Write your name here:

Math 2270-1 — Fall 2019 — Final Exam

1	2	3	4	5	6	7	8	9	10	11	12	13	Total

Instructions

1. **This exam is closed books and notes. Do not use a calculator or another electronic device. Do not use scratch paper.**
2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it's obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.
4. To avoid distraction and disruption **I am unable to answer questions during the exam.** If you believe there is something wrong with a problem state so, and if you are right you will receive generous credit. I will also be unable to discuss individual problems and grading issues with you after you are done while the exam is still in progress.
5. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to **check your answers.**
6. When you are done hand in your exam, pick up an answer sheet, and leave the room. Do not return to your seat.
7. All questions have equal weight.
8. Clearly indicate (for example, by circling or boxing) your final answers.

-1- (LU factorization.) Compute the LU factorization of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}.$$

-2- (Determinants.) For what value of t does

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & t \end{bmatrix} = 0?$$

-3- (Reduced Row Echelon Form.) Suppose the reduced row echelon form of a matrix A is

$$R = \begin{bmatrix} 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Compute the rank of A, the dimensions of its column and null spaces, and give a basis of the null space.

-4- (Inner Product Spaces and the Pythagorean Theorem.) Let V be an inner product space with an inner product $< \cdot, \cdot >$. List the defining properties of $< \cdot, \cdot >$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V. Define what we mean by $\|\mathbf{u}\|$. Also define what we mean when we say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Finally, prove the Pythagorean Theorem:

$$\|u + \mathbf{v}\|^2 = \|u\|^2 + \|\mathbf{v}\|^2 \quad \Longleftrightarrow \quad \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

-5- (Invertibility and Diagonalizability.) Give examples—as simple as possible—for matrices that are

a. invertible and diagonalizable:

b. invertible and non-diagonalizable:

c. singular and diagonalizable:

d. singular and non-diagonalizable:

-6- (Positive Definiteness.) Let

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 6 & 3 & 1 \\ 1 & 3 & 10 & 5 \\ 1 & 1 & 5 & 12 \end{bmatrix}$$

Show that A is positive definite.

-7- (Linear Transformation.) Suppose we write a cubic polynomial as

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

as usual. Find the 4×4 matrix M that maps the coefficient vector of p with respect to the basis $\{1, x, x^2, x^3\}$ onto the vector of the coefficients of the Taylor expansion about $x = 1$. In other words, find M such that

$$M \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1)/2 \\ p'''(1)/6 \end{bmatrix}$$

Note: I assume you are familiar with Taylor series, but for reference, the Taylor expansion of a function f about a point $x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \dots .$$

-8- (Linear Spaces.) Let S be the set of polynomials p of degree 2 that satisfy $p(0) = 0$. Show that S is a subspace of the space of quadratic polynomials, compute the dimension of S, and give a basis of S.

-9- (Orthogonal Matrices.) Give an example of an orthogonal matrix with complex (non-real) eigenvalues. Your example should be as simple as possible.

-10- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. T F A linear system may have exactly 2 solutions.

2. T F A linear system with fewer equations than unknowns always has at least one solution.

3. T F Every vector in a linearly dependent set can be written as a linear combination of the other vectors.

4. T F If a set of more than one vectors is linearly dependent then at least one of those vectors can be written as a linear combination of the others.

5. T F The matrix transformation $\mathbf{y} = A\mathbf{x}$ is one-to-one if and only if the columns of A are linearly independent.

6. T F Any vector in the span of a linearly dependent set can be written in more than one way as a linear combination of the given vectors.

7. T F The solution set of $A\mathbf{x} = \mathbf{0}$ is a linear space

8. T F If $\mathbf{u}$ and $\mathbf{v}$ are solutions of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ then $\mathbf{u} - \mathbf{v}$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

9. T F The homogeneous linear system $A\mathbf{x} = \mathbf{0}$ may be inconsistent.

10. T F The general solution of any consistent linear problem is any particular solution, plus the general solution of the associated homogeneous problem.

-11- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers. This question deals with matrix multiplication. Throughout assume that A and B are matrices.

1. T F The matrix product $C = AB$ can be formed only if A has as many rows as B.

2. T F If A and B are two non-square matrices it is impossible for AB and BA both to be defined.

3. T F The j-th column of AB equals the product of A and the j-th column of B.

4. T F The i-th row of AB equals the product of A and the i-th row of B.

5. T F The product AB of a non-zero $m \times 1$ matrix A and a non-zero $1 \times n$ matrix B is an $m \times n$ matrix of rank 1.

6. T F Every column of AB is in the column space of A.

7. T F Every row of AB is in the row space of A.

8. T F Assuming A is $m \times p$ and B is $p \times n$ then $C = AB$ is $m \times n$ and

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \qquad i = 1, \ldots m, \quad j = 1, \ldots, n.$$

9. T F If A and B are both $n \times n$ then $AB = BA$.

10. T F Assuming the $m \times p$ matrix A is the standard matrix of a function $f : \mathbb{R}^p \longrightarrow \mathbb{R}^m$, and the $p \times n$ matrix B is the standard matrix of a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, the product AB is the standard matrix of the function $g \circ f$.

-12- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. T F The determinant of an $n \times n$ matrix A equals the product of its eigenvalues.

2. T F A symmetric matrix A is positive definite if and only if all its eigenvalues are positive.

3. T F The eigenvalues of a triangular matrix are its diagonal entries.

4. T F Multiplying a rank 1 matrix A with a non-singular matrix B gives a rank 1 matrix $C = AB$

5. T F Suppose A is an $m \times n$ matrix. Then AA^T and A^TA are both symmetric and positive semidefinite.

6. T F A square matrix A is invertible if and only if A^T is invertible.

7. T F The processes of inverting and transposing a matrix commute.

8. T F If A and B are invertible $n \times n$ matrices then $(AB)^{-1} = A^{-1}B^{-1}$.

9. T F Suppose A is an $n \times n$ matrix and k is a scalar. Then $\det(kA) = k \det A$.

10. T F Interchanging two columns of a matrix does not change its determinant.

-13- (True or False.) Mark the following statements as true or false by circling **T** or **F**, respectively. You need not give reasons for your answers.

1. T F Suppose A, B, and P are $n \times n$ matrices and

$$B = P^{-1}AP.$$

The A and B have the same eigenvalues.

2. T F The eigenvectors of a triangular matrix are the standard basis vectors.

3. T F Suppose $\mathbf{a}$ is orthogonal to $\mathbf{b}$ and $\mathbf{b}$ is orthogonal to $\mathbf{c}$. Then $\mathbf{a}$ is orthogonal to $\mathbf{c}$.

4. T F The normal equations for the Least Squares problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min$$

are always consistent.

5. T F The zero vector in $\mathbb{R}^n$ is orthogonal to all vectors in $\mathbb{R}^n$.

6. T F A singular matrix may be orthogonal.

7. T F The matrix Σ in the singular value decomposition of an invertible square matrix is positive definite.

8. T F The singular values of a negative definite matrix are negative.

9. T F Suppose f is a scalar valued function of several variables. Then it assumes a minimum at a point if the gradient at that point is zero and the matrix of second derivatives at that point is positive definite.

10. T F Linear Algebra is cool.

I enjoyed meeting you, and it's been great fun teaching this class. I hope you found it a worthwhile experience. Best wishes to you, and perhaps I'll meet you again in some future class!

Peter Alfeld