Math 2270-1

Notes of 11/1/19

6.2 Orthogonal Sets

- Recall: two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if
$$\mathbf{u} \bullet \mathbf{v} = 0.$$

- A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ from $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors from that set is orthogonal, i.e.,
$$i \neq j \quad \Longrightarrow \quad \mathbf{u}_i \bullet \mathbf{u}_j = 0.$$

- Examples:
  - The standard basis of $\mathbb{R}^n$.
  - The set $\{\mathbf{u}, \mathbf{0}\}$.
  - Example 1, textbook, the set
$$S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

- **Theorem 4**, p. 340, textbook. If

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$$

is an orthogonal set of **nonzero** vectors in $\mathbb{R}^n$, then S is linearly independent. (Hence S is a basis of span(S).)

- Naturally, an **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
- For example, the set in Example 1 is an orthogonal basis of $\mathbb{R}^3$.
- Orthogonal Bases are nice! You can compute coefficients without solving a linear system.
- Suppose

$$\mathrm{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$$

is a basis of a subspace W of $\mathbb{R}^n$,

$$B = [\,\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\,],$$

and $\mathbf{y}$ is a vector in W. Then, in general, computing the coordinate vector

$$[\mathbf{y}]_\mathrm{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

of $\mathbf{y}$ requires the solution of the linear system

$$B[\mathbf{y}]_\mathrm{B} = \mathbf{y}.$$

- However, if B is an orthogonal basis we can compute the components of $[\mathbf{y}]_\mathrm{B}$ directly:

$$c_j = \frac{\mathbf{y} \bullet \mathbf{u}_j}{\mathbf{u}_j \bullet \mathbf{u}_j}.$$

- Example 2, p. 341. Express the vector

$$\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

as a linear combination of the vectors in the set

$$S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

in Example 1.

Orthogonal Projections onto a Line

- Again, this is a review and generalization from Math 2210.
- Given a non-zero vector $\mathbf{u}$ in $\mathbb{R}^n$ we wish to write $\mathbf{y}$ in $\mathbb{R}^n$ as a multiple of $\mathbf{u}$ and a vector orthogonal to $\mathbf{u}$.
- That is we wish to write

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \alpha\mathbf{u} + \mathbf{z}$$

  where

$$\mathbf{z} \bullet \mathbf{u} = 0.$$

- We want formulas for α and $\mathbf{z}$. They are easy to obtain.

- Example 3, pg. 342, textbook. Let

$$\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

  Write $\mathbf{y}$ as a linear combination of a vector in Span$\{\mathbf{u}\}$ and a vector that is orthogonal to $\mathbf{u}$.

Figure 1. Example 2.

- An orthogonal set is called an **orthonormal set** is all of its vectors are unit vectors.

- Example: The standard basis, and any (nonempty) subset of it.

- **Theorem 6**, p. 345. An $m \times n$ matrix U has orthonormal columns if and only if

$$U^T U = I$$

(where I is the $n \times n$ identity matrix.).

- **Theorem 7**, p. 345. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and y be vectors in $\mathbb{R}^n$. Then:

  a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

  b. $(U\mathbf{x}) \bullet (U\mathbf{y}) = \mathbf{x} \bullet \mathbf{y}$

  c. $(U\mathbf{x}) \bullet (U\mathbf{y}) = 0 \quad$ if and only if $\quad \mathbf{x} \bullet y = 0$

The Pythagorean Theorem

- Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

- More precisely, we should say that

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad \Longleftrightarrow \quad \mathbf{u} \bullet \mathbf{v} = 0.$$