

Math 2270-6

Notes of 11/13/2019

Announcement

- **Undergraduate Colloquium**

today, 12:⁵⁵~~45~~-1:45, LCB 225

- Ryleigh Moore will present:

Adventures in the Arctic

Mathematics Ph.D. student, Ryleigh Moore, was one of three American graduate students invited to participate in the Multidisciplinary drifting Observatory for the Study of Arctic Climate (MOSAiC) expedition out of Tromsø, Norway, from September 20 - October 28, 2019. The flagship German icebreaker, RV Polarstern, will be frozen in ice and drift for a full year through the Central Arctic following in the footsteps of an earlier 19th century expedition under Norwegian explorer, Fridtjof Nansen. MOSAiC will mark the first time a modern research icebreaker will study near the North Pole throughout the polar winter; it is hailed as the largest Central Arctic expedition ever, with 19 countries, over 600 people, and a budget exceeding 155 million US dollars.

In this talk, Ryleigh will discuss the science goals of MOSAiC and her experiences while on

the Russian research vessel, Akademik Fedorov. She will discuss life on a research vessel, how the expedition identified ice floes that would be used for instrument deployments, and her role of leading the installation of three seasonal ice mass balance (SIMB3) buoys in the Central Arctic.

More on Least Squares

- Suppose we have the overdetermined linear system

$$A\mathbf{x} = \mathbf{b}$$

where A is $m \times n$ and $m > n$.

- We discussed solving instead the Least Squares Problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min \quad (1)$$



Note that any solution \mathbf{x} of (1) is also a solution of the problem

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \min \quad (2)$$

- We saw that we can find \mathbf{x} by solving the **normal equations**

$$A^T A\mathbf{x} = A^T \mathbf{b}.$$

- Moreover, we saw that
0. Geometrically we get $A\mathbf{x}$ by projecting \mathbf{b} into the column space of A and we get the normal equations by observing that $A\mathbf{x} - \mathbf{b}$ is in the orthogonal complement of the column space of A .
 1. $A\mathbf{x}$ exists and is unique for every \mathbf{b} .
 2. \mathbf{x} is unique if the columns of A are linearly independent.
 3. $A^T A$ is invertible if and only if the columns of A are linearly independent.
 4. In that case we get

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

although we would not normally actually compute the inverse of $A^T A$

- Example: Suppose you have some data (x_i, y_i) , $i = 1, 2, \dots, n$ and you have reason to believe that the answer is some constant plus a 2π -periodic function that looks something like a sine curve. In other words,

$$y_i \approx f(x_i) = \alpha + \beta \sin x_i + \gamma \cos x_i.$$

- In equation form this is

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} 1 & \sin x_1 & \cos x_1 \\ 1 & \sin x_2 & \cos x_2 \\ \vdots & \vdots & \vdots \\ 1 & \sin x_n & \cos x_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- This is an overdetermined linear system. As we discussed in class yesterday, its least squares solution \mathbf{x} is given by the normal equations

$$A^T A \mathbf{x} = A^T \mathbf{b}$$

where, in this case, with all sums running from 1 to n ,

$$A^T A = \begin{bmatrix} \sum 1 & \sum \sin x_i & \sum \cos x_i \\ \sum \sin x_i & \sum \sin^2 x_i & \sum \sin x_i \cos x_i \\ \sum \cos x_i & \sum \cos x_i \sin x_i & \sum \cos^2 x_i \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} \sum y_i \\ \sum y_i \sin x_i \\ \sum y_i \cos x_i \end{bmatrix}$$

- Example: Temperatures in Salt Lake City.
- You'd expect (low and high daily) temperatures $T(t)$ in SLC to follow (roughly) some kind of sin curve such as

$$T(t) = a + b \sin \frac{2\pi t}{12} + c \cos \frac{2\pi t}{12} \quad (3)$$

where t is the time of the year, measured in months. a is the mean temperature over the whole year. b and c are suitable coefficients. I found the following data at

<http://www.rssweather.com/climate/Utah/Salt%20Lake%20City/>

Month	Low	High
Jan	21.0F	37.0F
Feb	25.5F	43.F
Mar	33.4F	52.8F
Apr	39.0F	60.9F
May	46.9F	70.6F
Jun	55.8F	82.2F
Jul	63.4F	90.6F
Aug	62.4F	88.7F
Sept	52.4F	77.6F
Oct	41.0F	64.0F
Nov	30.4F	48.7F
Dec	22.4F	38.0F

- Suppose you want to find the Least Squares approximation of the form (3) that best approximates the data.

- It would be unreasonable to do this problem by hand. Numerous facilities (e.g., matlab, maple, wolfram alpha, any number of programming languages) exist to get a computer to do the necessary calculations.
- The following matlab code and its outputs show the results of doing a Least Squares Approximation.

```

1 % approximate low and high temperatures
in SLC by sin and cos
2 % curve
3 % see http://www.rssweather.com/climate/Utah/Salt%20Lake%20City
4
5 low=[21.3;25.5;33.4;39.0;46.9;55.8;63.4;62.4;52.4;41.0;30.0;21.3];
6
7 pi = 2.0*acos(0.0);
8
9 for i =1:12
10     O(i,1) = 1.0;
11     S(i,1) = sin(2*i/12*pi);
12     C(i,1) = cos(2*i/12*pi);
13 end
14
15 A=[O'*O,O'*S,O'*C;S'*O,S'*S,S'*C;C'*O,C'*S,C'*C]
16 B=[O'*low;S'*low;C'*low];
17
18 X= A\B
19
20 for i = 1:12
21     app = X(1) + X(2)*S(i,1) + X(3)*C(i,1);
22     err = low(i)-app;
23     res = [i,low(i), app, err];
24     disp(res)
25 end

```

$$A^T A = A$$


```

1  A =
2
3      12.0000    -0.0000    -0.0000
4      -0.0000     6.0000     0.0000
5      -0.0000     0.0000     6.0000
6
7  X =
8
9      41.1583
10     -10.9147
11     -16.9332
12
13      1.0000     21.3000     21.0364     0.2636
14      2.0000     25.5000     23.2393     2.2607
15      3.0000     33.4000     30.2436     3.1564
16      4.0000     39.0000     40.1725    -1.1725
17      5.0000     46.9000     50.3655    -3.4655
18      6.0000     55.8000     58.0915    -2.2915
19      7.0000     63.4000     61.2803     2.1197
20      8.0000     62.4000     59.0774     3.3226
21      9.0000     52.4000     52.0731     0.3269
22     10.0000     41.0000     42.1442    -1.1442
23     11.0000     30.4000     31.9511    -1.5511
24     12.0000     22.4000     24.2252    -1.8252
25

```

```

1 % approximate low and high temperatures
in SLC by sin and cos
2 % curve
3 % see http://www.rssweather.com/climate/Utah/Salt%20Lake%20City
4
5 high=[37.0;43.4;52.8;60.9;70.6;82.2;90.6;88.7;77.6;64.0;48.0;37.0];
6
7 pi = 2.0*acos(0.0);
8
9 for i =1:12
10     O(i,1) = 1.0;
11     S(i,1) = sin(i/6*pi);
12     C(i,1) = cos(i/6*pi);
13 end
14
15 A=[O'*O,O'*S,O'*C;S'*O,S'*S,S'*C;C'*O,C'*S,C'*C]
16 B=[O'*high;S'*high;C'*high];
17
18 X= A\B
19
20 for i = 1:12
21     app = X(1) + X(2)*S(i,1) + X(3)*C(i,1);
22     err = high(i)-app;
23     res = [i,high(i), app, err];
24     disp(res)
25 end

```

```

1
2 A =
3
4     12.0000    -0.0000    -0.0000
5     -0.0000     6.0000     0.0000
6     -0.0000     0.0000     6.0000
7
8
9 X =
10
11     62.8750
12    -13.7609
13    -21.7808
14
15     1.0000    37.0000    37.1318    -0.1318
16     2.0000    43.4000    40.0673     3.3327
17     3.0000    52.8000    49.1141     3.6859
18     4.0000    60.9000    61.8481    -0.9481
19     5.0000    70.6000    74.8573    -4.2573
20     6.0000    82.2000    84.6558    -2.4558
21     7.0000    90.6000    88.6182     1.9818
22     8.0000    88.7000    85.6827     3.0173
23     9.0000    77.6000    76.6359     0.9641
24    10.0000    64.0000    63.9019     0.0981
25    11.0000    48.7000    50.8927    -2.1927
26    12.0000    38.0000    41.0942    -3.0942
27

```

Perhaps more impressive is a graphic display
of our data and its approximations:

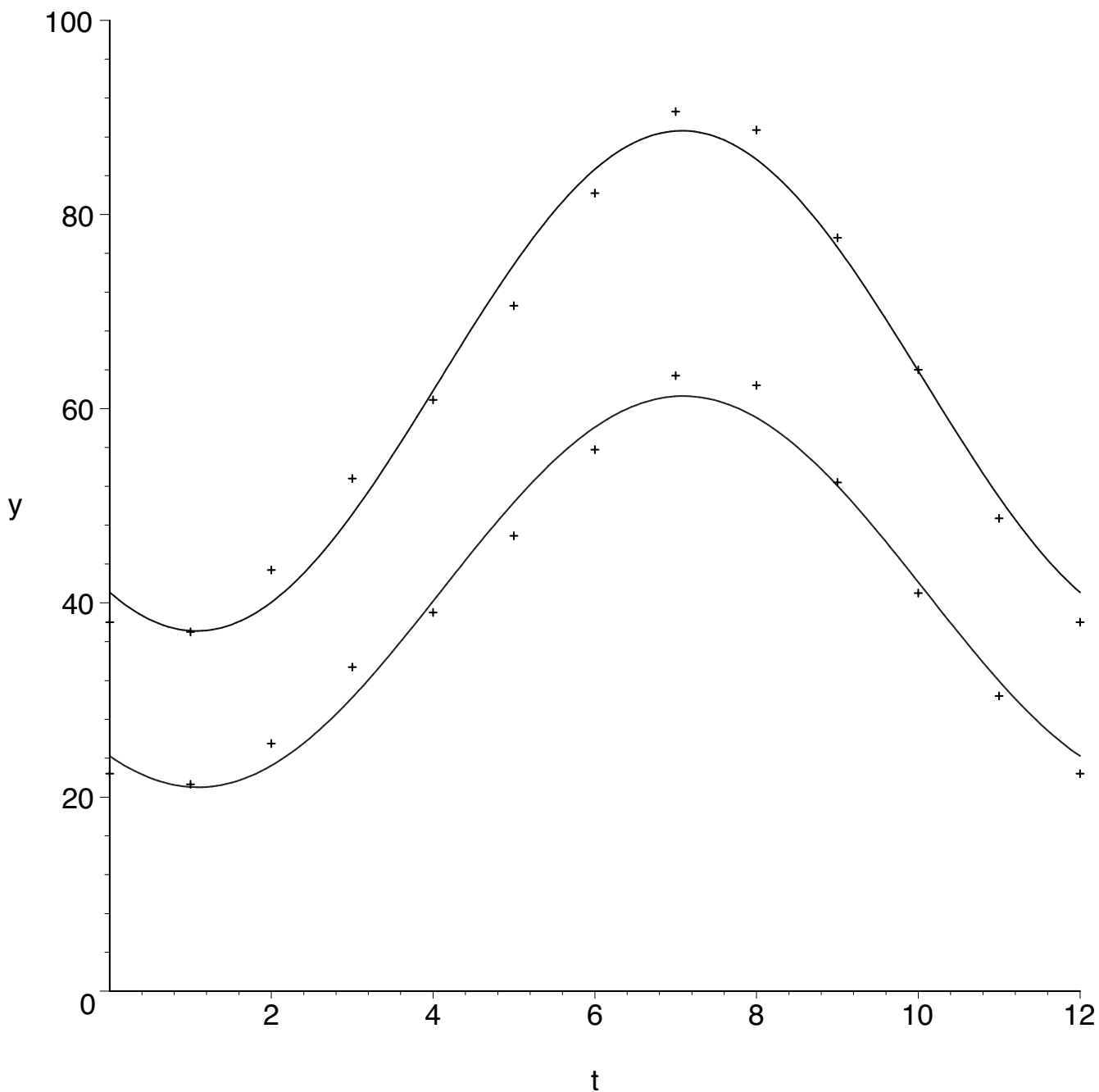


Figure 1. Temperature in Salt Lake City.

- As we've seen, things get more convenient if we have matrices with orthonormal columns. Let's first look at the case that A itself have orthonormal columns.

$$A = [a_1, \dots, a_n]$$

$$A^T A = [a_i^T a_j] = \underline{I}$$

- Next let's see what happens if we have a QR factorization of A .
- Suppose

$$A = QR$$

where R is upper triangular and Q has orthonormal columns.