small change in syllabus

Math 2270-1

Notes of 10/04/2019

4.3, 4.5, Vector Spaces, Bases, and Dimension

- Note: we focus on finite dimensional vector spaces.
- Let's use the space of quadratic polynomials as a running example.

$$V = \{p : p(x) = ax^2 + bx + c\}$$

- Recall the definition:

  A **vector space**[-1-] is a nonempty set V of objects, called vectors, on which are defined two operations, called **addition** and **multiplication by scalars (real numbers)**, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.

2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

4. There is a zero vector $\mathbf{0}$ in V such that

$$\mathbf{u}+\mathbf{0}=\mathbf{u}.$$

5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $u+(-\mathbf{u})=\mathbf{0}$.

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.

8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.

9. $c(d\mathbf{u})=(cd)\mathbf{u}$.

10. $1\mathbf{u}=\mathbf{u}$.

[-1-] Also called a **linear space**

- A **linear combination** of a set of vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$$

of vectors is a vector of the form

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

where the α_i are constants.

- A vector space is closed under linear combinations.

- A finite set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$$

of vectors is **linearly independent** if

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0} \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

- The **span** of a set S of vectors is the set of all linear combinations of vectors in S.

Usually S is finite. If S is infinite we assume that the linear combination has only finitely many non-zero terms.

$\{1, x, x^2\}$ l.i.

$\{1, x, x+1\}$ l.d.

span(S) vector space

- A **spanning set** of a vector space V is a set S such that V equals the span of S.

$\{1, x, x^2\}$ linearly independent
basis

$\{1, x, x^2, x^2+x+1, x^2-1\}$

V is a spanning set of (or for) V

- A **basis** of a vector space V is a linearly independent spanning set for V.

- Major Fact: All Bases of a given vector space V have the same number of elements (vectors). That number is the **dimension** of V.
- This is a consequence of Theorem 9 on page 227 of the textbook:
- If a vector space V has a basis

  $$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

  then any set in V containing more than n vectors is linearly dependent.
- The textbook has a proof using coordinate vectors, but those are not needed.

  Proof: Suppose

$$\mathcal{B} = \{\mathbf{b}_1, \quad \mathbf{b}_2, \quad \ldots, \quad \mathbf{b}_n\}$$

is a basis of V and

$$\mathcal{S} = \{\mathbf{v}_1, \quad \mathbf{v}_2, \quad \ldots, \quad \mathbf{v}_m\}$$

is a larger set of

$$m > n$$

vectors.

Since $\mathcal{B}$ is a basis we can write every vector $\mathbf{v}_i$ as a linear combination of the basis vectors $\mathbf{b}_j$:

$$\mathbf{v}_i = \sum_{j=1}^{n} c_{ij}\mathbf{b}_j, \qquad i = 1, \ldots, m. \tag{1}$$

Now consider a linear combination of the $\mathbf{v}_i$. We want to show that there are non-trivial such linear combinations that equal zero.

$$\sum_{i=1}^{m} \alpha_i \mathbf{v}_i = \mathbf{0} \quad \text{and} \quad \alpha_i \neq 0$$

for some i.

Using (1) we get

$$\begin{aligned} \mathbf{0} &= \sum_{i=1}^{m} \alpha_i \mathbf{v}_i \\ &= \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} c_{ij} \mathbf{b}_j \\ &= \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} \alpha_i c_{ij} \right)}_{=0} \mathbf{b}_j. \end{aligned}$$

Hence $\sum_{i=1}^{m} \alpha_i \mathbf{v}_i$ is zero so long as

$$\sum_{i=1}^{m} \alpha_i c_{ij} = 0 \tag{2}$$

and we are done if we can show that there are coefficients α_i, some of which are non-zero, such (2) is satisfied.

Now let

$$C = \left[c_{ij} \right]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}$$

In matrix form the equations (2) can be written as

$$C^T \mathbf{a} = \mathbf{0}. \tag{3}$$

C is an $m \times n$ matrix with $m > n$. Thus C^T has more columns than rows and the homogeneous system (3) does have a non-trivial solution. The set $\mathcal{S}$ is linearly dependent.

$$Ax = 0$$

$V = \text{span}\{1, x, x^2\}$ $\qquad B_1$

$B_2 = \{2, 2x, 2x^2\}$

$B_3 = \{1, 2x, 3x^2\}$

$\{1, 1+x, 1+x+x^2\}$

$ax^2 + bx + c = \alpha + \beta(1+x) + \gamma(1+x+x^2)$

$c = \alpha$

$b = \alpha + \beta$

$a = \alpha + \beta + \gamma$

Examples

- Find bases (and dimensions) for the following vector spaces.
- The set of all quadratic polynomials.

- The set of all upper triangular 2×2 matrices.

$$\mathcal{B}_1 = \left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\} \qquad \dim = 3$$

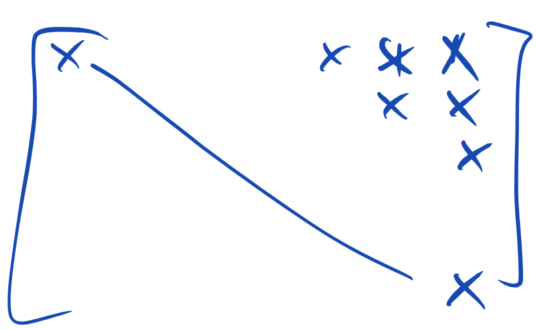
$$\begin{bmatrix}a&b\\0&c\end{bmatrix} = a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$\mathcal{B}_2 = \left\{ \underset{\alpha}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}, \underset{\beta}{\begin{bmatrix}1&1\\0&0\end{bmatrix}}, \underset{\gamma}{\begin{bmatrix}1&1\\0&1\end{bmatrix}} \right\}$$

$$\begin{bmatrix}a&b\\0&c\end{bmatrix} = \begin{bmatrix}\alpha+\beta+\gamma & \beta+\gamma\\ 0 & \gamma\end{bmatrix} \qquad \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

- The set V of all upper triangular $n \times n$ matrices.

$$\dim V = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$



- The set of all solutions of the differential equation

$$y'' = k^2 y$$

$$\mathcal{B} = \{ e^{kx}, e^{-kx} \}$$

$$0 = 0 \cdot e^{kx} + 0 \cdot e^{-kx}$$

- The set of all sequences that satisfy the Fibonacci[−2−] equation

$$y_{n+2} = y_n + y_{n+1}, \qquad n = 0, 1, 2, \ldots$$

form a linear space. Find a basis.

[−2−] Fibonacci, approx. 1170-1250, came up with this equation (where $y_0 = y_1 = 1$) to model the growth of a rabbit population.