Math 2270-1

Notes of 10/30/19

6.1 Inner Product, Length, Orthogonality

- Today's topic is familiar from Math 2210 where we discussed the dot product, the norm of a vector, and orthogonality of vectors.
- In our context, the terminology is slightly different, and we consider the space $\mathbb{R}^n$ for general n, instead of mostly, or just, $\mathbb{R}^2$ and $\mathbb{R}^3$.
- The **inner product**, previously called the **dot product**, of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, is defined to be

$$\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i.$$

- Examples: $u = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 4 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 9 \end{bmatrix}$

$$u^T v = 4 + 14 + 2 + 36 = 56$$

$$u^T v = (u, v) = \langle u, v \rangle = ip(u, v)$$

"outer product"

uv^T

$$
\begin{array}{cc}
 & \begin{bmatrix} 1 & 7 & 2 & 9 \end{bmatrix} \\
\begin{bmatrix} 4 \\ 2 \\ 1 \\ 4 \end{bmatrix} & \begin{bmatrix} 4 & 28 & 8 & 36 \\ 2 & 14 & 4 & 18 \\ 1 & 7 & 2 & 9 \\ 4 & 28 & 8 & 36 \end{bmatrix}
\end{array}
$$

- It's straightforward to verify the following algebraic properties of the inner product:

- **Theorem 1, p. 333.** Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$, and c be a scalar. Then

  a. $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$

  b. $(\mathbf{u}+\mathbf{v}) \bullet \mathbf{w} = \mathbf{u} \bullet \mathbf{w} + \mathbf{v} \bullet \mathbf{w}$

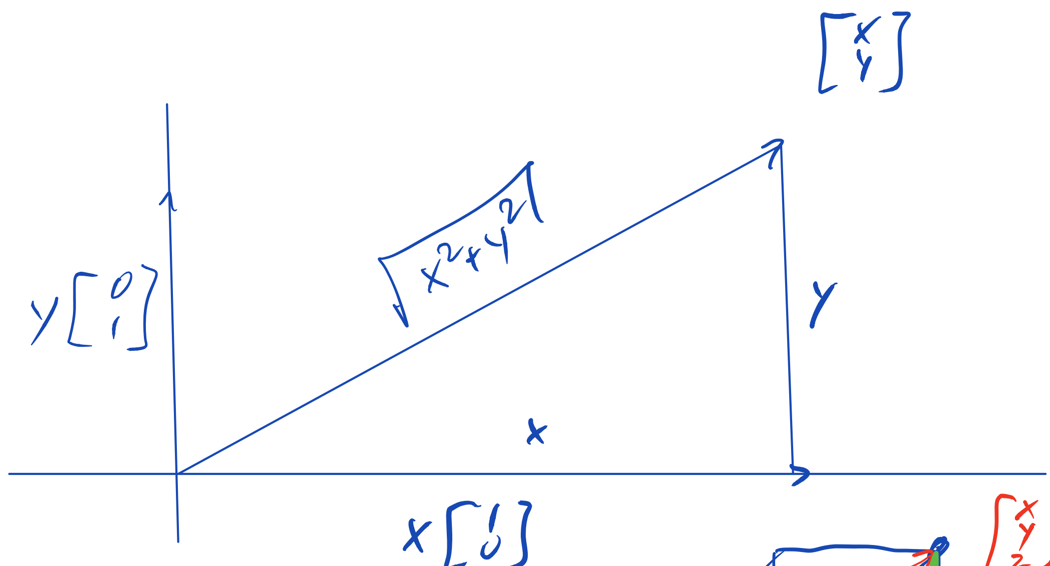
  c. $(c\mathbf{u}) \bullet \mathbf{v} = c(\mathbf{u} \bullet \mathbf{v})$ $= (u \bullet (cv))$

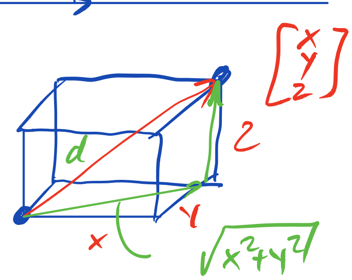
  d. $\mathbf{u} \bullet \mathbf{u} \geq 0$, and $\mathbf{u} \bullet \mathbf{u} = 0 \Longrightarrow \mathbf{u} = \mathbf{0}$.

- The **length** or **norm**[−1−] of a vector **v** is defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}. = \sqrt{v^T v}$$

- Examples. $\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \sqrt{9+16} = 5$

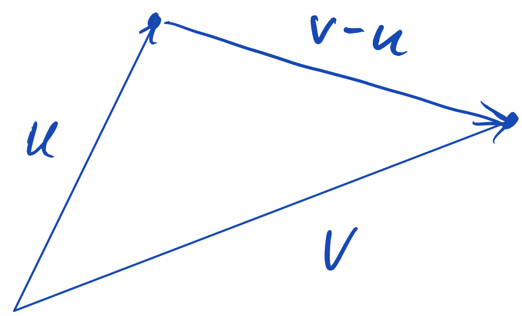




−1− also called **Standard Norm**, **Euclidean Norm**, or **2-norm**.

$d = \sqrt{\sqrt{x^2+y^2}^2 + z^2} = \sqrt{x^2+y^2+z^2}$

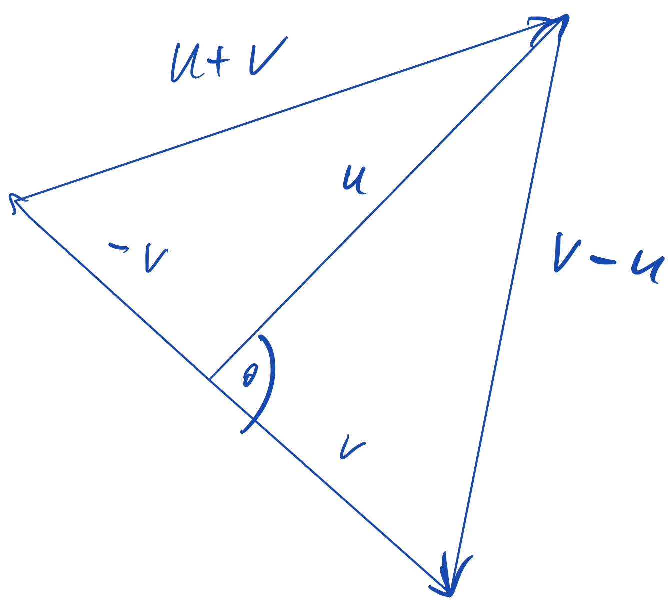
- Identifying points and vectors as usual, the distance between two vectors (points) $\mathbf{u}$ and $\mathbf{v}$ is given by $\|\mathbf{u} - \mathbf{v}\|$.



- If $\mathbf{u}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$ then $\|\mathbf{u}\|$ agrees with our ordinary concept of the length of a vector.

- The concept of orthogonality in $\mathbb{R}^2$ and $\mathbb{R}^3$ generalized to orthogonality in $\mathbb{R}^n$.
- Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if

$$\mathbf{u} \bullet \mathbf{v} = 0.$$



$$\|u+v\|^2 = \|u-v\|^2$$

$$(u+v)^T(u+v) = (u-v)^T(u-v)$$

$$u^Tu + 2u^Tv + v^Tv = u^Tu - 2u^Tv + v^Tv$$

$$2u^Tv = -2u^Tv$$

$$\Longrightarrow \quad u^Tv = 0$$

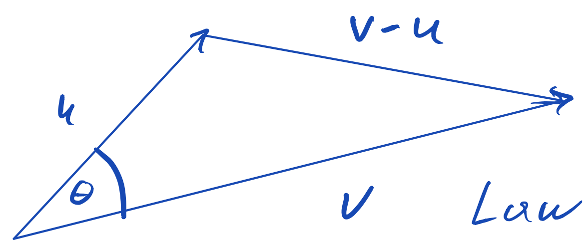
- In 2210 we learned that

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta) \tag{1}$$

  where θ is the angle formed by $\mathbf{u}$ and $\mathbf{v}$.

- This works also in $\mathbb{R}^n$. You can take (1) as the **definition** of θ.

the zero vector is orthogonal to all vectors in $\mathbb{R}^n$.



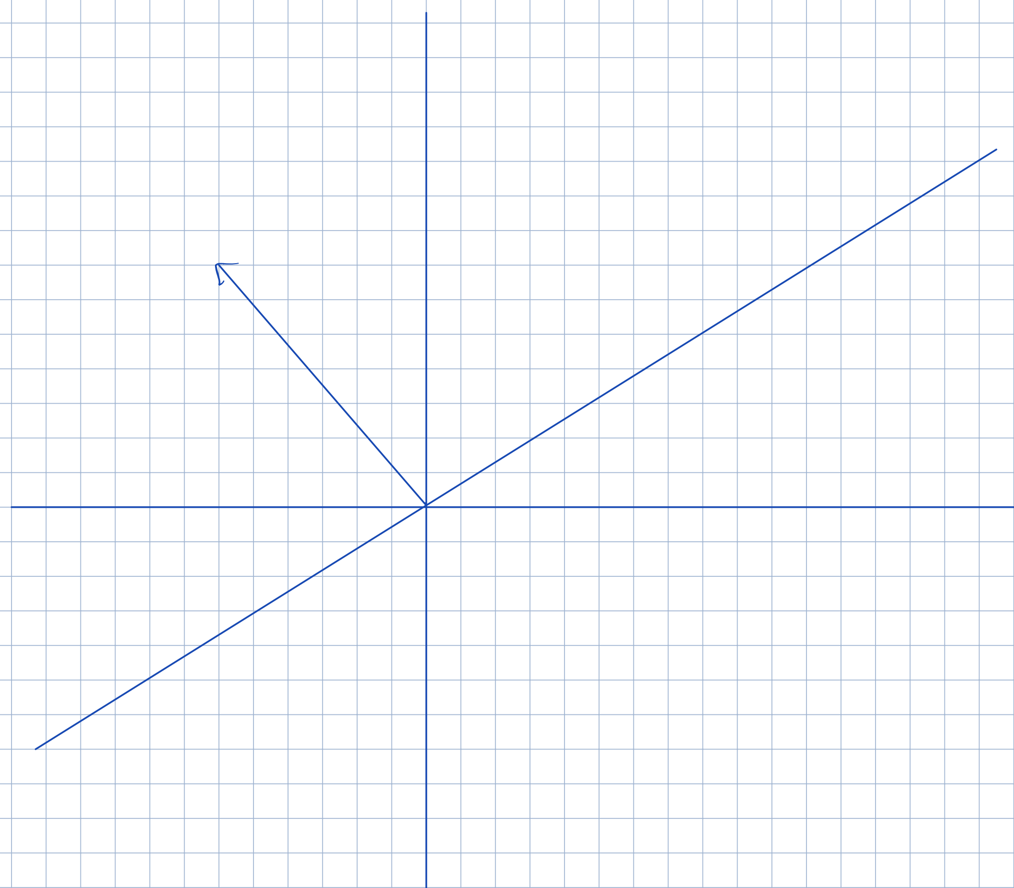
Law of Cosines

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$$

$$(v-u)^T(v-u) = u^Tu + v^Tv - 2\|u\|\|v\|\cos\theta$$

$$v^Tv - 2u^Tv + u^Tu = u^Tu + v^Tv - 2\|u\|\|v\|\cos\theta$$

$$u^Tv = \|u\|\|v\|\cos\theta$$



Orthogonal Complements

- Suppose W is a subspace of $\mathbb{R}^n$. Then the set

  $$W^\perp = \{\mathbf{z} : \mathbf{z}\text{is orthogonal to all vectors in } W\}$$

  is a linear space, called the **orthogonal complement** of W.

- $W^\perp$ is read as "W-perpendicular" or, more commonly, just "W-perp".

- Example: line and plane in $\mathbb{R}^3$.

$u, v \in W^\perp \Rightarrow v + w \in W^\perp$

$v^T w = 0$ for all $w \in W$

$u^T w = 0$ " "

$v^T w + u^T w = 0$

$(v + u)^T w = 0$

$v \in W^\perp$ $v^T w = 0$ f.a. $w \in W$

$c v^T w = 0$

$(cv)^T w = 0$

- **Theorem 3**, p. 337: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T

A $m \times n$

$$(\text{Row} A)^\perp = \text{Nul}\, A \text{ and } \quad (\text{Col} A)^\perp = \text{Nul} A^T.$$

$\text{Nul}\, A = \{x : Ax = 0\} \subset \mathbb{R}^n$