# Math 2270-1 — Fall 2019 — Exam 2 Answers

**Note that this answer set contains (quite a bit) more information than you needed to provide on the exam.**

**-1- (Matrix Multiplication.)** In one sentence, state why we multiply matrices the way we do.

## Discussion:

Here is one answer:

*Matrices represent linear functions, and the product of two matrices is the matrix representing the composition of the corresponding functions.*

Of course, many variations of that sentence are possible. For example, you could state more concisely

*Matrix Multiplication equals Function Composition*

The key word in any answer is "composition".

**-2- (Matrix Multiplication.)** Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Compute $AA^T$ and $A^TA$.

## Discussion:

Writing our matrices as usual we get

$$\begin{array}{cc} & \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 3 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 2 & 14 \end{bmatrix} \end{array} = AA^T \quad \text{and} \quad \begin{array}{cc} & \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 4 \\ 1 & 5 & 5 \\ 4 & 5 & 10 \end{bmatrix} \end{array} = A^TA$$

**-3- (Block Matrices.)** Suppose $A$ is the invertible upper triangular block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}$$

where $A_{11}$ is $p \times p$ and $A_{22}$ is $q \times q$. (Of course, $A_{12}$ is $p \times q$ and $\mathbf{0}$ is the $q \times p$ zero matrix.) Compute the inverse block matrix

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $B_{11}$ is $p \times p$, $B_{12}$ is $p \times q$, $B_{21}$ is $q \times p$, and $B_{22}$ is $q \times q$. Express the blocks of $A^{-1}$ in terms of the blocks of $A$.

## Discussion:

Note that our assumption that $A$ is invertible implies that $A_{11}$ and $A_{22}$ are invertible (why?). We want

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = I_{p+q} = \begin{bmatrix} I_p & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}$$

We thus get the 4 equations

$$A_{11}B_{11} + A_{12}B_{21} = I_p, \tag{1}$$

$$A_{11}B_{12} + A_{12}B_{22} = \mathbf{0} \tag{2}$$

$$A_{22}B_{21} = \mathbf{0} \tag{3}$$

$$A_{22}B_{22} = I_q \tag{4}$$

You want to check that the matrix sizes conform in each case. Also think about the sizes of those zero matrices. Equation (4) gives

$$B_{22} = A_{22}^{-1}.$$

Equation (3) gives

$$B_{21} = A_{22}^{-1}\mathbf{0} = \mathbf{0}.$$

This simplifies equation (1) which turns into

$$A_{11}B_{11} = I_p, \qquad \text{i.e.,} \qquad B_{11} = A_{11}^{-1}.$$

Equation (2) becomes

$$A_{11}B_{12} + A_{12}A_{22}^{-1} = 0$$

which implies that

$$B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}.$$

We get

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ \mathbf{0} & A_{22}^{-1} \end{bmatrix}$$

Again, we check by multiplying $A$ and its proposed inverse:

$$\begin{array}{cc} & \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ \mathbf{0} & A_{22}^{-1} \end{bmatrix} \\ \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix} & \begin{bmatrix} A_{11}A_{11}^{-1} & -A_{11}A_{11}^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ A_{22}\mathbf{0} & A_{22}A_{22}^{-1} \end{bmatrix} \end{array} = \begin{bmatrix} I_p & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix} = I_{p+q}.$$

See the notes of 9/17 for an alternative discussion.

**-4-** (**$LU$-factorization.**) Compute the $LU$-factorization of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -3 \end{bmatrix}$$

## Discussion:

We saw in class that computing the $LU$ factorization is equivalent to applying Gaussian elimination and reducing the matrix to row echelon form. Proceeding as we did in class, and storing the multipliers below the diagonal (and boxing them), we get:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ \boxed{2} & 1 & 2 \\ \boxed{-2} & 1 & -1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ \boxed{2} & 1 & 2 \\ \boxed{-2} & \boxed{1} & -3 \end{bmatrix}$$

Thus

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Of course we check by multiplying that we actually get $A = LU$:

$$\begin{array}{ccc} & \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} & = U \\ L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -3 \end{bmatrix} & = A = LU \end{array}$$

**-5- (Inverse Matrix.)** Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

## Discussion:

In class we derived the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In our example this gives

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \times 4 - 2 \times 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}.$$

Of course, you may not have remembered that formula. However, $A^{-1}$ can also be computed directly as we discussed in class, by augmenting $A$ with the identity matrix, and reducing the augmented matrix to reduced row echelon form. This gives

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix} \end{aligned}$$

This gives

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

which is the same answer as we obtained before. Of course, we check our result by multiplication of $A$ and its proposed inverse:

$$\begin{array}{cccc} & & \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} & = A^{-1} \\ A = & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & = I \end{array}$$

**-6- (Cramer's Rule.)** State and Prove Cramer's Rule.

## Discussion:

Cramer's Rule states that for the square linear system

$$A\mathbf{x} = \mathbf{b}$$

the $i$-th entry $x_i$ of $\mathbf{x}$ is given by

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained from $A$ by replacing the $i$-th column with $\mathbf{b}$. To see that this is true let $I_i(\mathbf{x})$ be the matrix obtained from the identity matrix by replacing the $i$-th column with $\mathbf{x}$. Thus

$$I_i(\mathbf{x}) = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_{i-1} \quad \mathbf{x} \quad \mathbf{e}_{i+1} \quad \ldots \quad \mathbf{e}_n\,]$$

Then, by the way we defined matrix multiplication,

$$AI_i(\mathbf{x}) = A_i(\mathbf{b}).$$

The determinant of the product equals the product of the determinants:

$$|A||I_i(\mathbf{x})| = |A_i(\mathbf{b})|, \quad \text{i.e.,} \quad |I_i(\mathbf{x})| = \frac{|A_i(\mathbf{b})|}{|A|}.$$

Moreover, by expanding about the $i$-th row we see that

$$|I_i(\mathbf{x})| = x_i.$$

Cramer's rule follows.

**-7- (Determinants.)** For which value of $t$ does

$$\det \begin{bmatrix} 1 & 2 & t \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} = 0?$$

## Discussion:

Expanding the determinant by cofactors about the third column gives

$$\det \begin{bmatrix} 1 & 2 & t \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} = t \det \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} - 5 \det \begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix} = -3t + 25 = 0.$$

This gives

$$t = \frac{25}{3}.$$

**-8- (True or False.)** Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers.

**1. T F** To form the matrix product $AB$ the matrix $B$ has to have as many columns as $A$ has rows.

$\boxed{\textbf{F}}$ $B$ has to have as many rows as $A$ has columns.

**2. T F** If $A$ and $B$ are matrices that are not square, and the product $AB$ can be formed, then the product BA cannot be formed.

$\boxed{\textbf{F}}$ See problem 1 of this exam.

**3. T F** Suppose $A$ is a square matrix. It is invertible if and only if there is no right hand side $\mathbf{b}$ for which the linear system $A\mathbf{x} = \mathbf{b}$ has several solutions.

$\boxed{\textbf{T}}$ This is item f. of the invertibility theorem in section 2.3 of the textbook.

**4. T F** The square matrix $A$ is invertible if and only if $A^T$ is invertible.

$\boxed{\textbf{T}}$ In fact, $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

**5. T F** Suppose $A$ and $B$ are both square matrices. Then if $AB$ is invertible, so is $B$.

$\boxed{\textbf{T}}$ If $B$ was not invertible there would be a vector $\mathbf{x} \neq 0$ such that $B\mathbf{x} = \mathbf{0}$. This would imply $AB\mathbf{x} = \mathbf{0}$, i.e., $AB$ is not invertible.

**6. T F** For two $n \times n$ matrices $A$ and $B$, the determinant of their product equals the product of their determinants.

$\boxed{\textbf{T}}$ We proved this in class using properties of elementary matrices.

**7. T F** For two $n \times n$ matrices $A$ and $B$, the determinant of their sum equals the sum of their determinants.

$\boxed{\textbf{F}}$ For example, if $A = B = I$, then $\det I = 1$ but

$$\det(I + I) = 2^n \neq 1 + 1 = \det I + \det I$$

(unless $n = 1$, of course.)

**8. T F** Suppose $c$ is a scalar, and $A$ is an $n \times n$ matrix. Then $\det(cA) = c \det A$.

$\boxed{\textbf{F}}$ Instead we have that $\det(cA) = c^n \det A$.

**9. T F** Suppose $T$ is a triangle formed by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ in $\mathbb{R}^2$. Then the area $a$ of $T$

is given by

$$a = |\det[\mathbf{u} \quad \mathbf{v}]|.$$

$\boxed{\textbf{F}}$ The absolute value of that determinant is **twice** the area of the triangle.

**10. T F** The inverse of an invertible matrix $A$ equals

$$A^{-1} = \frac{1}{\det A} C$$

where $C$ is the matrix of cofactors of $A$.

$\boxed{\textbf{F}}$ Actually,

$$A^{-1} = \frac{1}{\det A} C^T$$