Math 2270-1

Notes of 9/23/2019

- Quick Review: We define the determinant of a square, $n \times n$ matrix A by

$$\det A = |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

where

$$C_{ij} = (-1)^{i+j} |A_{ij}|$$

and A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and the j-th column.

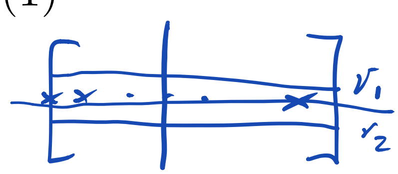
- C_{ij} is the (ij)**-cofactor** and the formula is the **cofactor expansion** of the determinant.

any choice of i or j will give the same answer!

- Theorem 1 in the textbook gives the more general formulas

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \tag{1}$$

for any choice of i and



$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \tag{2}$$

for any choice of j.

- **Exercise** Show that the above $2n$ formulas all give the same value.
- Hint: Show that any of those formulas gives rise to the more symmetric form

$$|A| = \sum_{\sigma} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

where the sum is over all $n!$ permutations

$$\sigma : \{1, 2, \ldots, n\} \longrightarrow \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

of the set $\{1, 2, \ldots, n\}$ and the sign of the permutation is 1 if the number of transpositions to get it is even, and -1 if that number is odd. A transposition is an interchange of two neighboring entries. For example, suppose σ reverses the sequence in the set $\{1, 2, 3\}$

$$\sigma : \{1, 2, 3\} \longrightarrow \{3, 2, 1\}\,.$$

Then the sign of σ is -1 since

$$\{1, 2, 3\} \longrightarrow \{1, 3, 2\} \longrightarrow \{3, 1, 2\} \longrightarrow \{3, 2, 1\}$$

since σ can be constructed from 3 transpositions.

3.2 Properties of Determinants

- Before we lose sight of the forest for all the trees, here are the main results in this section:

- The first describes the effects of row operations on the determinant:

Theorem 3. (Row Operations) Let A be a square matrix. Then

a. If a multiple of a row of A is added to another row to produce a matrix B, then

$$\det B = \det A.$$

b. If two rows of A are interchanged to produce B then

$$\det B = -\det A.$$

c. If one row of A is multiplied by a scalar k to produce B then

$$\det B = k \det A.$$

A 5×5

$|2A| = 32|A|$

$T(ku) = kT(u)$

Theorem 3 allows us to compute a determinant by reducing the matrix to triangular form!

- We also have an addition to our invertibility Theorem:

Theorem 4.

s. A square matrix A is invertible if and only if

$$\det A \neq 0.$$

- The determinant of a matrix equals the determinant of its transpose:

Theorem 5. If A is a square matrix then

$$\det A = \det A^T.$$

- Finally, the determinant of a product equals the product of the determinants:

Theorem 6. If A and B are both $n \times n$ matrices then

$$\det(AB) = \det A \times \det B.$$

$A = B = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$\det(A+B) \neq \underbrace{\det(A)+\det(B)}_{2}$$

$$4 = \det \begin{bmatrix}2&0\\0&2\end{bmatrix}$$

- The textbook also shows, without stating the fact as a Theorem, that the determinant is linear in each column separately (and by Theorem 5, also in each row). More specifically, Suppose

$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_{j-1} \quad \overset{u+v}{\mathbf{x}} \quad \mathbf{a}_{j+1} \dots \mathbf{a}_n\,]$$

where the j-th column $\mathbf{x}$ is variable and the other columns are constant. Then define

$$T(\mathbf{x}) = \det A$$

T is a linear transformation, i.e.,

$$T(k\mathbf{u}) = kT(\mathbf{u}) \quad \text{and} \quad T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ and scalars k.

- Some of these results follow easily from others:
- The first property of the linearity of the determinant follows from Theorem 3(c) (and Theorem 5) and the second property follows from the cofactor expansion and the distributive law.
- Theorem 5 is an immediate consequence of the fact that the determinant can be computed by cofactor expansion about any row or column. Transposition of a matrix just changes the rows into columns, and vice versa.
- Theorem 4 can be established by using row operations to reduce the matrix A to triangular form. The row operations do not change invertibility, and they modify the determinant only by a non-zero factor. The triangular form is non-singular if and only if all the diagonal entries are non-zero. The determinant of a triangular matrix is the product of the diagonal entries, and so the triangular matrix

is non-singular if and only if the determinant is non-zero.

- The proofs of Theorems 3 and 6 are a little more tricky. The textbook proceeds by writing the row operations in terms of elementary matrices which we discussed briefly in chapter 2.
- Theorem 3 is proved by induction.
- It's easy to check that the statements made there are true for 2×2 matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A| = ad - bc$$

$$\left| \begin{bmatrix} c & d \\ a & b \end{bmatrix} \right| = bc - ad = -|A|$$

$$\left| \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} \right| = kad - kbc = k(ad-bc) = k|A|$$

$$\left| \begin{bmatrix} a & b \\ c+ta & d+tb \end{bmatrix} \right| = a(d+tb) - b(c+ta)$$

$$= ad - bc + \cancel{tab} - \cancel{tab}$$

$$= ad - bc = |A|$$

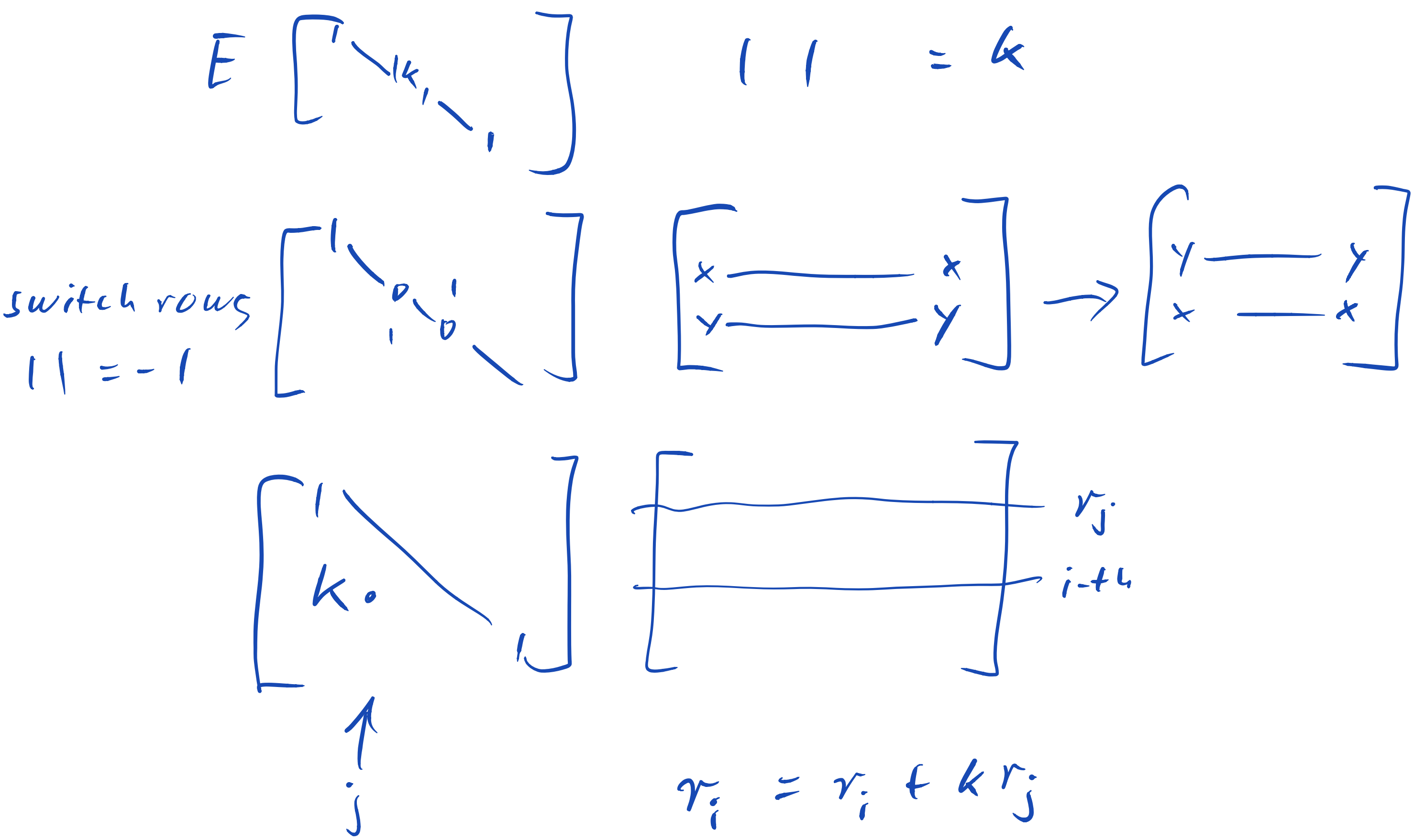
$n > 2$

- For the induction step assume that $n > 2$ and Theorem 3 is true for all smaller matrices. Note that the elementary row operations affect at most 2 rows, so there must be one, say the i-th one, that is unchanged by the row operation. Expand the determinant by cofactors about the i-th row and apply the induction hypothesis and the distributive law.

- For Theorem 6, first note that the statement

$$\det AB = \det A \times \det B \qquad (3)$$

is true if A is an elementary matrix, i.e., it performs one of the three elementary row operations.



- To see the general result first note that if A or B is singular then the result holds trivially. So suppose both A and B are invertible. This implies that the reduced row echelon form of A is the identity matrix. Since each elementary row operation is invertible, and the inverse is also an elementary row operation, we can write

  $$A = E_p E_{p-1} \dots E_1 I$$

  where the E_i are elementary matrices. This implies that

  $$|A| = |E_p||E_{p-1}| \dots |E_1||I|.$$

  Since the determinant of I is 1 and elementary matrices satisfy (3) we get that

  $$\begin{aligned} |AB| &= |E_p E_{p-1} \dots E_1 I B| \\ &= |E_p||E_{p-1}| \dots |E_1||I||B| \\ &= |A||B| \end{aligned}$$

- Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

- Compute the determinant of A by reducing it to lower triangular form and show that A is invertible.

$$|A| = \begin{vmatrix} 5 & 6 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= -7 + 4 - 3 = -6$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & -2 \end{vmatrix} = -6$$

n^3 vs $n!$

Δ form   cofactor expansion

LU

- More examples: