Math 2270-1

Notes of 9/24/2019

Review

- A is square, $n \times n$.
- The determinant of A is given by

  $$\det A = |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

  where
  $$C_{ij} = (-1)^{i+j} |A_{ij}|$$

  and A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and the j-th column.

- C_{ij} is the (ij)**-cofactor** and the formula is the **cofactor expansion** of the determinant.

$$|A| = \sum_{\sigma} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

- The determinant is linear in each row and column
- In particular, multiplying a row with a scalar multiplies the determinant with that scalar.

- Interchanging two rows of A changes the sign of the determinant.
- Adding a multiple of a row to another row does not change the determinant.
- A is invertible if and only if $\det A \neq 0$.
- Transposing does not change the determinant:

$$\det A^T = \det A.$$

- The determinant is multiplicative:

$$|AB| = |A||B|$$

3.3 Cramer's Rule etc.

- Cramer's Rule (named after Gabriel Cramer, 1704–1752) is useful for some theoretical calculations. It's not a competitive numerical method!
- Consider the linear system

$$A\mathbf{x} = \mathbf{b}$$

where once again A is an invertible square $(n \times n)$ matrix.

- define

$$A_i(\mathbf{b}) = [\, \mathbf{a}_1 \quad \dots \quad \mathbf{a}_{i-1} \quad \mathbf{b} \quad \mathbf{a}_{i+1} \quad \dots \quad \mathbf{a}_n \,]$$

- In other words, $A_i(\mathbf{b})$ is the matrix formed by replacing the i-th column of A by $\mathbf{b}$.
- Cramer's rule states that

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}$$

- Examples:

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = x$$

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 52 \\ 69 \end{bmatrix} = b$$

$$9 \cdot 52 - 7 \cdot 69$$

$$= 468 - 483$$

$$= -15$$

$$69 - 104 = -35$$

$$Ax = b$$

$$x_1 = \frac{\begin{vmatrix} 52 & 7 \\ 69 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 7 \\ 2 & 9 \end{vmatrix}} = \frac{-15}{-5} = 3$$

$$x_2 = \frac{\begin{vmatrix} 1 & 52 \\ 2 & 69 \end{vmatrix}}{-5} = \frac{-35}{-5} = 7$$

$$\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}$$

- Proof of Cramer's Rule:

$$Ax = b \qquad x_i = ?$$

$$I_i(x) = \begin{array}{c} \text{i-th} \\ \begin{bmatrix} 1 & & \vert & & \\ & \ddots & \vert & & \\ & & \vert & & 1 \end{bmatrix} \\ x \end{array} \qquad |I_i(x)| = x_i$$

exercise

$$AI_i(x) = A_i(b)$$

$$|A|\underbrace{|I_i(x)|}_{x_i} = |A_i(b)|$$

$$x_i = \frac{|A_i(b)|}{|A|}$$

A Formula for A^{-1}

- remember our definition of cofactors:

$$C_{ij} = (-1)^{i+j}|A_{ij}|$$

- Next remember that the j-th column of A^{-1} is the solution of

$$A\mathbf{x}_j = \mathbf{e}_j$$

  where, as usual, $\mathbf{e}_j$ is the j-th column of the identity matrix.

- By Cramer's Rule, the (i, j)-entry x_{ij} of A^{-1} is

$$x_{ij} = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

i-th

$$A_i(e_j) = \begin{bmatrix} a_1 & \cdots & 1 & \cdots & a_n \end{bmatrix} \leftarrow j\text{-th}$$

e_j

- Finally, expand $A_i(e_j)$ by cofactors about the i-th column to see that

$$x_{ij} = \frac{\det A_i(\mathbf{e}_j)}{\det A} = \frac{C_{ji}}{\det A}.$$

$|A_i(e_j)| =$

- We get the formula

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- The matrix

$$\mathrm{adj}A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

is called the **adjugate** of A.

note that the adjugate is the **transpose** of the matrix of cofactors!

Aside: $|[a_{11}]| = a_{11}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |A| = ad - bc = D$

C = matrix of cofactors

$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\quad C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \overset{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}}$

✓

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

- Examples

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & -20 \\ \text{etc.} & \end{bmatrix}$$

Geometric Interpretation of the determinant

- Recall these three properties:

1. The determinant of the identity matrix is 1.

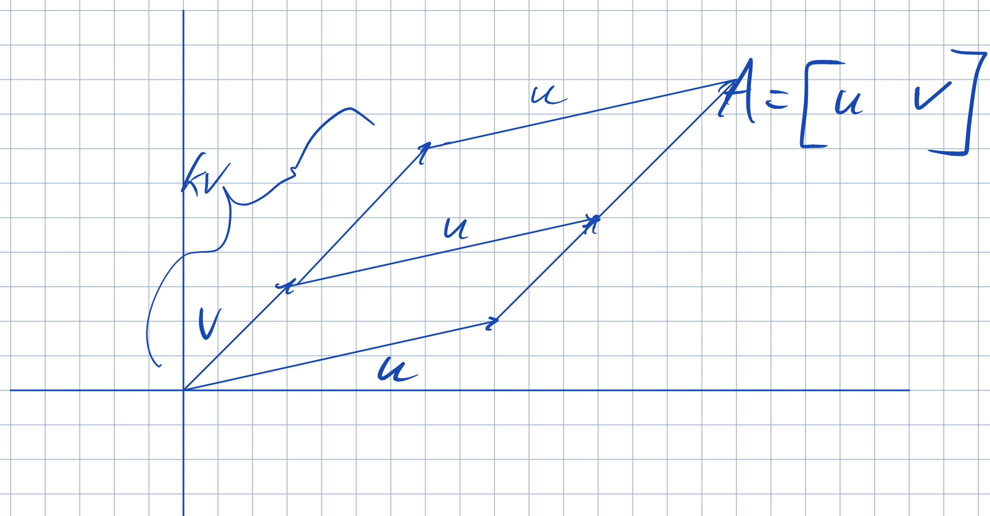
2. If you interchange two rows of the matrix the determinant changes it sign.

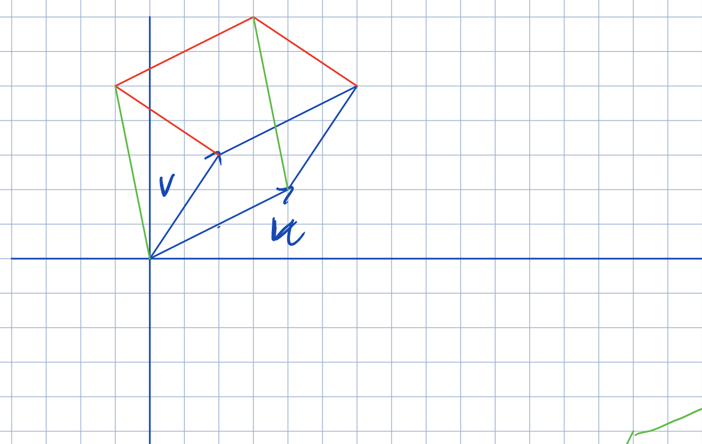
3. The determinant is a function that is linear in each row separately. (In other words, if you think of the determinant as a function of a specific row, keeping everything else constant you get a linear function.)

- Aside: Strang uses these three properties to define determinants.

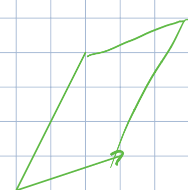
- Ignoring the sign change, these properties define the area or volume of the rectangle (in R^2) or parallelepiped (in R^3) formed by the 2 or 3 columns of the given matrix.

- The corresponding object in higher dimensions is called a **parallelotope**.





$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$



- The volume V of a parallelotope defined by the columns of A is therefore given by

$$V = |\det A|$$

where the vertical bars in this case do denote the absolute values.

- Examples

Linear Transformations

- The textbook discusses in some detail that applying a linear transformation to a geometric object multiplies the volume of the object by the absolute value of the determinant of the linear transformation.
- You actually already saw this principle in action when discussing the change of variable formula in multivariable calculus.