Math 2270-1

Matrix Multiplication

- The composition $f \circ g$ of two linear functions f and g is linear, and its matrix is the product of the matrices of the constituent functions.

$$\begin{array}{c} f \circ g \\ \\ \mathbb{R}^p \; \underset{B}{\overset{g}{\longrightarrow}} \; \mathbb{R}^n \; \underset{A}{\overset{f}{\longrightarrow}} \; \mathbb{R}^m \\ n \times p \qquad m \times n \\ C = AB \\ C \text{ is } m \times p \end{array}$$

- Note the switch in the sequence. B comes first in the diagram and second in the product, just like g comes first in the diagram and second in the composition.

Six Views of $C = AB$

- We'll look at six different ways of thinking about matrix multiplication. All of them are useful!

- For any matrix A let $\mathbf{r}_i(A)$ denote the i-th row of A, interpreted as a matrix with one row, and let $\mathbf{c}_i(A)$ denote the i-th column, interpreted as matrix with one column. We also identify 1×1 matrices with their single scalar entry.

- Let's illustrate the descriptions with the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix}.$$

1. The Formula. Here is what you might find in a textbook or mathematical dictionary: The product of an $m \times n$ matrix A and an $n \times p$ matrix B is an $m \times p$ matrix $C = AB$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, p.$$

- For our example,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times 1 + 2 \times 2 \\ 2 \times 3 + 1 \times 4 & 1 \times 3 + 2 \times 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix}. \end{aligned}$$

2. Writing it. We saw in class that it is advantageous to write the second factor to the upper right of the first factor. The product fits into the corner made by the two factors, and the $i-j$ entry of C sits at the intersection of the i-th row of A and the j-th column of B. In our example we get

$$\begin{array}{cc} & \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \\ \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} & \begin{bmatrix}4 & 5\\ 10 & 11\end{bmatrix}\end{array}$$

- More generally, we get:

$$
\begin{array}{cc}
 & B \quad n \times p \\
 & \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{i1} & \dots & b_{ij} & \dots & b_{ip} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix} \\
\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} & \begin{bmatrix} & & & & \\ & & \vdots & & \\ & \dots & c_{ij} & \dots & \\ & & \vdots & & \\ & & & & \end{bmatrix} \\
A \quad m \times n & C = AB \quad m \times p
\end{array}
$$

It is evident from this picture that

- the $i - j$ entry of C is the (dot) product of the i-th row of A and the j-th column of B,
- the j-th column of C is the product of A and the j-th column of B,

– the i-th row of C is the product of the i-th row of A and B.

- Following is an elaboration of these views. But first, here is a clean copy of the same picture:

$$\begin{array}{cc}
 & B \quad n \times p \\
 & \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{i1} & \dots & b_{ij} & \dots & b_{ip} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix} \\
\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} & \begin{bmatrix} & & & & \\ & & \vdots & & \\ & \dots & c_{ij} & \dots & \\ & & \vdots & & \\ & & & & \end{bmatrix} \\
A \quad m \times n & C = AB \quad m \times p
\end{array}$$

3. The entry by entry view.

$$c_{ij} = \mathbf{r}_i(A)\mathbf{c}_j(B).$$

In our example

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix}. \end{aligned}$$

4. The Column View. This is actually how we first derived our formula for matrix multiplication: the j-th column of C equals A multiplied with the j-column of B. As a formula:

$$\mathbf{c}_j(C) = A\mathbf{c}_j(B), \qquad j = 1, \ldots, p.$$

In our example:

$$\begin{aligned} C &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} A \begin{bmatrix} 2 \\ 1 \end{bmatrix} & A \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix}. \end{aligned}$$

- Note that in this view every column of the product is a linear combination of the columns of A. The coefficients of the linear combination are in the corresponding column of B.

5. The Row View. The i-th row of C is the i-th row of A multiplied with B:

$$r_i(C) = r_i(A)B.$$

In our example:

$$\begin{aligned} C &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} [1 \quad 2]\, B \\ [3 \quad 4]\, B \end{bmatrix} \\ &= \begin{bmatrix} [1 \quad 2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ [3 \quad 4] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \end{aligned}$$

- Note that in this view every row of the product is a linear combination of the rows of B. The coefficients of the linear combination are in the corresponding row of A.

6. The matrix view. Note that the product of the k-th column of A and the k-th row of B is an $m \times p$ matrix, the product of an $m \times 1$ matrix and a $1 \times p$ matrix. The $i - j$ entry of $\mathbf{c}_k(A)\mathbf{r}_k(B)$ is $a_{ik}b_{kj}$. So we get, by our formula

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

that

$$C = \sum_{k=1}^{n} \mathbf{c}_k(A)\mathbf{r}_k(B).$$

In our example

$$\begin{aligned} C &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} \end{aligned}$$

in general, the product of an $m \times 1$ matrix A and a $1 \times p$ matrix B is an $m \times p$ matrix C which has rank 1. Every row of C is a multiple of A and every column of C is a multiple of B.