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Math 2270-1 — Fall 2019 — Exam 2

1	2	3	4	5	6	7	8	Total
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Instructions

1. **This exam is closed books and notes. Do not use a calculator or other electronic devices. Do not use scratch paper.**
2. Use these sheets to record your work and your results. Use the space provided, and the back of these pages if necessary. **Show all work.** Unless it's obvious, indicate the problem each piece of work corresponds to, and for each problem indicate where to find the corresponding work.
4. To avoid distraction and disruption **I am unable to answer questions during the exam.** If you believe there is something wrong with a problem state so, and if you are right you will receive generous credit. I will also be unable to discuss individual problems and grading issues with you after you are done while the exam is still in progress.
5. If you are done before the allotted time is up I recommend strongly that you stay and use the remaining time to **check your answers.**
6. When you are done hand in your exam, pick up an answer sheet, and leave the room. Do not return to your seat.
7. All questions have equal weight.
8. Clearly indicate (for example, by circling or boxing) your final answers.

Note that this answer set contains (quite a bit) more information than you needed to provide on the exam.

-1- (Matrix Multiplication.) In one sentence, state why we multiply matrices the way we do.

Discussion:

Here is one answer:

Matrices represent linear functions, and the product of two matrices is the matrix representing the composition of the corresponding functions.

Of course, many variations of that sentence are possible. For example, you could state more concisely

Matrix Multiplication equals Function Composition

The key word in any answer is “composition”.

-2- (Matrix Multiplication.) Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Compute AA^T and $A^T A$.

Discussion:

Writing our matrices as usual we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 3 \end{bmatrix} = AA^T \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = A^T A$$

-3- (Block Matrices.) Suppose A is the invertible upper triangular block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}$$

where A_{11} is $p \times p$ and A_{22} is $q \times q$. (Of course, A_{12} is $p \times q$ and $\mathbf{0}$ is the $q \times p$ zero matrix.) Compute the inverse block matrix

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $p \times p$, B_{12} is $p \times q$, B_{21} is $q \times p$, and B_{22} is $q \times q$. Express the blocks of A^{-1} in terms of the blocks of A .

Discussion:

Note that our assumption that A is invertible implies that A_{11} and A_{22} are invertible (why?). We want

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = I_{p+q} = \begin{bmatrix} I_p & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix}$$

We thus get the 4 equations

$$A_{11}B_{11} + A_{12}B_{21} = I_p, \quad (1)$$

$$A_{11}B_{12} + A_{12}B_{22} = \mathbf{0} \quad (2)$$

$$A_{22}B_{21} = \mathbf{0} \quad (3)$$

$$A_{22}B_{22} = I_q \quad (4)$$

You want to check that the matrix sizes conform in each case. Also think about the sizes of those zero matrices. Equation (4) gives

$$B_{22} = A_{22}^{-1}.$$

Equation (3) gives

$$B_{21} = A_{22}^{-1}\mathbf{0} = \mathbf{0}.$$

This simplifies equation (1) which turns into

$$A_{11}B_{11} = I_p, \quad \text{i.e.,} \quad B_{11} = A_{11}^{-1}.$$

Equation (2) becomes

$$A_{11}B_{12} + A_{12}A_{22}^{-1} = \mathbf{0}$$

which implies that

$$B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}.$$

We get

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ \mathbf{0} & A_{22}^{-1} \end{bmatrix}$$

Again, we check by multiplying A and its proposed inverse:

$$\begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ A_{22}\mathbf{0} & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I_p & \mathbf{0} \\ \mathbf{0} & I_q \end{bmatrix} = I_{p+q}.$$

See the notes of 9/17 for an alternative discussion.

-4- (LU-factorization.) Compute the LU -factorization of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -3 \end{bmatrix}$$

Discussion:

We saw in class that computing the LU factorization is equivalent to applying Gaussian elimination and reducing the matrix to row echelon form. Proceeding as we did in class, and storing the multipliers below the diagonal (and boxing them), we get:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{2} & 1 & 1 \\ \boxed{-2} & 1 & -1 \\ & & \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boxed{2} & 1 & 1 \\ \boxed{2} & 1 & 2 \\ \boxed{-2} & \boxed{1} & -3 \end{bmatrix}$$

Thus

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

Of course we check by multiplying that we actually get $A = LU$:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = U$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & -1 & -3 \end{bmatrix} = A = LU$$

-5- (Inverse Matrix.) Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Discussion:

In class we derived the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In our example this gives

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \times 4 - 2 \times 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}.$$

Of course, you may not have remembered that formula. However, A^{-1} can also be computed directly as we discussed in class, by augmenting A with the identity matrix, and reducing the augmented matrix to reduced row echelon form. This gives

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix} \\ &\longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{bmatrix} \end{aligned}$$

This gives

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

which is the same answer as we obtained before. Of course, we check our result by multiplication of A and its proposed inverse:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = A^{-1} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

-6- (Cramer's Rule.) State and Prove Cramer's Rule.

Discussion:

Cramer's Rule states that for the square linear system

$$A\mathbf{x} = \mathbf{b}$$

the i -th entry x_i of \mathbf{x} is given by

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained from A by replacing the i -th column with \mathbf{b} . To see that this is true let $I_i(\mathbf{x})$ be the matrix obtained from the identity matrix by replacing the i -th column with \mathbf{x} . Thus

$$I_i(\mathbf{x}) = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \quad \mathbf{e}_{i-1} \quad \mathbf{x} \quad \mathbf{e}_{i+1} \quad \dots \quad \mathbf{e}_n]$$

Then, by the way we defined matrix multiplication,

$$AI_i(\mathbf{x}) = A_i(\mathbf{b}).$$

The determinant of the product equals the product of the determinants:

$$|A||I_i(\mathbf{x})| = |A_i(\mathbf{b})|, \quad \text{i.e.,} \quad |I_i(\mathbf{x})| = \frac{|A_i(\mathbf{b})|}{|A|}.$$

Moreover, by expanding about the i -th row we see that

$$|I_i(\mathbf{x})| = x_i.$$

Cramer's rule follows.

-7- (Determinants.) For which value of t does

$$\det \begin{bmatrix} 1 & 2 & t \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} = 0?$$

Discussion:

Expanding the determinant by cofactors about the third column gives

$$\det \begin{bmatrix} 1 & 2 & t \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} = t \det \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} - 5 \det \begin{bmatrix} 1 & 2 \\ 6 & 7 \end{bmatrix} = -3t + 25 = 0.$$

This gives

$$t = \frac{25}{3}.$$

-8- (True or False.) Mark the following statements as true or false by circling **F** or **T**, respectively. You need not give reasons for your answers.

1. **T** **F** To form the matrix product AB the matrix B has to have as many columns as A has rows.

F B has to have as many rows as A has columns.

2. **T F** If A and B are matrices that are not square, and the product AB can be formed, then the product BA cannot be formed.

F

See problem 1 of this exam.

3. **T F** Suppose A is a square matrix. It is invertible if and only if there is no right hand side \mathbf{b} for which the linear system $A\mathbf{x} = \mathbf{b}$ has several solutions.

T

This is item f. of the invertibility theorem in section 2.3 of the textbook.

4. **T F** The square matrix A is invertible if and only if A^T is invertible.

T

In fact, $(A^T)^{-1} = (A^{-1})^T$.

5. **T F** Suppose A and B are both square matrices. Then if AB is invertible, so is B .

T

If B was not invertible there would be a vector $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$. This would imply $AB\mathbf{x} = \mathbf{0}$, i.e., AB is not invertible.

6. **T F** For two $n \times n$ matrices A and B , the determinant of their product equals the product of their determinants.

T

We proved this in class using properties of elementary matrices.

7. **T F** For two $n \times n$ matrices A and B , the determinant of their sum equals the sum of their determinants.

F

For example, if $A = B = I$, then $\det I = 1$ but

$$\det(I + I) = 2^n \neq 1 + 1 = \det I + \det I$$

(unless $n = 1$, of course.)

8. **T F** Suppose c is a scalar, and A is an $n \times n$ matrix. Then $\det(cA) = c \det A$.

F

Instead we have that $\det(cA) = c^n \det A$.

9. **T F** Suppose T is a triangle formed by the vectors \mathbf{u} , \mathbf{v} and $\mathbf{u} + \mathbf{v}$ in \mathbb{R}^2 . Then the area a of T is given by

$$a = |\det [\mathbf{u} \quad \mathbf{v}]|.$$

F

The absolute value of that determinant is **twice** the area of the triangle.

10. **T F** The inverse of an invertible matrix A equals

$$A^{-1} = \frac{1}{\det A} C$$

where C is the matrix of cofactors of A .

F

Actually,

$$A^{-1} = \frac{1}{\det A} C^T$$