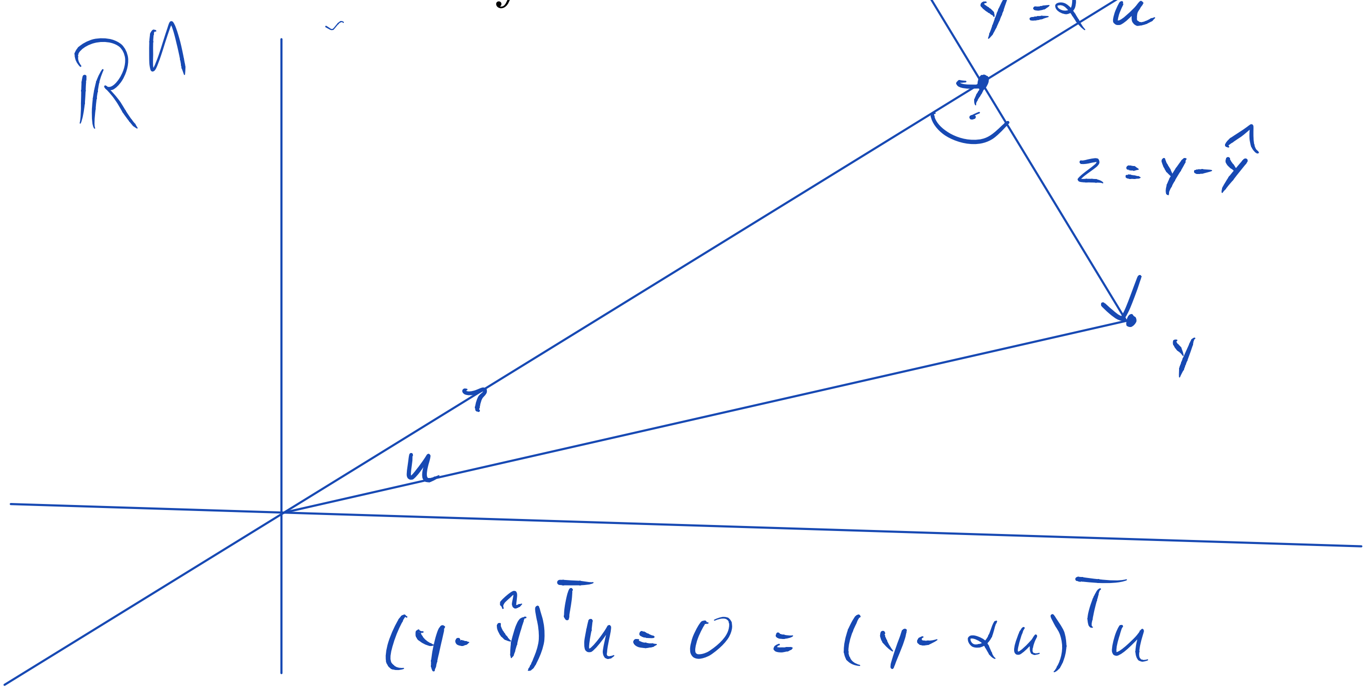
Math 2270-1

Notes of 11/8/19

Orthogonal Projections

- We start with revisiting the idea of projecting a point $\mathbf{y}$ in $\mathbb{R}^2$ onto a line through the origin.
- The projection of $\mathbf{y}$ is the point on the line that is closest to $\mathbf{y}$.



$$(y-\hat{y})^T u = 0 = (y-\alpha u)^T u$$

$$0 = y^T u - \alpha u^T u$$

$$\alpha = \frac{y^T u}{u^T u}$$

$$\hat{y} = \frac{y^T u}{u^T u} u = \frac{y \cdot u}{u \cdot u} u = \frac{(y,u)}{(u,u)} u$$

- This idea can be generalized to projecting a point $\mathbf{y}$ in $\mathbb{R}^n$ onto a subspace of R^n.

- **Theorem 8.** Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$

where $\hat{y}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

- This is the **orthogonal Decomposition theorem.** The vector $\hat{\mathbf{y}}$ in (1) is called the **orthogonal projection of y onto** W.

- The textbook uses the notation

$$\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}.$$

- The textbook proves the Theorem by actually computing $\hat{\mathbf{y}}$ and $\mathbf{z}$:

- In fact, if $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_p}\}$ is any orthogonal basis of W, then

$\checkmark$

$$\begin{aligned}\hat{y} &= \frac{\mathbf{y} \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \bullet \mathbf{u}_2}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 + \ldots + \frac{\mathbf{y} \bullet \mathbf{u}_p}{\mathbf{u}_p \bullet \mathbf{u}_p}\mathbf{u}_p \\ &= \sum_{i=1}^{p} \frac{\mathbf{y} \bullet \mathbf{u}_i}{\mathbf{u}_i \bullet \mathbf{u}_i}\mathbf{u}_i\end{aligned}$$

$u_i \cdot u_k = 0$ if $i \neq k$

$$\left(y - \sum \frac{y \cdot u_i}{u_i \cdot u_i} u_i\right) \cdot u_k$$

$$= y \cdot u_k - \frac{y \cdot u_k}{u_k \cdot u_k} u_k \cdot u_k = 0$$

and

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

- However, the uniqueness of the decomposition (1) shows that the orthogonal projections depends only on W and not on its basis.

$W = \operatorname{span}\{u_1, \dots, u_p\}$ $i \neq j \Rightarrow u_i \cdot u_j = 0$

$u_i \neq 0$ $u_i \cdot u_i > 0$

$(y - \hat{y}) \cdot w = 0$ for all w in W

$z \cdot u_i = 0 \Rightarrow z \cdot w = 0$ for all w in W

$i = 1, \dots, p$

$$\hat{y} = \sum_{i=1}^{p} \alpha_i u_i$$

$$0 = (y - \hat{y}) \cdot u_k = \left(y - \sum_{i=1}^{p} \alpha_i u_i\right) \cdot u_k$$

$$= y \cdot u_k - \sum_{i=1}^{p} \alpha_i u_i \cdot u_k$$

$$= y \cdot u_k - \alpha_k u_k \cdot u_k$$

$$\alpha_k = \frac{y \cdot u_k}{u_k \cdot u_k}$$

- Example 2, textbook. Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\hat{y} = \frac{u_1 \cdot y}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot y}{u_2 \cdot u_2} u_2$$

$$= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Geometric Interpretation of Orthogonal Projection

- Note that if the dimension of W is one, and $\{\mathbf{u}\}$ is a basis of W then the orthogonal projection of $\mathbf{y}$ onto W is just

$$\hat{\mathbf{y}} = \frac{\mathbf{u} \bullet \mathbf{y}}{\mathbf{u} \bullet \mathbf{u}}\mathbf{u}.$$

- Thus the terms $\frac{\mathbf{u}_i \bullet \mathbf{u}_i}{\mathbf{u}_i \bullet \mathbf{u}_i}\mathbf{u}_i$ in

$$\hat{y} = \sum_{i=1}^{p} \frac{\mathbf{u}_i \bullet \mathbf{y}}{\mathbf{u}_i \bullet \mathbf{u}_i}\mathbf{u}_i$$

are just the projections of $\mathbf{y}$ onto the spaces $\operatorname{span}\{\mathbf{u}_i\}$.

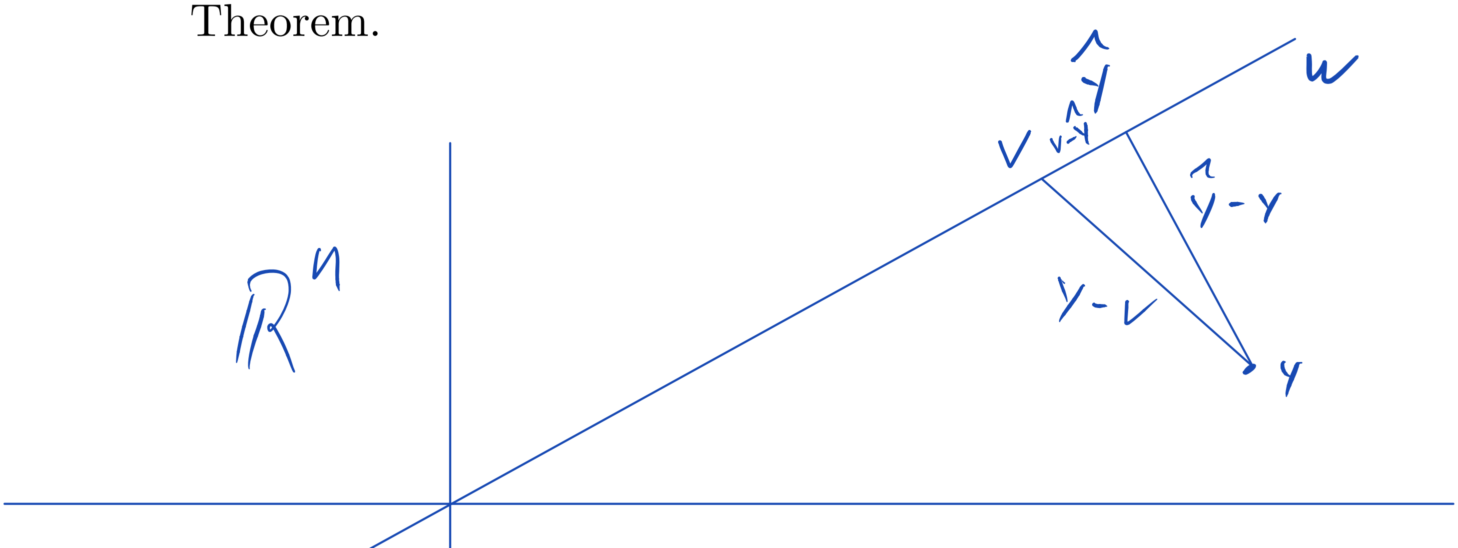
- As in our initial example, the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto W is the point in W that is closest to $\mathbf{y}$. This is the contents of

- **Best Approximation Theorem** (Theorem 9, p. 352) Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \mathbf{v}\| > \|\mathbf{y} - \hat{\mathbf{y}}\|$$

  for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$.

- This is a simple consequence of the Pythagorean Theorem.



$$\|y-v\|^2 = \|\hat{y}-y\|^2 + \|v-\hat{y}\|^2$$

$$\|y-v\| > \|\hat{y}-y\|$$

- Example 4, p. 353, textbook. Let

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$$

Compute the point $\hat{y}$ in W that is closest to $\mathbf{y}$ and its distance from $\mathbf{y}$.

$$\hat{y} = \underset{1/2}{\frac{15}{30}} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \underset{-7/2}{\frac{-21}{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\| y - \hat{y} \| = \left\| \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \right\| = \sqrt{45}$$

- Formulas simplify if our basis is orthonormal, rather than just orthogonal.
- **Theorem 10**, p. 353. If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\hat{y} = \text{proj}_W \mathbf{y} = \sum_{i=1}^{p} (\mathbf{y} \bullet \mathbf{u}_i)\mathbf{u}_i.$$

$U = [u_1, \ldots, u_p]$

$U^T y = \begin{bmatrix} u_1^T \\ \vdots \\ u_p^T \end{bmatrix} y = \begin{bmatrix} u_1^T y \\ \vdots \\ u_p^T y \end{bmatrix}$

If $U = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \mathbf{u}_p\,]$, then

$$\text{proj}_W \mathbf{y} = UU^T \mathbf{y}$$

for all $\mathbf{y}$ in $\mathbb{R}^n$.

- Next question: How do we get orthonormal bases?

$\mathcal{B} = \{v_1, v_2, \dots, v_p\}$ Basis of W

Want:

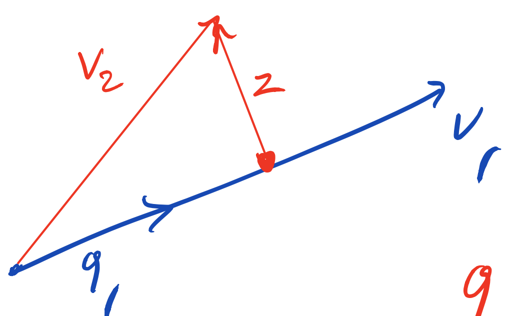
$\mathcal{Q} = \{q_1, q_2, \dots, q_p\}$

$$q_i \cdot q_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and:

$$\text{span}\{v_1, \dots, v_k\} = \text{span}\{q_1, \dots, q_k\}$$

$k = 1, \dots, p$



$q_2^T q_1 = 0$

$$q_2 = \frac{z}{\|z\|}$$