Today: 12:55 LCB 225

Alumni Panel

career paths for U of U Math Majors

(Undergraduate Colloquium)

$$C_{ij} = (-1)^{i+j} |A_{ij}|$$

$$C = [C_{ij}]$$

$$A^{-1} = \frac{C^T}{\det|A|}$$

Math 2270-1

Notes of 9/25/2019

- Today's subject: generalize the notion of a vector space.
- General procedure:

1. Start with a notion or a specific example.

2. Come up with a precise definition.

3. Investigate what objects satisfy the definition, and what properties are implied by the definition.

- In the past we applied the same idea in Calculus, to
  - limits
  - derivatives
  - integrals

- The definition often consists of a list of required properties, sometimes called **axioms**. That's the approach we'll take today.

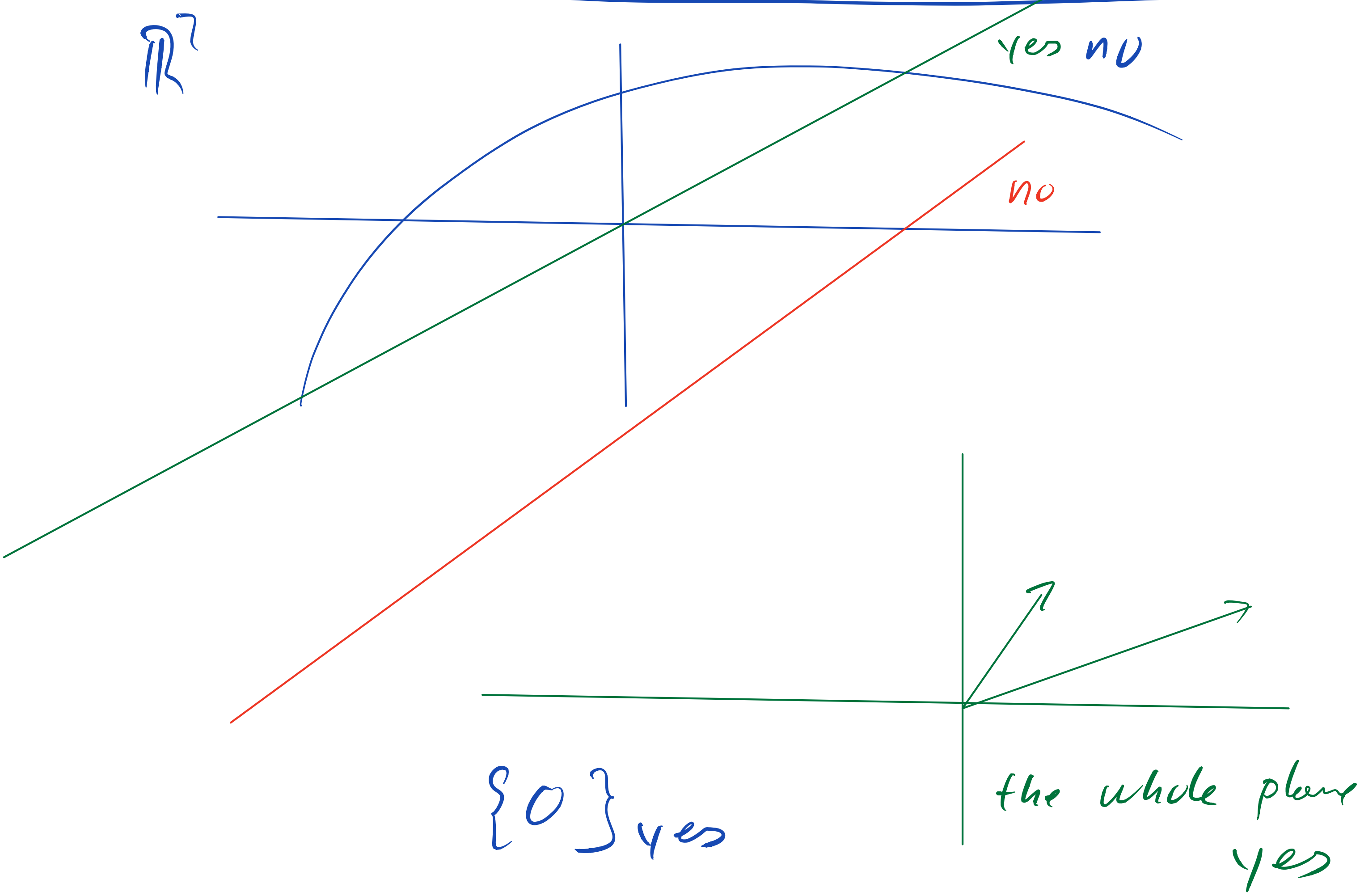
- We start with $\mathbb{R}^n$. What key properties does $\mathbb{R}^n$ have?

$0 \qquad u + 0 = u \qquad u, v \in \mathbb{R}^n$

$u + v = v + u$

$(u+v) + w = u + (v + w)$

$u + v \in \mathbb{R}^n$



4.1 Vector Spaces and Subspaces

- **Definition:** (p. 192, textbook): A **vector space** is a nonempty set V of objects, called vectors, on which are defined two operations, called **addition** and **multiplication by scalars (real numbers)**, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

4. There is a zero vector $\mathbf{0}$ in V such that

$$\mathbf{u} + \mathbf{0} = \mathbf{u}.$$

5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $u + (-\mathbf{u}) = \mathbf{0}$.

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.

10. $1\mathbf{u} = \mathbf{u}$.

- Exercises: **Using only the axioms**, prove that $\mathbf{0}$ is unique, $-\mathbf{u} = (-1)\mathbf{u}$, $0\mathbf{u} = \mathbf{0}$, and $c\mathbf{0} = \mathbf{0}$.

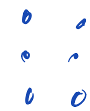
A phrase that is synonymous with "vector space" is **linear space.** The textbook does not use that phrase, but I believe it is used more commonly than "vector space".

We only consider scalars that are real numbers. One can build a very similar theory of vector spaces based on the set of complex numbers.

If you've studied algebra you are familiar with the concept of a **field**. (The sets of real numbers and complex numbers are special cases of fields.) One can develop a theory of linear spaces over any field.

Examples

- $\mathbb{R}^n$
- Arrows in the plane or 3 dimensional space. (Of course we then identify the arrows with vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.)



$\mathbb{R}^2$ $\quad \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \quad x, y \text{ integer} \right\}$ no

rational no

Sequences

$a = $ " $a_0, a_1, a_2, \dots$ "

$b = $ " $b_0, b_1, b_2, \dots$ "

$a + b = $ " $a_0 + b_0, a_1 + b_1, + \dots$ "

$ca = ca_0, ca_1, \dots$

f. b.

x " $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$ "

$a_{2k} = 0 \qquad k = 1, 2, \dots$ yes

$a_{2k} = 1$ no

$\{ \{a_n\} : \lim_{n \to \infty} a_n \text{ exists} \}$ yes

Polynomials

$$p(x) = \sum_{j=0}^{n} \alpha_j x^j$$

$$\{ \text{even polynomials} \}$$

$$\{ \text{odd polynomials} \}$$

Functions

$$\{ f \mid f : D \to \mathbb{R} \}$$

$\mathbb{R}^{m \times n}$ = set of all $m \times n$ matrices

$m = n$

$\{A : A^T = A\}$

$\{A : A \text{ diagonal}\}$

$\{A : A \text{ upper } \Delta\text{ar}\}$

lower Δar

$\{A : A \text{ invertible}\}$ no

- A **subspace** of a vector space V is a non-empty subset of V that is closed under addition and scalar multiplication.

- The textbook requires three properties (see p. 195).

- Definition: A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. H is closed under vector addition. That is if $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{h}$ are in H.

c. H is closed under scalar multiplication. That is if $\mathbf{u}$ is in H and c is a scalar, then $c\mathbf{u}$ is in H.

- If H is non empty, and closed under addition and scalar multiplication, then

$$\mathbf{0} = \mathbf{v} + (-1)\mathbf{v}$$

  is in H.

- Thus the only reason to have the requirement **a.** is to make sure H is non-empty.

every subspace is a vector space itself.

An equivalent (exercise!) definition of "subspace" is: A subspace of a vectors space V is a subset H that is itself a vector space.

- For every vector space V, the set $\{\mathbf{0}\}$ is a subspace, called the **zero subspace**.

- Find examples ~~of~~ subspaces for the previous examples of vector spaces.

Span

- the span of a set of vectors is a subspace.