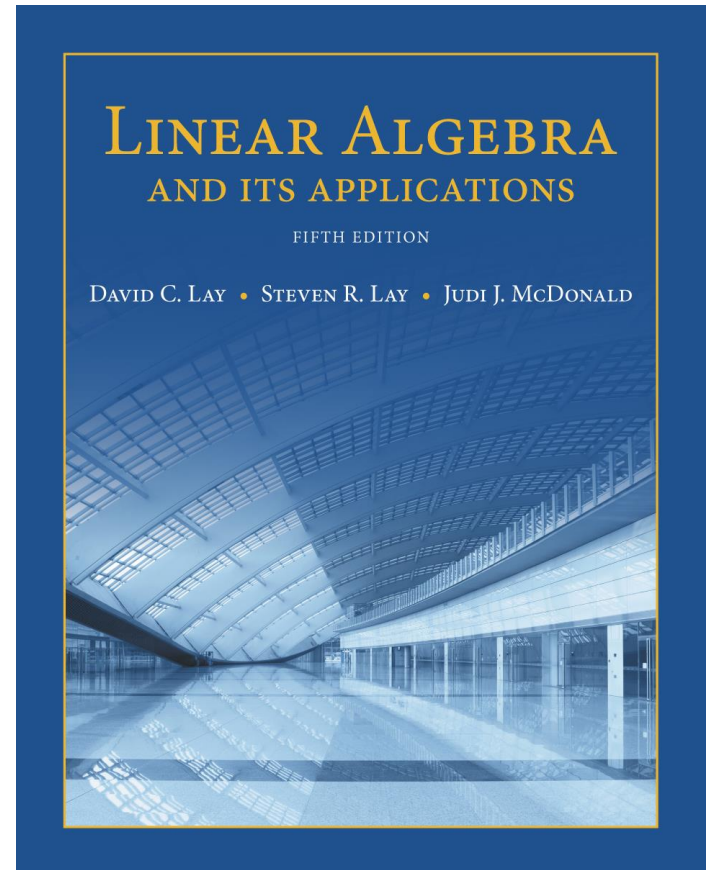


# 2

## Matrix Algebra

### 2.3

#### CHARACTERIZATIONS OF INVERTIBLE MATRICES



# THE INVERTIBLE MATRIX THEOREM

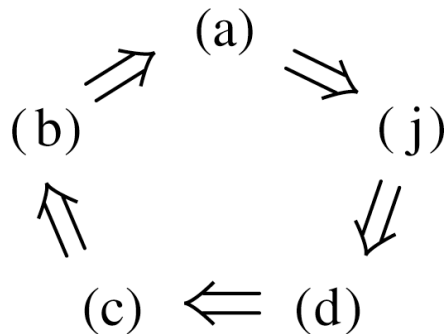
- **Theorem 8:** Let  $A$  be a square  $n \times n$  matrix. Then the following statements are equivalent. That is, for a given  $A$ , the statements are either all true or all false.
  - a.  $A$  is an invertible matrix.
  - b.  $A$  is row equivalent to the  $n \times n$  identity matrix.
  - c.  $A$  has  $n$  pivot positions.
  - d. The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
  - e. The columns of  $A$  form a linearly independent set.

# THE INVERTIBLE MATRIX THEOREM

- f. The linear transformation  $x \mapsto Ax$  is one-to-one.
- g. The equation  $Ax = b$  has at least one solution for each  $b$  in  $\mathbb{R}^n$ .
- h. The columns of  $A$  span  $\mathbb{R}^n$ .
- i. The linear transformation  $x \mapsto Ax$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- j. There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
- k. There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
- l.  $A^T$  is an invertible matrix.

# THE INVERTIBLE MATRIX THEOREM

- First, we need some notation.
- If the truth of statement (a) always implies that statement (j) is true, we say that (a) *implies* (j) and write  $(a) \Rightarrow (j)$ .
- The proof will establish the “circle” of implications as shown in the following figure.



- If any one of these five statements is true, then so are the others.

# THE INVERTIBLE MATRIX THEOREM

- Finally, the proof will link the remaining statements of the theorem to the statements in this circle.
- **Proof:** If statement (a) is true, then  $A^{-1}$  works for  $C$  in (j), so  $(a) \Rightarrow (j)$ .
- Next,  $(j) \Rightarrow (d)$ .
- Also,  $(d) \Rightarrow (c)$ .
- If  $A$  is square and has  $n$  pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of  $A$  is  $I_n$ .
- Thus  $(c) \Rightarrow (b)$ .
- Also,  $(b) \Rightarrow (a)$ .

# THE INVERTIBLE MATRIX THEOREM

- This completes the circle in the previous figure.
- Next,  $(a) \Rightarrow (k)$  because  $A^{-1}$  works for  $D$ .
- Also,  $(k) \Rightarrow (g)$  and  $(g) \Rightarrow (a)$ .
- So  $(k)$  and  $(g)$  are linked to the circle.
- Further,  $(g)$ ,  $(h)$ , and  $(i)$  are equivalent for any matrix.
- Thus,  $(h)$  and  $(i)$  are linked through  $(g)$  to the circle.
- Since  $(d)$  is linked to the circle, so are  $(e)$  and  $(f)$ , because  $(d)$ ,  $(e)$ , and  $(f)$  are all equivalent for *any* matrix  $A$ .
- Finally,  $(a) \Rightarrow (l)$  and  $(l) \Rightarrow (a)$ .
- This completes the proof.

# THE INVERTIBLE MATRIX THEOREM

- Theorem 8 could also be written as “The equation  $A\mathbf{x} = \mathbf{b}$  has a *unique* solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .”
- This statement implies (b) and hence implies that  $A$  is invertible.
- The following fact follows from Theorem 8.  
Let  $A$  and  $B$  be square matrices. If  $AB = I$ , then  $A$  and  $B$  are both invertible, with  $B = A^{-1}$  and  $A = B^{-1}$ .
- The Invertible Matrix Theorem divides the set of all  $n \times n$  matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

# THE INVERTIBLE MATRIX THEOREM

- Each statement in the theorem describes a property of every  $n \times n$  invertible matrix.
- The *negation* of a statement in the theorem describes a property of every  $n \times n$  singular matrix.
- For instance, an  $n \times n$  singular matrix is *not* row equivalent to  $I_n$ , does *not* have  $n$  pivot position, and has linearly *dependent* columns.



# THE INVERTIBLE MATRIX THEOREM

- **Example 1:** Use the Invertible Matrix Theorem to decide if  $A$  is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**

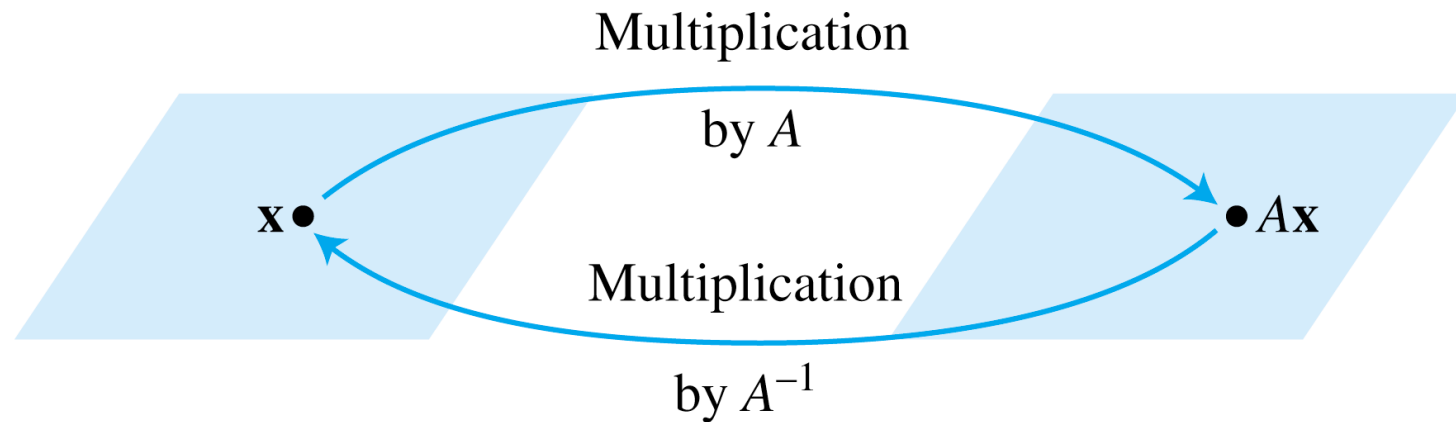
$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

# THE INVERTIBLE MATRIX THEOREM

- So  $A$  has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).
- The Invertible Matrix Theorem *applies only to square matrices*.
- For example, if the columns of a  $4 \times 3$  matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form  $Ax = b$ .

# INVERTIBLE LINEAR TRANSFORMATIONS

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix  $A$  is invertible, the equation  $A^{-1}Ax = x$  can be viewed as a statement about linear transformations. See the following figure.



$A^{-1}$  transforms  $Ax$  back to  $x$ .

# INVERTIBLE LINEAR TRANSFORMATIONS

- A linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be **invertible** if there exists a function  $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (1)$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (2)$$

- **Theorem 9:** Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation and let  $A$  be the standard matrix for  $T$ . Then  $T$  is invertible if and only if  $A$  is an invertible matrix. In that case, the linear transformation  $S$  given by  $S(\mathbf{x}) = A^{-1}\mathbf{x}$  is the unique function satisfying equation (1) and (2).

# INVERTIBLE LINEAR TRANSFORMATIONS

- **Proof:** Suppose that  $T$  is invertible.
- Then (2) shows that  $T$  is onto  $\mathbb{R}^n$ , for if  $\mathbf{b}$  is in  $\mathbb{R}^n$  and  $\mathbf{x} = S(\mathbf{b})$ , then  $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$ , so each  $\mathbf{b}$  is in the range of  $T$ .
- Thus  $A$  is invertible, by the Invertible Matrix Theorem, statement (i).
- Conversely, suppose that  $A$  is invertible, and let  $S(\mathbf{x}) = A^{-1}\mathbf{x}$ . Then,  $S$  is a linear transformation, and  $S$  satisfies (1) and (2).
- For instance,  $S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$ .
- Thus,  $T$  is invertible.