z = V^T$

$c = U^T b$

AB = Function Composition

$f(x) = Bx$

$g(y) = Fy$

$C = AB$ standard matrix of $g \circ f$

$f: \mathbb{R}^n \to \mathbb{R}^p$ $\qquad g\ \mathbb{R}^p \to \mathbb{R}^m$

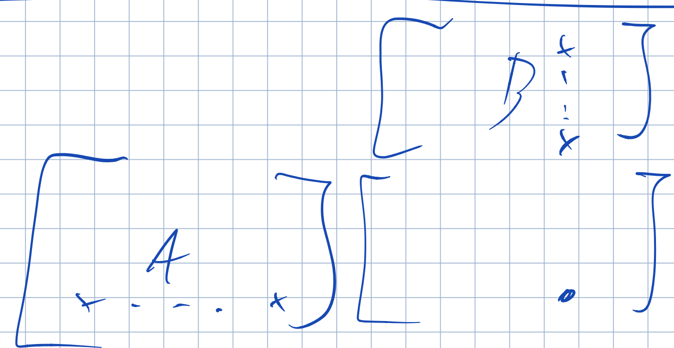
$B \quad p \times n$ $\qquad m \times p$

$$g \circ f \quad \mathbb{R}^n \xrightarrow{f} \mathbb{R}^p \xrightarrow{g} \mathbb{R}^m \qquad m \times n$$

$$\underset{B}{p \times n} \qquad \underset{A}{m \times p}$$

$$C = A\,B = (m \times p) \cdot (p \times n) = m \times n$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$$



$$c_{ij} = r_i(A) \cdot c_j(B)$$

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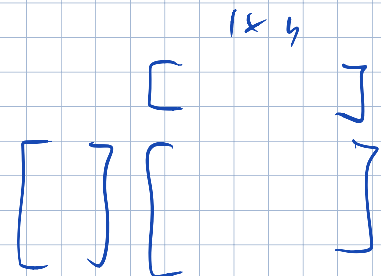
r_i i-th row

c_i i-th column

$$c_i(C) = A\, c_i(B)$$

$$r_i(C) = r_i(A)\, B$$

$$C = \sum_{i=1}^{p} c_i(A)\, r_i(B)$$



$$A = U \Sigma V_T = \sum_{i=1}^{n} \sigma_i u_i v_i^T = \sum_{i=1}^{p} \sigma_i u_i v_i^T \qquad p \text{ suitable}$$

$[U, S, V] = svd(A)$ matlab

$[V, D] = eig(A)$

$Ax = \lambda x \qquad x \neq 0$

$z = \gamma x \qquad \gamma \neq 0$

$Az = \lambda z$

$Ax = \lambda x$

$Ax - \lambda x = 0 = (A - \lambda I)x = 0 \qquad x \neq 0$

$\det(A - \lambda I) = 0$

$\det(A - \lambda I) = (-\lambda)^n + LOT$

$\rightsquigarrow$ evalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

$x_1, x_2, \ldots, x_n$ evectors

$(A - \lambda_i I)x = 0$

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[1]

Ex.:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0$$

$$2 - \lambda = \pm 1$$
$$\lambda = 2 \pm 1$$
$$\lambda = 1$$
$$\lambda = 3$$

$\lambda = 1$ $\quad x = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad Ax = 1 \cdot x$ $\quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

3

$$2a + b = 3a$$
$$a + 2b = 3b$$

$$A \begin{bmatrix} a \\ b \end{bmatrix} = 3 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$-a + b = 0$$
$$a + b = 0$$

$a = b \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$a = -b$$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\langle \cdot , \cdot \rangle$ IP $\qquad \langle u, v \rangle \in \mathbb{R}$

Diagonalizability

A: similar to a diagonal matrix

$\iff$ A diagonalizable

A similar to B

if $B = S^{-1}AS$ for some non-singular matrix S

$$Ax = \lambda x$$

$$\begin{aligned} B(S^{-1}x) &= S^{-1}ASS^{-1}x \\ &= S^{-1}Ax \\ &= S^{-1}\lambda x \\ &= \lambda(S^{-1}x) \end{aligned}$$

B diagonal $\quad B = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$$

$$S^{-1}AS = D$$

$$AS = SD \qquad \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \cdots \lambda_i e_i \cdots \end{bmatrix}$$

$$Ac_i(S) = \lambda_i c_i(S)$$

$Q = Q^T \qquad Q^T A Q = D$

$$A = Q D Q^T$$
$$A^T = Q D^T Q^T = Q D Q^T$$

orthogonally diagonalizable

A non-defective $\iff$ diagonalizable

$\iff$ a complete set of eigenvectors

$\iff$ span$\{e\,v\} = \mathbb{R}^n$

non-singular	non-defective	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
singular	non-defective	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
non-singular	defective	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
singular	defective	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \lambda = 1 \qquad \underline{Ax = x}$

$x = \begin{bmatrix} a \\ b \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$a + b = a \qquad b = 0$

$b = b$

defective $\Longleftrightarrow$ non-diagonalizable

singular $\Longleftrightarrow$ non-invertible

Linear Space, Vector Space

A $m \times n$ 4 spaces

column space	$\{y : y = Ax\}$	$\subset \mathbb{R}^m$
row space	$\{z : z^T = x^T A\}$	$\subset \mathbb{R}^n$
null space (kernel)	$\{x : Ax = 0\}$	$\subset \mathbb{R}^n$
left null space	$\{y : y^T A = 0\}$	$\subset \mathbb{R}^m$

$P_2 = \text{span}\{1, x, x^2\}$

$$p(x) = (2x-1)(x-3) = 2x^2 - 7x + 3 \rightarrow \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}$$

$D : P_2 \to P_1$ $\qquad Dp = p'$

$\underset{B_2}{\{1, x, x^2\}}$ $\qquad \underset{B_1}{\{1, x\}}$

$$[p']_{B_1} = M\,[p]_{B_2}$$

$M \quad 2\times 3$

$D\,1 = 0$

$D\,x = 1$

$D\,x^2 = 2x$

$$[\qquad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} \beta \\ 2\gamma \end{bmatrix}$$

$$D(\alpha + \beta x + \gamma x^2) = \beta + 2\gamma x$$

P_{20}

$$\langle p, q \rangle = \sum_{i=1}^{40} p(i)\, q(i)$$

$\langle p, q \rangle = \langle q, p \rangle$ ✓

$\langle p+q, r \rangle = \langle p, r \rangle + \langle q, r \rangle$ ✓

$\langle \alpha p, q \rangle = \alpha \langle p, q \rangle$ ✓

$\langle p, p \rangle = \geq 0$ ✓

$\langle p, p \rangle = 0 \Rightarrow p = 0$ ✓

yes it is an i.p.

$$\|p\| = \sqrt{\langle p, p \rangle}$$

$\langle p, q \rangle$ orthogonal $\Longleftrightarrow \langle p, q \rangle = 0$

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \Longleftrightarrow \langle u, v \rangle = 0$$

$$\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle$$
$$= \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle$$