Math 2270-6

Notes of 1/29/19

- Tomorrow we will have our first exam.
- It will have 6 questions covering chapter 1. (Matrix multiplication and inverses are part of chapter 2 and will be covered in exam 2 on February 13.)
- You answer the questions on the exam itself. Sufficient space for your answers will be provided. The exam will be printed one-sided and you can also use the backs of the pages if necessary.
- The exam will be closed books and notes, no scratch paper, no electronics. All you have to bring to the exam is a dependable writing utensil.
- You should have plenty of time. Accuracy is more important than speed. Proceed slowly and deliberately.
- The questions are deliberately simple. If you find yourself engaged in a length calculation chances are you missed something and you are on the wrong track.
- Read the entire exam when you get it, then do the easiest problems first.
- If you get stuck on a particular problem set it aside and return to it only after you've answered the other questions.

- **Check your answers.**
- **To avoid distraction and disruption I will not be able to talk with you during the exam!**
- If you believe there is mistake in one of the problems write a note on the exam and if you are correct you will receive generous credit.
- All questions have equal weight.
- Explain what you are doing and write clearly. Obviously you have a strong interest in me understanding your work.

The subject

This list is neither complete nor self contained. Rather, the individual points should stir your memory of related concepts, facts, and connections. If you draw a blank you should review the relevant parts of your notes or the textbook.

In class we started by discussing computational procedures, based on row echelon forms, to solve a linear system. We then discussed more theoretical facts and concepts. In these notes, we go the other way, starting with the facts and concepts, and discussing computational procedures at the end.

- The focus of chapter 1 is on linear systems

  $$A\mathbf{x} = \mathbf{b} \tag{1}$$

  where A is an $m \times n$ matrix, $\mathbf{x}$ is in $\mathbb{R}^n$, and $\mathbf{b}$ is in $\mathbb{R}^m$.

- A vector $\mathbf{x}$ satisfying the equation $A\mathbf{x} = \mathbf{b}$ is a **solution** of the linear system.

- **Solving** the linear system means figuring out whether there is a solution, and if so, how many, and what they are.

- If $m = n$ then the matrix A, and the system $A\mathbf{x} = \mathbf{b}$, are said to be **square.**

- $A\mathbf{x}$ is a **linear combination** of the columns of A with the coefficients being given by the entries of $\mathbf{x}$. If

$$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n\,] \tag{2}$$

where the $\mathbf{a}_i$ are the columns of A, and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \tag{3}$$

then

$$A\mathbf{x} = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n\,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{b}. \tag{4}$$

- In general, a **linear combination** of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\dots$ $\mathbf{v}_n$ is an expression of the form

$$\sum_{i=1}^{n} c_i \mathbf{v}_i \tag{5}$$

where the **coefficients** or **weights** c_i are real numbers.

- A function

$$\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \tag{6}$$

is a function whose domain is $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$. We also write it in a more familiar form as

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \tag{7}$$

where $\mathbf{x}$ is in $\mathbb{R}^n$ and $\mathbf{y}$ is in $\mathbb{R}^m$,

- $\mathbf{f}$ is **linear** if

$$\mathbf{f}(\mathbf{u}+\mathbf{v}) = \mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v}) \quad \text{and} \quad \mathbf{f}(c\mathbf{u}) = c\mathbf{f}(\mathbf{u}) \tag{8}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and **scalars** (real numbers) c.

- The matrix transformation

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x} \tag{9}$$

is linear.

- Actually, and most amazingly, given a linear function $\mathbf{T}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, there is a matrix A such that

$$\mathbf{T}(\mathbf{x}) = A\mathbf{x}. \tag{10}$$

- To construct the matrix A we use the columns of the identity matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}. \tag{11}$$

- The standard notation for the i-th column of I is $\mathbf{e}_i$ which is the vector in $\mathbb{R}^n$ all of whose entries are zero, except that the i-th entry equals 1.

- The $\mathbf{e}_i$ are called the **standard basis vectors** in $\mathbb{R}^n$. The value of n is not part of the notation, it should be clear from the context.

- For any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{12}$$

we have

$$\mathbf{x} = I\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i. \tag{13}$$

- Suppose now that we are given a linear function (or transformation) T from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let

$$\mathbf{a}_i = T(\mathbf{e}_i), \quad i = 1, 2, \ldots, n \tag{14}$$

Moreover, let A be the $m \times n$ matrix

$$A = [\, \mathbf{a}_1 \quad \mathbf{a}_2 \quad \ldots \quad \mathbf{a}_n \,] \tag{15}$$

Then

$$
\begin{aligned}
T(\mathbf{x}) &= T\left(\sum_{i=1}^{n} x_i \mathbf{e}_i\right) \\
&= \sum_{i=1}^{n} T\left(x_i \mathbf{e}_i\right) \qquad \text{(by part 1 of linearity!)} \\
&= \sum_{i=1}^{n} x_i T(\mathbf{e}_i) \qquad \text{(by part 2 of linearity!)} \\
&= \sum_{i=1}^{n} x_i \mathbf{a}_i \\
&= A\mathbf{x}.
\end{aligned}
\tag{16}
$$

- In other words, with our choice of A,

$$T(\mathbf{x}) = A\mathbf{x} \tag{17}$$

- In this context, A is called the **standard matrix** of the transformation T.

Simplifying things slightly we can say that **linear functions are synonymous with matrices**.

- That's why matrices are so important!
- A set

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \tag{18}$$

of vectors is **linearly independent** if the only way the zero vector can be written as a linear combination of the given vectors is to make all coefficients zero.

- In other words:

$$\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0} \qquad \Longrightarrow \qquad c_1 = c_2 = \ldots = c_n = 0. \tag{19}$$

- Here are a number of true statements about linear independence. (We assume that all sets contain only vectors from the same space, i.e, they all have the same number of entries.)
  - A set containing just one vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v}$ is nonzero.
  - A set containing exactly two vectors is linearly independent if and only if neither vector is a multiple of the other.
  - If a set of (more than one) vectors is linearly dependent than at least one of those vectors can be written as a linear combination of the others.
  - No vector in a linearly independent set can be written as a linear combination of the others.
  - A linearly dependent set may (or may not) contain vectors that cannot be written as a linear combination of vectors.
  - Any set that contains the zero vector is linearly dependent.

- The matrix transformation

$$\mathbf{y} = A\mathbf{x} \tag{20}$$

  is one-to-one if and only if the columns of A are linearly independent.

- The columns of A are linearly independent if and only if the homogeneous problem

$$A\mathbf{x} = \mathbf{0} \tag{21}$$

  has only the trivial solution $\mathbf{x} = \mathbf{0}$.

- The **span** of a set of vectors is the set of all linear combinations of those vectors.

- Here are some true statements about the span of a set of vectors.

  - Any vector in the span of a linearly independent set can be written in only one way as a linear combination of the given vectors.

  - Any vector in the span of a linearly dependent set can be written in more than one way as a linear combination of the given vectors.

  - The linear system

$$A\mathbf{x} = \mathbf{b} \tag{22}$$

    has a solution if and only if $\mathbf{b}$ is in the span of the columns of A.

- The linear system has a solution for all right hand sides $\mathbf{b}$ if and only if the span of the columns of A is all of $\mathbb{R}^m$.
- The linear system has a unique solution if $\mathbf{b}$ is in the span of the columns of A and those columns are linearly independent.
- The linear system has a unique solution for all right hand sides $\mathbf{b}$ if the span of the columns of A is all of $\mathbb{R}^m$ and the columns are linearly independent.

- The linear system $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = 0$.

- Here are some more true statements about linear systems:
  - if $\mathbf{u}$ and $\mathbf{v}$ are solutions of $A\mathbf{x} = \mathbf{0}$ then so are $\mathbf{u}+\mathbf{v}$ and any other linear combination of $\mathbf{u}$ and $\mathbf{v}$.
  - If $\mathbf{u}$ and $\mathbf{v}$ are solutions of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ then $\mathbf{u} - \mathbf{v}$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
  - A linear system may have no solutions, a unique solution, or infinitely many solutions. (It may not have precisely 17 solutions, for example.)
  - **The general solution of the linear system**

$$A\mathbf{x} = \mathbf{b} \tag{23}$$

can be written as any particular solution plus the general solution of the

homogeneous system

$$A\mathbf{x} = \mathbf{0}. \tag{24}$$

- The last statement is one of the most central principles in mathematics. It applies to all linear problems, not just linear algebraic equations.

Computations

- Given a matrix A you want to understand clearly how to answer the following questions about the linear system $A\mathbf{x} = \mathbf{b}$:
  - Given $\mathbf{b}$ is there at least one solution? In other words, is the system **consistent** for that vector $\mathbf{b}$?
  - Is there a solution for all right hand sides $\mathbf{b}$? In other words, is the system consistent for all possible $\mathbf{b}$? In yet other words, is the matrix transformation $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$ **onto**?
  - If the system is consistent, is there only one solution? If there is we call the solution **unique**. In other words, is the matrix transformation $\mathbf{T}(\mathbf{x}) = A\mathbf{x}$ **one-to-one**?
- There are many ways to compute the answers to these problems. Those we discussed are based on the reduced and unreduced **row echelon forms** of the **augmented matrix**

$$M = [\, A \quad \mathbf{b} \,] \tag{25}$$

of the linear system $A\mathbf{x} = \mathbf{b}$.

- These are obtained by applying **elementary row operations** to the augmented matrix. There are three such operations:

1. **Add a multiple of one row to another row.**
2. **Interchange two rows.**
3. **Multiply a row by a non-zero constant.**

- It is clear that these operations do not change the solution of the linear system and generate augmented matrices corresponding to **equivalent linear systems.**

- Two linear systems are **equivalent** if they have the same solution sets.

- A rectangular matrix is in **echelon form** if it has the following three properties:

1. **All nonzero rows are above any rows of all zeros.**
2. **Each leading entry of a row is in a column to the right of the leading entry of the row above it.**
3. **All entries in a column below a leading entry are zeros.**

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**.

4. **The leading entry in each non-zero row**

is 1.

5. Each leading 1 is the only nonzero entry in its column.

- A **pivot position** in a matrix M is a location in M that corresponds to a leading 1 in the reduced row echelon from of M. A **pivot row** of A is a row that contains a pivot position and a **pivot column** of A is a column that contains a pivot position.
- The **reduced row echelon form** of a matrix is unique. (We will see precisely why this is true when we get to chapter 4.)
- A variable is **basic** if it corresponds to a pivot column.
- Otherwise it is **free**. The free variables can assume any values in the solutions of the linear systems. The values of the basic variables are determined uniquely by those of the free variables.
- Here are some true statements about the solution of linear systems

$$A\mathbf{x} = \mathbf{b} \tag{26}$$

  - The linear system has a solution if and only if the last column of the augmented matrix is not a pivot column.
  - The linear system has infinitely many solutions only if it has free variables. (For solutions to exist the last column of the

augmented matrix still must not be a pivot column.)

- The general solution of the linear system can be written down most easily given the **reduced row echelon form** of the augmented matrix.
- However, some effort can be saved by just computing the (unreduced) row echelon form and then applying backward substitution.
- The computations of row echelon forms in the textbook are written as sequences of matrices, for clarity. Doing the calculations that way involves a lot of copying, however.
- If the purpose of the calculation is strictly the solution of the linear system, rather than theoretical insight, a more streamlined procedure can be used. Everything is written down only once. However, I recommend that to guard against errors you keep track of row sums and compute them redundantly in two ways. As long as they are equal you can be reasonably confident that your calculations so far are accurate.
- Example: Solve the linear system:

$$\begin{array}{ccccccccccc} a & + & b & + & c & & d & = & 2^1 & = & 2 \\ a & + & 2b & + & 4c & + & 8d & = & 2^2 & = & 4 \\ a & + & 3b & + & 9c & + & 27d & = & 2^3 & = & 8 \\ a & + & 4b & + & 16c & + & 64d & = & 2^4 & = & 16 \end{array} \tag{27}$$

- Here is the detailed calculation

$$
\begin{array}{cccccccc}
\# & a & b & c & d & \textbf{RHS} & \textbf{RS} \\
 & & & & & & \\
\mathbf{1} & 1 & 1 & 1 & 1 & 2 & 6 \\
\mathbf{2} & 1 & 2 & 4 & 8 & 4 & 19 \\
\mathbf{3} & 1 & 3 & 9 & 27 & 8 & 48 \\
\mathbf{4} & 1 & 4 & 16 & 64 & 16 & 101 \\
 & & & & & & \\
\mathbf{5}=\mathbf{2}-\mathbf{1} & & 1 & 3 & 7 & 2 & 13 \\
\mathbf{6}=\mathbf{3}-\mathbf{1} & & 2 & 8 & 26 & 6 & 42 \\
\mathbf{7}=\mathbf{4}-\mathbf{1} & & 3 & 15 & 63 & 14 & 95 \\
 & & & & & & \\
\mathbf{8}=\mathbf{6}-2\times\mathbf{5} & & & 2 & 12 & 2 & 16 \\
\mathbf{9}=\mathbf{7}-3\times\mathbf{5} & & & 6 & 42 & 8 & 56 \\
 & & & & & & \\
\mathbf{10}=\mathbf{9}-3\times\mathbf{8} & & & & 6 & 2 & 8
\end{array}
\tag{28}
$$

$$
\begin{array}{cccccccc}
\mathbf{10} & \Longrightarrow & 6d & = & 2 & \Longrightarrow & d & = & \frac{1}{3} \\
\mathbf{8} & \Longrightarrow & 2c+4 & = & 2 & \Longrightarrow & c & = & -1 \\
\mathbf{5} & \Longrightarrow & b-3+\frac{7}{3} & = & 2 & \Longrightarrow & b & = & \frac{8}{3} \\
\mathbf{1} & \Longrightarrow & a+\frac{8}{3}-1+\frac{1}{3} & = & 2 & \Longrightarrow & a & = & 0
\end{array}
\tag{29}
$$