Math 2270-1

Notes of 11/25/2019

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Announcements

- hw 14 opens today. It's mostly a review, consisting of old T/F questions.
- this week, discuss parts of chpt 7
- next week, review
- Final Exam, Thursday, 12/12 8:00-10:00 am, LCB 219
- Additional and optional Q&A session, Wednesday, 12/11, 10:30-12:30, LCB 215. HW 14 will close December 4.

Chapter 7: Symmetric Matrices

- The main result today is that a (real square) matrix has an **orthogonal similarity transform to diagonal form** if and only if it is symmetric. Getting there will be a little technical!

- Throughout this chapter let A be a square real matrix. Unless stated otherwise, the number of rows and columns is n.

- A square matrix A is symmetric if

$$A = A^T.$$

  This means that

$$a_{ij} = a_{ji}$$

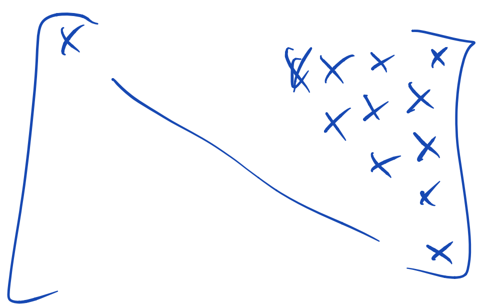
- The entries along the diagonal are arbitrary, but those off the diagonal occur in pairs.

- The concept of symmetry only applies to square matrices.

- Examples

$$\begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$



$$\begin{bmatrix} 0 & \times \\ \times & 0 \end{bmatrix}$$

- The set of symmetric $n \times n$ matrices forms a linear space. What is its dimension?

$$\frac{n(n+1)}{2}$$

A antisymmetric if $-A^T = A$ $\qquad \frac{n(n-1)}{2}$

Review of Eigenvalues and Similarity

- A non-zero vector $\mathbf{x}$ is an **eigenvector** of A corresponding to the eigenvalue λ if

$$A\mathbf{x} = \lambda\mathbf{x}.$$

- An eigenvector is determined only up to a non-zero factor.
- The eigenvalues of A are the roots of the characteristic equation

$$\det(A - \lambda I) = 0.$$

- A has precisely n eigenvalues, properly counting multiplicity.
- The eigenvalues of A may be complex. If there are any complex eigenvalues they occur in conjugate complex pairs.
- The eigenvalues of a symmetric matrix are real. (see notes of 10/25/19).
- For any symmetric matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal.

Similarity

- Two matrices A and B are similar if there is a non-singular matrix P such that

$$B = P^{-1}AP.$$

- P is sometimes referred to as a **similarity transform**.
- Similar matrices have the same eigenvalues and the eigenvectors of B are of the form $P^{-1}\mathbf{x}$ where $\mathbf{x}$ is an eigenvector of A:

$$B(P^{-1}\mathbf{x}) = (P^{-1}AP)P^{-1}\mathbf{x} = P^{-1}A\mathbf{x} = \lambda(P^{-1}\mathbf{x}).$$

- A matrix A is **diagonalizable** if it is similar to a diagonal (but not necessarily real) matrix.
- A is diagonalizable if and only if it has a linearly independent set of n eigenvectors.
- Suppose there is such a set, satisfying

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \qquad i = 1, \ldots, n. \tag{1}$$

- Then let

$$P = [\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \ldots \mathbf{p}_n\,] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

- Equation (1) is the column version of the matrix equation

$$AP = PD$$

- Hence

$$D = P^{-1}AP$$

and so D is similar to A and the similarity transform P is the matrix of eigenvectors.

- this suggests the following procedure for diagonalizing a matrix:

1. Find n linearly independent eigenvectors.
2. Collect those vectors into the matrix P.
3. compute P^{-1}
4. Then

$$D = P^{-1}AP$$

is a diagonal matrix with the eigenvalues along the diagonal.

- **Example 3**, textbook. The (symmetric) matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has the characteristic equation

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2)$$

Thus -2 is a single eigenvalue and 7 is a double eigenvalue.

- The standard technique of solving the linear system
$$A\mathbf{x} = \lambda x$$
gives the eigenvectors
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$
for $\lambda = 7$ and
$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$
for $\lambda = -2$

- These eigenvectors are linearly independent. Collecting them into matrix
$$P = \begin{bmatrix} 1 & -1/2 & -1 \\ 0 & 1 & -1/2 \\ 1 & 0 & 1 \end{bmatrix}$$
gives the similarity transform
$$P^{-1}AP = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- We saw in chapter 6 that orthogonal matrices are particularly nice.

- Two matrices A and B are **orthogonally similar** if there is an orthogonal matrix P such that
$$B = P^{-1}AP = P^T AP.$$

- Suppose A has an orthonormal basis of n eigenvectors.

- Then we can collect those into an orthogonal matrix P where
$$P^{-1} = P^T$$

- We get $D = P^T DP$ and therefore
$$A = PDP^T.$$

- The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in Example 3 are linearly independent, but not orthogonal. However, they span a space of dimension 2, the eigenspace associated with the eigenvector 7, and we can c construct an orthogonal basis of that space by the Gram-Schmidt Process. We can also normalize $\mathbf{v}_3$ to be a unit vector. This gives the modified orthogonal similarity transform
$$Q = \begin{bmatrix} 1/\sqrt{2} & \frac{-1}{\sqrt{18}} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$$
with
$$Q^T AQ = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Returning to the general orthogonal similarity transform, note that the matrix

$$PAP^{-1} = A = PDP^T$$

$$P^{-1} = P^T$$

  is symmetric. Thus we see that **a matrix that is orthogonally diagonalizable** is symmetric.

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

- The converse is also true: Every symmetric (real) matrix is **orthogonally diagonalizable.**

- We get **Theorem 2.** A matrix is orthogonally diagonalizable if and only if it is symmetric.

- We saw the "only if" part, the "if" part is trickier. The textbook gives more information, referring to a bunch of exercises, but it does not have a proof.

- However, here is a proof by induction.

- Induction: We show than a statement is true for $n = 1$. Then we show that it is true for n if it is true for $n-1$, for all $n = 1, 2, 3, \ldots$.

- There is nothing to show for $n = 1$. A is symmetric and P is the 1×1 identity.

- So suppose that any symmetric $(n-1)\times(n-1)$ matrix A_1 does have an orthogonal similarity transform to diagonal form:

$$P_1^{-1} A_1 P_1 = D_1$$

  where P_1 is orthogonal, i.e.,

$$P_1^T P_1 = I_1$$

where I_1 is the $(n-1) \times (n-1)$ identity, and D_1 is $(n-1) \times (n-1)$ diagonal.

- Now suppose A is symmetric and $n \times n$. Then it has a real eigenvalue λ_1 and corresponding real eigenvector $\mathbf{v}_1$, i.e.,

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1.$$

- Since eigenvectors are determined only up to a constant factor we may assume that

$$\|\mathbf{v}_1\| = 1.$$

- We construct an orthogonal $n \times n$ matrix

$$\bar{P} = [\, \mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n \,]$$

- This is always possible. It can be accomplished, for example, by rotating the standard coordinate system so that the first standard basis vector $\mathbf{e}_1$ lines up with $\mathbf{v}_1$, or we could add vectors to $\{\mathbf{v}_1\}$ to get a basis of R^n, and then apply the Gram-Schmidt process.

- Then

$$A\bar{P} = [\, \lambda_1 \mathbf{v}_1 \quad A\mathbf{v}_2 \quad \dots \quad A\mathbf{v}_n \,]$$

and

$$\begin{aligned} \bar{P}^T A \bar{P} &= [\, \lambda_1 \bar{P}^T \mathbf{v}_1 \quad \bar{P}^T A\mathbf{v}_2 \quad \dots \quad \bar{P}^T A\mathbf{v}_n \,] \\ &= [\, \lambda_1 \mathbf{e}_1 \quad \bar{P}^T A\mathbf{v}_2 \quad \dots \quad \bar{P}^T A\mathbf{v}_n \,] \end{aligned}$$

since $\bar{P}$ is orthogonal.

- Moreover, $\bar{P}^T A\bar{P}$ is symmetric since A is symmetric:

$$(\bar{P}^T A\bar{P})^T = \bar{P}^T A^T \bar{P} = \bar{P}^T A\bar{P}.$$

- Thus $\bar{P}^T A\bar{P}$ has the block structure

$$\tilde{A} = \bar{P}^T A\bar{P} = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & A_1 \end{bmatrix}$$

where $\mathbf{0}$ is the zero vector in $\mathbb{R}^{n-1}$ and A_1 is $(n-1)\times(n-1)$ and symmetric.

- By our induction hypothesis A_1 has an orthogonal similarity transform to diagonal form:

$$D_1 = P_1^T A_1 P_1.$$

- We now apply a standard trick and embed the matrix A_1 in the $n \times n$ identity $\begin{bmatrix} 1 & 0^T \\ 0 & P_1^T \end{bmatrix}$ matrix. This gives

$$\tilde{P} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & P_1 P_1^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

- Note that $\tilde{P}$ is an orthogonal matrix!

- Now observe that

$$\begin{aligned} \tilde{P}^T \tilde{A}\tilde{P} &= \tilde{P}^T \bar{P}^T A\bar{P}\tilde{P} \\ &= (\bar{P}\tilde{P})^T A(\bar{P}\tilde{P}) \\ &= \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & D_1 \end{bmatrix} \end{aligned}. \qquad (2)$$

- This may perhaps be seen more clearly by multiplying the matrices involved in the form we use for multiplication:

$$
\begin{array}{rcccl}
 & & & \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P_1 \end{bmatrix} & = \tilde{P} \\
\tilde{A} & = & \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & A_1 \end{bmatrix} & \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & A_1 P_1 \end{bmatrix} & = A\tilde{P} \\
\tilde{P}^T & = & \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & P_1^T \end{bmatrix} & \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & P_1^T A_1 P_1 \end{bmatrix} & = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & D_1 \end{bmatrix}
\end{array}
$$

- We define

$$P = \bar{P}\tilde{P}.$$

P is the product of two orthogonal matrices and hence orthogonal:

$$P^T P = (\bar{P}\tilde{P})^T (\bar{P}\tilde{P}) = \tilde{P}^T \bar{P}^T \bar{P}\tilde{P} = \tilde{P}^T \bar{P} = I.$$

- The matrix

$$D = \begin{bmatrix} \lambda_1 & \mathbf{0}^T \\ \mathbf{0} & D_1 \end{bmatrix}$$

is diagonal, and so we get the required orthogonal transform

$$P^T A P = D.$$

- **End of Proof!**

- Recall that a matrix may be defective, i.e., it may not have a complete set of n linearly independent eigenvectors.
- However, the existence of an orthogonal similarity transform implies that symmetric matrices are never defective.
- More precisely:

Spectral Theorem for Symmetric Matrices An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace corresponding to λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to distinct eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

Put a little more simply: As far as the eigenvalue problem, **symmetric matrices** are as nice as can be.

We don't know enough yet to appreciate it, but the corresponding statement for linear system is that **orthogonal matrices** are as nice as can be.