Math 2270-1

Notes of 08/28/2019

1.5 Solution Sets of Linear Systems

- first: review of yesterday
- Suppose A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar. Then it can be verified straight from the definition that

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Linearity

- Let $\mathbf{f}$ be a function whose domain is $\mathbb{R}^n$ and whose range is (a subset of $\mathbb{R}^m$), i.e.,

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

  where $\mathbf{y}$ is in $\mathbb{R}^m$ and $\mathbf{x}$ is in $\mathbb{R}^n$.

- We encountered functions like this in Math 2210.
- The function $\mathbf{f}$ is said to be **linear** if

$$\mathbf{f}(\mathbf{u}+\mathbf{v}) = \mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v}) \quad \text{and} \quad \mathbf{f}(c\mathbf{u}) = c\mathbf{f}(\mathbf{u}),$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and scalars c.

the properties a. and b. above say that the function

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

is linear!

Major Principle

- Before we loose sight of the forest for all the trees: this section describes the solution set of linear systems

$$A\mathbf{x} = \mathbf{b}.$$

- That description is an example for one of the most important principles in mathematics:

Principle: The solution set of any linear problem is any particular solution of that problem, plus the general solution of the homogeneous version of that problem.

- More specifically, suppose $\mathbf{p}$ is a solution of the equation

$$A\mathbf{x} = \mathbf{b}.$$

Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors $\mathbf{w}$ of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h$$

where $\mathbf{v}_h$ is any solution of the homogeneous problem

$$A\mathbf{x} = \mathbf{0}.$$

- To see this suppose we have two solutions of our non-homogeneous problem:

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad A\mathbf{y} = b$$

  Then

$$\mathbf{v}_h = \mathbf{x} - \mathbf{y}$$

  is a solution of the homogeneous system:

$$A\mathbf{v}_h = A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

- So $\mathbf{x}$ can be obtained from $\mathbf{y}$ by adding $\mathbf{v}_h$, a solution of the homogeneous system.
- Also observe that if $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{0}$ then $\mathbf{x} + \mathbf{v}$ also solves the linear system:

$$A(\mathbf{x} + \mathbf{v}) = A\mathbf{x} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Note that we did not actually use the fact that $A\mathbf{x}$ means multiplication of a matrix and a vector. All that matters is linearity of $A\mathbf{x}$.

- The principle stated in bold face above applies to all linear problems, including, for example, ordinary and partial differential equations, difference equations, and integral equations.
- It's OK not to know fully what those things are, but you should appreciate the power of the above **Principle**.