Math 2270-1

Notes of 10/21/2019

5.1 Eigenvalues and Eigenvectors

- We'll start with an example. Suppose we have a population of individuals (think mice, for example) with 4 cohorts, those 0, 1, 2, 3 units of time (years) old. Every year all those 3 years old die. Of the other three cohorts, half survive. The population reproduces. Cohort 2 (1 year old) produces 2/3 of their number in offspring, cohort 3 produce 4/3 their number, and cohort 4 produce 8/3 their number. Is there a population that is stable, which means that the total number of individuals and the age distribution do not change over time.
- Suppose the population is given by

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

where p_i is the number of individuals $i-1$ years old.

- We are looking for a vector $\mathbf{p}$ such that

$$\begin{bmatrix} 0 & 2/3 & 4/3 & 8/3 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}.$$

- It is easy to check that

$$\mathbf{p} = \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2/3 & 4/3 & 8/3 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \quad \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

solves the problem.

- Here is a modification of this problem. Suppose cohorts 2 and 3 produce twice their number in offspring, and cohorts 1 and 4 produce no offspring at all. We would want a vector **p** such that

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}.$$

- There is no such vector! However, using suitable software (matlab) we can check that within the given accuracy

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 9.081 \\ 3.811 \\ 1.599 \\ 0.671 \end{bmatrix} = \underset{1.195}{\cancel{1.915}} \begin{bmatrix} 9.081 \\ 3.811 \\ 1.599 \\ 0.671 \end{bmatrix}$$

- So here we have a population that is stable in the sense that its age distribution does not change. However, the total population grows by a little more than 19 percent each year.

note that if **p** is a population that satisfies our linear system then of course any scalar multiple of **p** also satisfies the equation.

- **Definition**: An **eigenvector** of a square ($n \times n$) matrix A is a **non-zero** vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda \mathbf{x}$$

for some scalar λ. λ is called the **eigenvalue** of A corresponding to the eigenvector $\mathbf{x}$. $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue λ.

- The pair $(\lambda, \mathbf{x})$ is sometimes called an **eigenpair** of A.

- The word "eigen" is German for "own". The eigenvalues of A are numbers "owned" by A, they are the matrix's very own numbers. With an eigenvector, the matrix acts like a number. Multiplying with the matrix is equivalent to multiplying with the eigenvalue.

Note than any non-zero scalar multiple of an eigenvector is also an eigenvector, with the same eigenvalue.

Why do we require the eigenvector to be non-zero?

In one word, what is the major difference between the two problems

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad A\mathbf{x} = \lambda \mathbf{x}?$$

- Some examples:
- What are eigenvalues and eigenvectors of the identity matrix?
- What are eigenvalues and eigenvectors of the zero matrix?
- What are eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}?$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$\lambda = 1$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th}$$

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- More insight can be gained by writing

$$A\mathbf{x} = \lambda\mathbf{x}$$

$x \neq 0$

as

$$A\mathbf{x} - \lambda x = (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

$Bx = 0$

$x \neq 0$

- Any eigenvector is a non-trivial solution of the homogeneous linear system

$$(A - \lambda I)\mathbf{x} = 0.$$

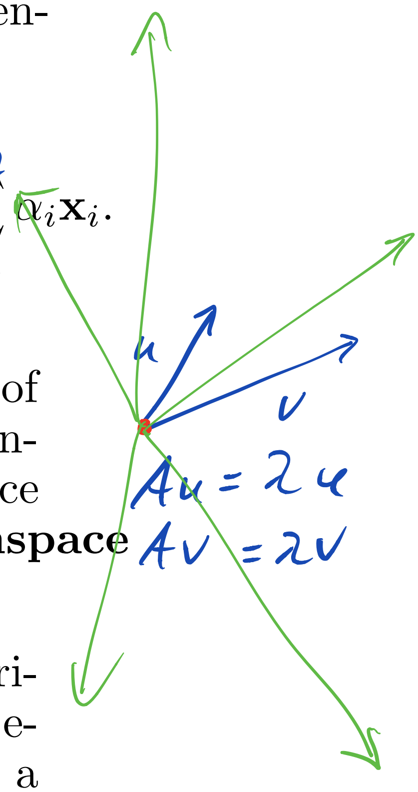
- Every eigenvector is in the nullspace of $A - \lambda I$.
- Every non-zero vector in the nullspace of $A - \lambda I$ is an eigenvector of A.
- A square homogeneous linear system has a non-trivial solution if and only if the coefficient matrix is singular.

thus λ is an eigenvalue of A if and only if $A - \lambda I$ is singular.

Upshot: one more characterization of singularity. A square matrix A is singular if and only if 0 is an eigenvalue of A. It is invertible if and only if all eigenvalues of A are non-zero.

- Suppose $\mathbf{x}_i, i = 1, \ldots, m$ are eigenvectors corresponding to the same eigenvalue λ. Then any (non-zero) linear combination of the eigenvectors is also an eigenvector:

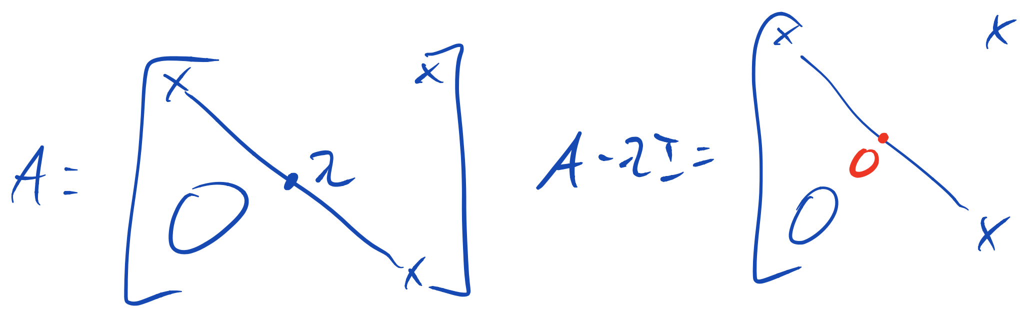
$$A\sum_{i=1}^{m}\alpha_i\mathbf{x}_i=\sum_{i=1}^{m}\alpha_i A\mathbf{x}_i=\sum_{i=1}^{m}\alpha_i\lambda\mathbf{x}_i=\lambda\sum_{i=1}^{m}\alpha_i\mathbf{x}_i.$$



- Thus, if we add the zero vector to the set of eigenvectors corresponding to a specific eigenvalue, that set is a linear space, the nullspace of $A-\lambda I$. That space is also called the **eigenspace of A corresponding to λ.**

- Important example: The eigenvalues of a triangular matrix are the diagonal entries, because if λ is on the diagonal then $A-\lambda I$ is a triangular matrix with at least one zero entry on the diagonal, and is thus singular.

A u.t.



- We are now approaching the question of how to find eigenvalues and eigenvectors.

Row operations do not preserve eigenvalues and eigenvectors!

- Examples

- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ versus $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} y \\ x \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ versus $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

- We need something else ...

- Observation: λ is an eigenvalue if and only if $A - \lambda I$ is singular.
- Recall that a matrix is singular if and only if its determinant is zero. Thus we get the key result:

$$\lambda \text{ is an e.v.} \quad \Leftrightarrow \quad \boxed{\det(A - \lambda I) = 0.}$$

- This is worth some study ....

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x+2y \\ 4y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \qquad 4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x + 2y = x$$
$$4y = y \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \cdot 2/3 \\ 4 \end{bmatrix}$$

$$y = 1$$

$$\begin{bmatrix} x+2y \\ 4y \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \end{bmatrix}$$

$$x + 2 = 4x$$
$$2 = 3x$$
$$x = \frac{2}{3}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$$= (1-\lambda)(4-\lambda) - 6$$