Math 2270-1

Notes of 9/20/2019

Determinant of a 2×2 Matrix

- Suppose
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- We want to compute
$$A^{-1}.$$

- To do this we augment A by the identity matrix and use row operations to reduce A to reduced row echelon form:

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \longrightarrow \begin{matrix} \mathbf{r}_1 \longrightarrow \mathbf{r}_1/a \\ \mathbf{r}_2 \longrightarrow \mathbf{r}_2/c \end{matrix} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 1 & \frac{d}{c} & 0 & \frac{1}{c} \end{pmatrix}$$

$$\longrightarrow \quad \mathbf{r}_2 \longrightarrow \mathbf{r}_2 - \mathbf{r}_1 \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{d}{c} - \frac{b}{a} & -\frac{1}{a} & \frac{1}{c} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{ac} & -\frac{1}{a} & \frac{1}{c} \end{pmatrix}$$

$$\longrightarrow \quad \mathbf{r}_2 \longrightarrow \tfrac{ac}{ad-bc}\mathbf{r}_2 \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\longrightarrow \quad \mathbf{r}_1 \longrightarrow \mathbf{r}_1 - \tfrac{b}{a}\mathbf{r}_2 \begin{pmatrix} 1 & 0 & \frac{1}{a} - \frac{b}{a} \times \frac{c}{ad-bc} & \frac{-b}{ad-ac} \\ 0 & 1 & \frac{c}{ad-bc} & -\frac{a}{ad-bc} \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} 1 & 0 & d & -b \\ 0 & 1 & -c & a \end{bmatrix}$$

- Thus

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{and} \quad \det A = ad - bc.$$

3.1 Determinants

- This is a very short chapter (3 sections).
- Determinants are numbers associated with **square matrices**.
- Every square matrix has a determinant, only square matrices have a determinant.
- For a square matrix A that number is usually written as $\det A$ or $|A|$. Note that in this context the vertical bars do not mean absolute values!
- We'll study determinants for two major reasons:

1. A (square) matrix is invertible if and only if its determinant is non-zero.
2. Determinants will be essential in studying **eigenvalues** in chapter 5.

- A very different discussion of determinants is in **Gilbert Strang, Introduction to Linear Algebra, 5th ed., Wellesley-Cambridge Press, 2016, ISBN-13: 978-0980232776 ISBN-10: 0980232775.**
- Let me also remind me of his highly popular video lectures on

`https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/`.

Determinants are introduced in Lecture 18.

- There is also a 764 page monograph: Thomas Muir, A Treatise on the Theory of Determinants", revised and enlarged by William Metzler, Dover, 1933.

- We'll follow the textbook.

- Recall that we found the inverse of a 2×2 matrix to be

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

- The number $a_{11}a_{22} - a_{12}a_{21}$ is defined to be the determinant of A.

- Clearly, a 2×2 matrix$A$ is invertible if and only if its determinant is zero.

- What happens for a 3×3 matrix?
- **Exercise:** Show (by using row operations) that the matrix

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  is invertible if and only if

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0.$$

- We call the expression D in that equation the determinant of A_3:

$$D = \det A_3$$

- Note the structure of A_3:
  - It's the sum of 6 products, with a plus or minus sign. Each term is obtained by picking exactly one factor from each row and column.
  - Moreover, D can be rewritten in several ways, including:

$$\begin{aligned} D &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= -a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} - a_{32} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} + a_{33} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{aligned}$$

- This illustrates (but of course does not prove) the general pattern.
- In general, for $n > 2$ the determinant of A is defined recursively. Suppose A is an $n \times n$ matrix where $n > 1$. We define A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and the j-th column.
- Example:

- With this notation, the textbook defines the determinant as follows: For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [\, a_{ij} \,]$ is

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned} . \tag{1}$$

- Examples

A major fact about determinants is that they can be computed using any row or column.

- The corresponding formula is usually expressed in terms of **cofactors**:

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

- The factor

$$(-1)^{i+j}$$

creates the familiar checkerboard pattern

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Theorem 1 in the textbook gives the more general formulas

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad (2)$$

for any choice of i and

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad (3)$$

for any choice of j.

- Of course knowing nothing of determinants it is not at all obvious that these formulas should give the same number for all choices of i and j, and all (square) matrices A.

- quote from the textbook: “We omit the proof of [this] fundamental theorem to avoid a lengthy digression.”

- By comparison, Strang starts out by defining the determinant to be the uniquely defined function associating a number with any square matrix that has these properties:

1. The determinant of the identity matrix is 1.

2. If you interchange two rows of the matrix the determinant changes it sign.

3. The determinant is a function that is linear in each row separately. (In other words, if you think of the determinant as a function of a specific row, keeping everything else constant you get a linear function.)

- Everything, including the formulas listed in Theorem 1 flow from there, starting with the proof that there is a unique function with these properties.

- In the textbook, we start with formula (1), state Theorem 1 without proof, and then go from there, deriving in particular the three properties listed above.

- We'll go along with that ...

- When using the cofactor formula we want to take advantage of zero entries.
- For example, compute the determinant of

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

- Theorem 2: The determinant of a triangular matrix equals the product of the diagonal entries.
- This follows easily from the cofactor expansion. We'll just look at a simple example that will illustrate the pattern. Let x denote arbitrary entries. Compute the determinant of

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ x & b & 0 & 0 \\ x & x & c & 0 \\ x & x & x & d \end{bmatrix}$$

- The cofactor expansion becomes very easy if there are many zero entries that are distributed in a beneficial pattern.

However, the cofactor expansion is prohibitively expensive in terms of computational effort if we cannot make use of the presence of zero entries.

- Assuming all entries are non-zero, computing the determinant of an $n \times n$ matrix requires the computation of

  $$n \times (n-1) \times (n-2) \times 3 \times 2 \times 1 = n!$$

  products.

- wikipedia has me acknowledge the source of the picture: Argonne National Laboratory's Flickr page.
- Currently[1], the worlds fastest supercomputer (see Figure 1) can perform about 100×10^{20} (a hundred quadrillion[2]) multiplications per second.
- How long does it take to compute the determinant of an $n \times n$ matrix by the cofactor expansion, on that computer?

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[1] see `https://en.wikipedia.org/wiki/Supercomputer`

[2] That's about 10 billion times as fast as a fast desktop computer, see `https://www.popsci.com/intel-teraflop-chip`

Figure 1. A fast computer.

- The following table lists the time T required to compute the determinant by the cofactor formula for four values of n.

n	T
20	0.00024 seconds
25	26 minutes
30	840 years
35	32 billion years

By comparison, the age of the Universe is estimated to be 14 billion years. Computing the determinant of a 35×35 matrix would therefore take more than twice as long as the Universe has existed.

a 35×35 matrix is small by today's standards.

On the other hand, computing the determinant of a 1000×1000 matrix takes less than a second in Matlab.

```
>> A = rand(1000,1000);
>> [Q,R] = qr(A);
>> t = cputime;
>> d = det(Q)

d =

   -1.0000

>> cputime-t

ans =

    0.5500

>>
```

- Evidently Matlab does not use the cofactor expansion. There must be a better way!