Math 2270-1

Notes of 8/27/2019

The Matrix Equation $A\mathbf{x} = \mathbf{b}$

- If A is an $m \times n$ matrix and $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$ is the linear combination of the columns of A, with the weights being given by the entries of $\mathbf{x}$.
- More explicitly:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

Note that the product $A\mathbf{x}$ is defined only if $\mathbf{x}$ has as many rows as A has columns!

- Examples:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$2x_1 + 3x_2 = b_1$$
$$3x_1 + 4x_2 = b_2$$

$\left([2,3]\begin{bmatrix}1&2\\3&4\end{bmatrix} = 2[1,2]+3[3,4] = [11,16] \right)$ IFTP Aside

- Clearly (Theorem 3, p. 36, textbook): If A is an $m \times n$ matrix, with columns $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^n$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$M = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \quad \mathbf{b}\,]$$

- Note that A is the **coefficient matrix** of the linear system described by M. A is an $m \times n$ matrix, and M is an $m \times (n+1)$ matrix.
- Sometimes M is also written as

$$M = [\,A \quad \mathbf{b}\,]\,.$$

- Example: Write the linear system

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & - & x_3 & = & 4 \\ & - & 5x_2 & + & 3x_3 & = & 1 \end{array}$$

as a vector equation and as a matrix equation (and solve it).

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$x_3 = 4$ (exmpl.

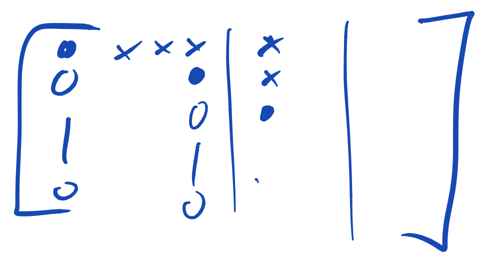
$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & 1 \end{bmatrix}$$



x_3 = arbitrary

$$-5x_2 + 3x_3 = 1$$

$$5x_2 = 3x_3 - 1$$

$$x_2 = \frac{1}{5}(3x_3 - 1)$$

$$x_1 + 2x_2 - x_3 = 4$$

$$x_1 = 4 - 2x_2 + x_3 = 4 - \frac{2}{5}\left(3x_3 - 1\right) + x_3$$

$$= 4 + \frac{2}{5} - \frac{1}{5}x_3$$

- An important question for any matrix A is if the linear system $A\mathbf{x} = \mathbf{b}$ has a solution **for all right hand sides b**.

- Example: Is the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

consistent for all right hand sides $\mathbf{b}$?

$$M = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 1 & b_3 \end{bmatrix}$$

- Example 3, textbook. Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Is the linear system $A\mathbf{x} = \mathbf{b}$ consistent for all right hand sides $\mathbf{b}$?

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & \underbrace{b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1)}_{z} \end{bmatrix}$$

ex. $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $z = 1$ no soln

$b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $z = 0$ consistent

∞ many solns

- Theorem 4 in the textbook tells us four equivalent conditions for consistency for all right hand sides.

- **Theorem 4**: Let A be an $m \times n$ matrix. Then the following statements are equivalent. (This means that for a particular A they are all true or all false.)

a. For each $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

- Note that Theorem 4: makes a statement about a matrix, not a specific linear system or augmented matrix.

- We defined the product of a matrix A and a vector as a linear combination of the columns of A.

- However, recall the definition of the **dot product**:

$$\mathbf{u} \bullet \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

We can thus think of the i-th entry of $A\mathbf{x} = \mathbf{b}$ as the dot product of the i-th row of A with $\mathbf{x}$.

- Example:

see above

The Identity Matrix

- Compute

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1r + 0s + 0t \\ 0r + 1s + 0t \\ 0r + 0s + 1t \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$= r \underset{i = e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + s \underset{j = e}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + t \underset{k = e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

- The matrix

$$I_n = I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad n \times n$$

is called the $(n \times n)$ identity matrix. It acts like the number 1, i.e., for all $\mathbf{x}$ in $\mathbb{R}^n$,

$$I\mathbf{x} = \mathbf{x}.$$

Algebraic Properties of the Matrix-Vector Product

- Suppose A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar. Then it can be verified straight from the definition that

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

$u_1 a_1 + v_1 a_1 + u_2 a_2 + v_2 a_2 + \dots$

$u_1 a_1 + u_2 a_2 + \dots + v_1 a_1 + v_2 a_2$

Linearity

- Let $\mathbf{f}$ be a function whose domain is $\mathbb{R}^n$ and whose range is (a subset of $\mathbb{R}^m$), i.e.,

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

where $\mathbf{y}$ is in $\mathbb{R}^m$ and $\mathbf{x}$ is in $\mathbb{R}^n$.

- We encountered functions like this in Math 2210.
- The function $\mathbf{f}$ is said to be **linear** if

$$\mathbf{f}(\mathbf{u}+\mathbf{v}) = \mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v}) \quad \text{and} \quad \mathbf{f}(c\mathbf{u}) = c\mathbf{f}(\mathbf{u}),$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and scalars c.

the properties a. and b. above say that the function

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

is linear!