Math 2270-1

Notes of 8/26/2019

- **Quote of the day:** *Linear Algebra is all I do these days. Tell your students to pay more attention than they think should be necessary.* Scott Alfeld, Assistant Professor of Computer Science, Amherst College, specializing in Machine Learning.
- reminder: Tomorrow we'll meet in in JFB 102.

1.3 Vector Equations

- We learned about vectors in Trigonometry and in Calc III.
- So part of this section is a review of past material.
- What will be new, however, is the connection between vectors and linear systems, and the major concept of a **linear combination.**
- We think of vectors as ordered lists of numbers, arranged in a **column**.
- A **(column) vector** is a matrix with just one column.
- A **row vector** would be a matrix with one row of numbers.

- We use the convention that when we say "vector" without specifying row or column we mean a column vector.
- We use bold face lower case letters to indicate vectors.
- For example, with s, t, w_1, w_2, and w_3 being real[-1-] numbers,

$$\mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} s \\ t \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

  are all vectors.

In Math 2210 we would have written these vectors as

$$< 4, -2 >, \qquad < s, t >, \qquad \text{or} \qquad < w_1, w_2, w_3 > .$$

For our class, that notation is now obsolete.

However, the row vector $[1, 2]$ is not the same as the column vector

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

[-1-] There is a corresponding theory of vectors and matrices with complex entries, but that is beyond the scope of our class.

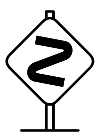
- The sets of all vectors with two, three, or n entries are denoted by

$$\mathbb{R}^2, \qquad \mathbb{R}^3, \qquad \text{or} \qquad \mathbb{R}^n,$$

respectively.

- These are pronounced r-2, r-3, and r-en, respectively.
- The sum of two vectors is obtained by adding corresponding pairs of entries. For example

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1-3 \\ 2+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

 you cannot add a vector in $\mathbb{R}^2$ to a vector in $\mathbb{R}^3$!

- In this context we also refer to real numbers as **scalars**.
- The **scalar multiple** of a vector $\mathbf{u}$ and a scalar c is obtained by multiplying every entry of $\mathbf{u}$ with c:

$$c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} \qquad \text{e.g.,} \qquad 3 \begin{bmatrix} \pi \\ 1/3 \\ e/6 \end{bmatrix} = \begin{bmatrix} 3\pi \\ 1 \\ e/2 \end{bmatrix}.$$

- These notions operations have the familiar geometric meanings.

Figure 1. Geometric meaning of vector operations..

- Vectors can be represented as arrows, with tails and tips.
- Vectors can also be identified with points. (The

tip is the point, the tail is the origin).

- The **Parallelogram Law** says that for all vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

Figure 2. $\mathbf{w} = v + (w - v)$.

- The vector with all entries zero is the **zero vector**, denoted by $\mathbf{0}$.

- For simplicity we write

$$-\mathbf{u} = (-1)\mathbf{u}$$

and

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

- Below are some properties of vectors and scalars. You should convince yourself that these are true, going back to the relevant definitions if necessary.

$$\begin{array}{clcl}
(i) & \mathbf{u}+\mathbf{v} & = & \mathbf{v}+\mathbf{u} \\
(ii) & (\mathbf{u}+\mathbf{v})+\mathbf{w} & = & \mathbf{u}+(\mathbf{v}+\mathbf{w}) \\
(iii) & \mathbf{u}+\mathbf{0} & = & \mathbf{u} \\
(iv) & \mathbf{u}+(-\mathbf{u}) & = & \mathbf{0} \\
(v) & c(\mathbf{u}+\mathbf{v}) & = & c\mathbf{u}+c\mathbf{v} \\
(vi) & (c+d)\mathbf{u} & = & c\mathbf{u}+d\mathbf{u} \\
(vii) & c(d\mathbf{u}) & = & (cd)\mathbf{u} \\
(viii) & 1\mathbf{u} & = & \mathbf{u}
\end{array}$$

- Glimpse ahead: Sets of objects that we can add and multiply with scalars, and that have the above listed properties, will be called **vector spaces**. The elements of these spaces are then called **vectors**. This is an alternative way to define vectors.

Linear Combinations

Given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ and scalars c_1, c_2, ..., c_n the vector

$$\mathbf{y} = c_1\mathbf{v}_1 + \mathbf{c}_2\mathbf{v}_2 + \ldots + \mathbf{c}_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i$$

is the **linear combination** of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ with **weights** (or **coefficients**) c_1, c_2, ..., c_n.

- Examples:

- Example 5: Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

Determine whether $\mathbf{b}$ can be written (or **generated**) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

So we can describe a linear system as: figure out whether the right hand side can be written as a linear combination of the columns of the coefficient matrix.

Span

- Definition (p. 30, textbook): If $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_p$ is denoted by

$$\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$$

and called the subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_p$. That is

$$\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\} = \left\{ \mathbf{y} : \mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{v}_i. \right\}$$

Geometric Interpretation of Span

Linear Systems

- In our new terminology we can say that a linear system is consistent if and only if the right hand side is contained in the span of the columns of the coefficient matrix.

- In general, the $m \times n$ linear system,

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}$$

becomes the vector equation

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or, more concisely,

$$\sum_{j=1}^{n} x_j \mathbf{a}_j = \mathbf{b}$$

where

$$\mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}, \quad i = 1, \dots, m, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$