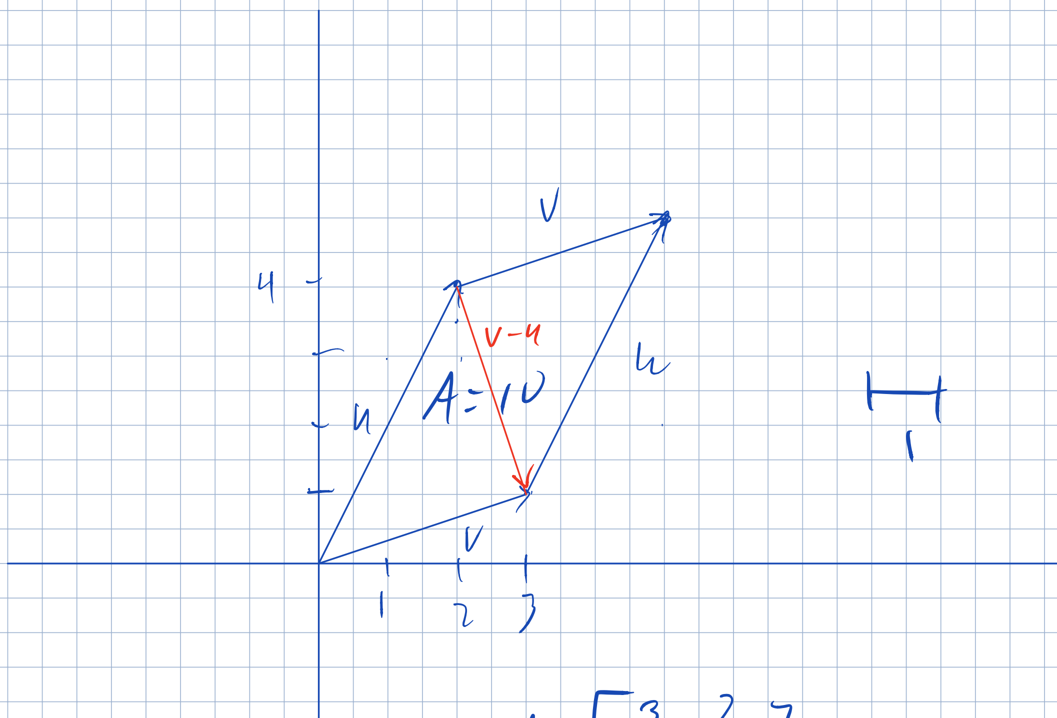

$$A = \begin{array}{c} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \\ \;\; v \quad u \end{array}$$

Math 2270-1

Notes of 9/27/2019

- Recall: a vector space is a set of objects that can be added and multiplied with scalars, such that those 10 axioms hold.
- A subspace of a vector~~s~~ space V is a non-empty subset H of V that is closed under addition and scalar multiplication.
- Loose end from Wednesday:

  Span

- the span of a set of vectors in a space V is a subspace of V.

4.2 Null and Column Spaces

- Today's topic: Vector Spaces associated with a matrix or a linear transformation.
- Suppose, as usual that A is an $m \times n$ matrix. We denote the columns of A by $\mathbf{a}_1$, ..., $\mathbf{a}_n$, i.e.,
$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n\,].$$
- We associate with A the linear (matrix) transformation
$$T(\mathbf{x}) = A\mathbf{x}$$
- There are four spaces associated with A.

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

$$x \in \mathbb{R}^n$$
$$T(x) \in \mathbb{R}^m$$

The Column Space of A

- The **column space** of A is the set of all linear combinations of columns of A. In other words,

$$\begin{aligned}\mathrm{Col}A &= \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \\ &= \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \quad \text{for some } \mathbf{x} \in \mathbb{R}^n\}.\end{aligned}$$

- Col$A$ being the span of a set of vectors clearly is a linear space.
- It's also obvious that the set Col$A$ contains the zero vector, and is closed under addition and scalar multiplication.
- Col$A$ is a subspace of $\mathbb{R}^m$.
- Col$A$ equals $\mathbb{R}^m$ if and only if the matrix transform T is onto.
- Col$A$ is also called the **range** of T.

The Null Space of A

- The **null space** of A is the set of all solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\mathrm{Nul}A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}\,.$$

$Ax = Ay = 0$

$0 = Ax + Ay$
$= A(x+y)$

- The set Nul$A$ contains the origin and is closed under addition and scalar multiplication. It is a vector space!

$A(cx) = cAx = 0$

- Nul$A$ is a subspace of $\mathbb{R}^n$.
- Nul$A$ is the zero space,

$$\mathrm{Nul}A = \{\mathbf{0}\}\,,$$

  if and only if the columns of A are linearly independent.
- Nul$A$ is the zero space if and only if the matrix transformation T is one-to-one.
- Nul$A$ is also called the **kernel** of T.

$A^T y = 0$

$y^T A = 0^T$

The Row Space of A

- The Row Space of A is the set of all linear combinations of rows of A.
- You can interpret it as the column space of A^T.
- The row space is a subset of $\mathbb{R}^n$.

subspace

The Left Null Space of A

- The left Null Space of A is the null space of A^T.
- It is a subspace of $\mathbb{R}^m$.

the textbook does not mention the row space or the left null space. I list them only for completeness.

- Consider Example 5 in the textbook. (modified to simplify the arithmetic)

$$A = \begin{bmatrix} 2 & 4 & -2 & 2 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & -1 & 5 & 5 \\ 0 & 1 & -5 & 3 \end{bmatrix}$$

(1)

$$\to \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & -1 & 5 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -5 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Ax = 0$

$x_4 = 0$

$x_3 \in \mathbb{R}$

$x_2 = 5x_3$

$x_1 = -2x_2 + x_3 = -9x_3$

$x_3 \begin{bmatrix} -9 & 5 & 1 & 0 \end{bmatrix}$

$$\mathrm{Col}\, A \neq \mathrm{COL}\ \mathrm{REF}(A)$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathrm{COL}\, A = \left\{ x\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x \in \mathbb{R} \right\}$$

$$\mathrm{ref}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathrm{col}\big(\mathrm{ref}(A)\big) = \left\{ x\begin{bmatrix} 1 \\ 0 \end{bmatrix} \; x \in \mathbb{R} \right\}$$

- Discuss the table on page 206 of the textbook

Contrast Between Nul *A* and Col *A* for an *m* x *n* Matrix *A*

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

DEFINITION

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W). The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. If T happens to arise as a matrix transformation—say, $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A—then the kernel and the range of T are just the null space and the column space of A, as defined earlier.

It is not difficult to show that the kernel of T is a subspace of V. The proof is essentially the same as the one for Theorem 2. Also, the range of T is a subspace of W. See Figure 2 and Exercise 30.

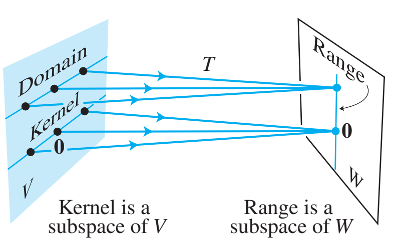

FIGURE 2 Subspaces associated with a linear transformation.

In applications, a subspace usually arises as either the kernel or the range of an appropriate linear transformation. For instance, the set of all solutions of a homogeneous linear differential equation turns out to be the kernel of a linear transformation.

Kernel and Range of a Linear Transformation

- Suppose we have a linear transformation

$$T : V \longrightarrow W$$

  from a vector space V to a vector space W.

- Recall that T is linear if

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

  for all $\mathbf{u}$ and $\mathbf{v}$ in V, and

$$T(c\mathbf{v}) = cT(\mathbf{v})$$

  for all scalars c and $\mathbf{v}$ in V.

- We define the **kernel** of the T to be the set of all vectors $\mathbf{v}$ in V such that

$$T(\mathbf{v}) = 0.$$

- As before, the **range of** T is the set of all $\mathbf{w}$ in W that can be written as $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V.

- If T is defined by a matrix transform,

$$T(\mathbf{x}) = A\mathbf{x},$$

  then the range of T is the column space of A, and the kernel of T is the null space of A.

- Example 8. Suppose V is the set of all continuously differentiable functions defined on some interval $I = [a, b]$. Let W be the space of all continuous functions defined on I. Then the differentiation operator $Df = f'$ is a linear operator from V to W. What is its kernel?

$$Df = f'$$

$$\ker(D) = \{ f(x) = c \mid c \in \mathbb{R} \}$$

D onto

- What if D denoted the second derivative (and V is the space of twice continuously differentiable functions)?

- Example 9: Suppose

$$Dy = y'' + \omega^2 y$$

for some fixed ω? Compute the kernel of D

- What about the kernel of

$$Dy = y'' - \omega^2 y?$$