Math 2270-1

Notes of 10/04/2019

4.3, 4.5, Vector Spaces, Bases, and Dimension

- Note: we focus on finite dimensional vector spaces.

- Let's use the space of quadratic polynomials as a running example.

$$V = \left\{ p : p(x) = ax^2 + bx + c \right\}$$

- Recall the definition:

A **vector space**[-1-] is a nonempty set V of objects, called vectors, on which are defined two operations, called **addition** and **multiplication by scalars (real numbers)**, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

4. There is a zero vector $\mathbf{0}$ in V such that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}.$$

5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $u + (-\mathbf{u}) = \mathbf{0}$.

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.

10. $1\mathbf{u} = \mathbf{u}$.

[-1-] Also called a **linear space**

- A **linear combination** of a set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$$

of vectors is a vector of the form

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

where the α_i are constants.

- A vector space is closed under linear combinations.

- A finite set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$$

  of vectors is **linearly independent** if

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0} \quad \implies \quad \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

- The **span** of a set S of vectors is the set of all linear combinations of vectors in S.

  Usually S is finite. If S is infinite we assume that the linear combination has only finitely many non-zero terms.

- A **spanning set** of a vector space V is a set S such that V equals the span of S.

- A **basis** of a vector space V is a linearly independent spanning set for V.

- Major Fact: All Bases of a given vector space V have the same number of elements (vectors). That number is the **dimension** of V.

- This is a consequence of Theorem 9 on page 227 of the textbook:

- If a vector space V has a basis

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

  then any set in V containing more than n vectors is linearly dependent.

- The textbook has a proof using coordinate vectors, but those are not needed.

  Proof: Suppose

$$\mathcal{B} = \{\mathbf{b}_1, \quad \mathbf{b}_2, \quad \ldots, \quad \mathbf{b}_n\}$$

is a basis of V and

$$\mathcal{S} = \{\mathbf{v}_1, \quad \mathbf{v}_2, \quad \ldots, \quad \mathbf{v}_m\}$$

is a larger set of
$$m > n$$

vectors.

Since $\mathcal{B}$ is a basis we can write every vector $\mathbf{v}_i$ as a linear combination of the basis vectors $\mathbf{b}_j$:

$$\mathbf{v}_i = \sum_{j=1}^{n} c_{ij}\mathbf{b}_j, \qquad i = 1, \ldots, m. \qquad (1)$$

Now consider a linear combination of the $\mathbf{v}_i$. We want to show that there are non-trivial such linear combinations that equal zero.

$$\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = \mathbf{0} \quad \text{and} \quad \alpha_i \neq 0$$

for some i.

Using (1) we get

$$\begin{aligned}
\mathbf{0} &= \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \\
&= \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} c_{ij} \mathbf{b}_j \\
&= \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} \alpha_i c_{ij} \right)}_{=0} \mathbf{b}_j.
\end{aligned}$$

Hence $\sum_{i=1}^{m} \alpha_i \mathbf{v}_i$ is zero so long as

$$\sum_{i=1}^{m} \alpha_i c_{ij} = 0 \qquad (2)$$

and we are done if we can show that there are coefficients α_i, some of which are non-zero, such (2) is satisfied.

Now let

$$C = \begin{bmatrix} c_{ij} \end{bmatrix}_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}$$

In matrix form the equations (2) can be written as
$$C^T \mathbf{a} = \mathbf{0}. \qquad (3)$$

C is an $m \times n$ matrix with $m > n$. Thus C^T has more columns than rows and the homogeneous system (3) does have a non-trivial solution. The set $\mathcal{S}$ is linearly dependent.

Examples

- Find bases (and dimensions) for the following vector spaces.

- The set of all quadratic polynomials.

- The set of all upper triangular 2×2 matrices.

The set of all upper triangular $n \times n$ matrices.

- The set of all solutions of the differential equation

$$y'' = k^2 y$$

- The set of all sequences that satisfy the Fibonacci^{-2-} equation

$$y_{n+2} = y_n + y_{n+1}, \qquad n = 0, 1, 2, \dots$$

form a linear space. Find a basis.

$^{-2-}$ Fibonacci, approx. 1170-1250, came up with this equation (where $y_0 = y_1 = 1$) to model the growth of a rabbit population.