Math 2270-1

Notes of 9/4/19

- Quick review:
- Let $\mathbf{f}$ be a function whose domain is $\mathbb{R}^n$ and whose range is (a subset of) $\mathbb{R}^m$, i.e.,

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

  where $\mathbf{y}$ is in $\mathbb{R}^m$ and $\mathbf{x}$ is in $\mathbb{R}^n$.

- The function $\mathbf{f}$ is said to be **linear** if

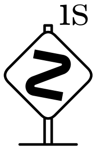
  $\mathbf{f}(\mathbf{u}+\mathbf{v}) = \mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})$ and $\mathbf{f}(c\mathbf{u}) = c\mathbf{f}(\mathbf{u})$,

  for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and scalars c.

- Given an $m \times n$ matrix A the function (or matrix transform)

$$T(\mathbf{x}) = A\mathbf{x}$$

  is linear.



$m \times n$

Given a linear Transform T from $\mathbb{R}^n$ to $\mathbb{R}^m$ the matrix

$$A = [\, T(\mathbf{e}_1) \quad T((\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n) \,]$$

is the **standard matrix** of T and

$$T(\mathbf{x}) = A\mathbf{x}$$

- Let's do an example.

$T: \mathbb{R}^3 \to \mathbb{R}^4$ linear

$T(x) = Ax$ $\quad A = ?$

$A: 4 \times 3$

$$e_i = \begin{bmatrix} 0 \\ | \\ 0 \\ 1 \\ 0 \\ | \\ 0 \end{bmatrix} \leftarrow \text{i-th row}$$

$A = [a_1, a_2, a_3]$

$$[a_1, a_2, a_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 a_1 + x_2 a_2 + x_3 a_3$$

$a_1 = T(e_1) \quad a_2 = T(e_2) \quad a_3 = T(e_3)$

$$Ax = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= T(x_1 e_1 + x_2 e_2 + x_3 e_3) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$$

$$= T(x)$$

$$T(e_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad T(e_2) = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 6 & 2 \\ 3 & 7 & 2 \\ 4 & 8 & 2 \end{bmatrix}$$

$$T(x) = Ax \qquad x \in \mathbb{R}^3$$

$$T(x) \in \mathbb{R}^4$$

$$A \quad 4 \times 3$$

$T: \mathbb{R}^n \to \mathbb{R}^m \iff m \times n$ matrix

Interpretation of Linear Systems

- A mapping

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

  is said to be **onto $\mathbb{R}^m$** (or just **onto**) if each $\mathbf{b}$ in $\cancel{\mathbb{R}^n}$ $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

- In the terminology used by the textbook an equivalent statement is that the range is all of the codomain of T.

- A mapping

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

  is called **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

- While we are at it: a mapping that is both onto and one-to-one is called a **bijection**.

- Using this terminology and the transformation

$$T(\mathbf{x}) = A\mathbf{x},$$

  discuss the issues of existence and uniqueness of the solution of the linear system.

$Ax = b$ has a solution if
b in range of T

$Ax = b$ has a unique solution if it has a solution at all $\iff$ T one-to-one

$Ax = b$ has a soln for all b in $\mathbb{R}^m$

T onto

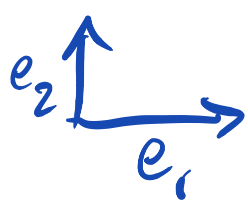
$Ax = b$ has a unique soln for all b

bijection

injective $\longleftrightarrow$ one-to-one
surjective $\longleftrightarrow$ onto
bijective $\longleftrightarrow$ one-to-one and onto

Caveats

- In the context of today's discussion we have assumed implicitly that functions are between finite dimensional vector spaces.
- For example, differentiation and integration are also linear, but they cannot in general be written in terms of matrices. (We will see later in the semester how to write those operators as matrices in special cases.)
- The matrix of a linear transformation depends on how we express vectors. We don't know yet how, but we don't need to use the standard vectors $\mathbf{e}_i$.



- Suppose T and S are linear transforms from $\mathbb{R}^m$ to $\mathbb{R}^n$.
- Then so is cT and $S+T$.
- Suppose A and B are the standard matrices of S and T.
- What are the matrices of cT and $S+T$?

cT: $cT(x) = Bx$ $\qquad T(x) = A$

$$= c(Ax)$$

$$= (cA)x \qquad c[a_{ij}] = [c\,a_{ij}]$$

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$S(x) = Bx \qquad T(x) = Ax$

$(T+S)(x) = S(x) + T(x)$

A, B $m \times n$ $\qquad = Ax + Bx$

$$= (A+B)x$$

$$= Cx$$

The composition of two linear transforms is a linear transform. What is its standard matrix, in terms of the original standard matrices?

$$\underbrace{\mathbb{R}^p \xrightarrow[\substack{B\\ n\times p}]{S} \mathbb{R}^n \xrightarrow[\substack{A\\ m\times n}]{T} \mathbb{R}^m}_{C \qquad m\times p}$$

$$L = T \circ S$$

$$\begin{aligned} L(u+v) &= T(S(u+v)) \\ &= T(S(u)+S(v)) \\ &= T(S(u)) + T(S(v)) \\ &= L(u) + L(v) \end{aligned}$$

$$A, B \longrightarrow C \quad ?$$

$$T(S(x)) = T(Bx) \qquad B = [b_1, \ldots, b_p]$$

$$= T\left(\sum_{i=1}^{p} x_i b_i\right)$$

$$= A \sum_{i=1}^{p} x_i b_i$$

$$= \sum_{i=1}^{p} x_i \underbrace{A b_i}_{c_i}$$

$$= \sum_{i=1}^{p} x_i \; c_i$$

$$= Cx \qquad C = [c_1, \ldots, c_p]$$

$$c_i = A b_i$$

$$C \quad m \times p$$

$$\underset{m \times p}{C} = \underset{m \times n}{A} \; \underset{n \times p}{B}$$

$$c_i \quad A b_i$$

ex.:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix}$$