

Math 2270-1

Notes of 8/23/2019

Announcements

- first home work (hw 2) is now open
- will close 9/3
- WeBWorK based
- Lots of true/false questions from textbook
- weight of home works increased (3% instead of 2%) increased.
- The purpose of the hw is to help you learn the subject. Make the best of the assignments and don't cut corners.
- I recommend you work with class mates.
- hw problems are mostly new. The one point contest applies.

How actually to solve a linear system



The purpose of our discussion of elementary row operations and (reduced or unreduced) row echelon forms is to gain insight into the structure of matrices and linear systems.



Most actual linear systems are solved by computer.



Those systems can be very large.

- But computers are very fast:

```
1 >> a=rand(1000,1000);
2 >> b=rand(1000,1);
3 >> tic;
4 >> x=a\b;
5 >> toc
6
7 Elapsed time is 0.299431 seconds.
8
9 >> r=norm(a*x-b)
10
11 r =
12
13     1.1345e-10
```

- However, there are occasions where you may want to solve a linear system by hand.
- In those rare cases you are concerned about **efficiency** (in terms of your own time) and

correct arithmetic.

- In the calculations we have seen so far we write down the entire (augmented) matrix at every step. Much of the information in that matrix does not change in the process. There is no point in rewriting it at every step.
- Moreover, if you make a single numerical mistake in the process everything following that mistake will be incorrect.



You want to detect mistakes when they occur!

- The key to guarding against errors is **redundancy!**
- Recall the three elementary row operations:
 1. Replace one row by the sum of itself and a multiple of another row. (Since the multiplying factor can be negative this includes the option of subtracting a multiple of a row.)
 2. Interchanging two rows.
 3. Multiplying a row with a non-zero factor.
- Clearly these operations do not change the solution of a linear system.
- We use the operations to obtain an equivalent triangular system, or a system in row echelon form, and then find the solution by backward substitution.



To guard against errors we use **row sums**, the sum of all entries in a row. We compute that sum in two ways: by summing the entries in the row, and by applying the relevant row operations. As long as the two computations give the same answer we can be confident that our calculation is correct.



at the end of the calculation we check our answer, by substituting in the original linear system, or going back to the problem giving rise to the linear system.

Example: Polynomial Interpolation

- Suppose we want to find a cubic polynomial

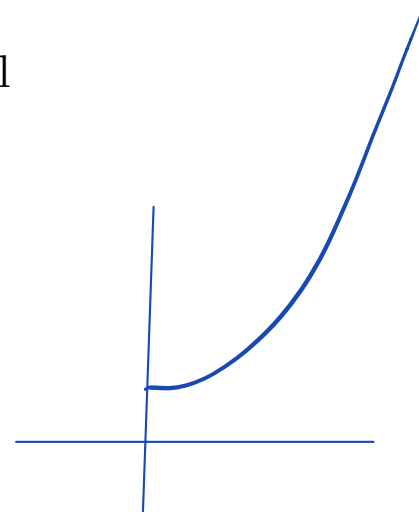
$$p(x) = a + bx + cx^2 + dx^3$$

that agrees with the exponential function

$$f(x) = 2^x$$

at the points

$$x = 1, 2, 3, 4.$$



- We get the linear system:

$$\begin{array}{rcccccccl} p(1) & = & a & + & b & + & c & + & d & = & 2^1 & = & 2 \\ p(2) & = & a & + & 2b & + & 4c & + & 8d & = & 2^2 & = & 4 \\ p(3) & = & a & + & 3b & + & 9c & + & 27d & = & 2^3 & = & 8 \\ p(4) & = & a & + & 4b & + & 16c & + & 64d & = & 2^4 & = & 16 \end{array}$$



| # | a | b | c | d | RHS | RS | RHS |
|------------|---|---|----|----|-----|-----|-----|
| ① | 1 | 1 | 1 | 1 | 2 | 6 | 1 |
| ② | 1 | 2 | 4 | 8 | 4 | 19 | 2 |
| ③ | 1 | 3 | 9 | 27 | 8 | 48 | 3 |
| ④ | 1 | 4 | 16 | 64 | 16 | 101 | 5 |
| ⑤ = ② - ① | | 1 | 3 | 7 | 2 | 13 | 1 |
| ⑥ = ③ - ① | | 2 | 8 | 26 | 6 | 42 | 2 |
| ⑦ = ④ - ① | | 3 | 15 | 63 | 14 | 95 | 4 |
| ⑧ = ⑥ - 2⑤ | | | 2 | 12 | 2 | 16 | 0 |
| ⑨ = ⑦ - 3⑤ | | | 6 | 42 | 8 | 56 | 1 |
| ⑩ = ⑨ - 3⑧ | | | | 6 | 2 | 8 | 1 |

$$\textcircled{10} \quad 6d = 2 \quad d = \frac{1}{3}$$

$$6d = 1 \quad d = \frac{1}{6}$$

$$\textcircled{8} \quad 2c + 12d = 2 = 2c + 4 \quad c = -1 \quad \begin{matrix} 2c + 2 = 0 \\ c = -1 \end{matrix}$$

$$\textcircled{5} \quad b = 2 - 3c - 7d = 2 + 3 - \frac{7}{3} = \frac{8}{3}$$

$$b + 3c + 7d = 2$$

$$b = 1 - 3c - 7d = 1 + 3 - \frac{7}{6} = \frac{17}{6}$$

$$\textcircled{1} \quad a = 2 - b - c - d = 2 - \frac{8}{3} + 1 - \frac{1}{3} = 0$$

$$p(x) = \frac{x^3}{3} - x^2 + \frac{8}{3}x$$

$$a = 1 - b - c - d$$

$$= 1 - \frac{17}{6} + 1 - \frac{1}{6} = -1$$

$$p(x) = \frac{x^3}{6} - x^2 + \frac{17}{6}x - 1$$

- Here is a printed version of the calculation we just carried out:

| # | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> | RHS | RS |
|-----------------------|-------------------------------------|----------|-----------------------|----------|----------------|---------------|
| 1 | 1 | 1 | 1 | 1 | 2 | 6 |
| 2 | 1 | 2 | 4 | 8 | 4 | 19 |
| 3 | 1 | 3 | 9 | 27 | 8 | 48 |
| 4 | 1 | 4 | 16 | 64 | 16 | 101 |
| 5 = 2 - 1 | | 1 | 3 | 7 | 2 | 13 |
| 6 = 3 - 1 | | 2 | 8 | 26 | 6 | 42 |
| 7 = 4 - 1 | | 3 | 15 | 63 | 14 | 95 |
| 8 = 6 - 2 × 5 | | | 2 | 12 | 2 | 16 |
| 9 = 7 - 3 × 5 | | | 6 | 42 | 8 | 56 |
| 10 = 9 - 3 × 8 | | | | 6 | 2 | 8 |
| 10 \implies | | | $6d$ | $= 2$ | $\implies d =$ | $\frac{1}{3}$ |
| 8 \implies | | | $2c + 4$ | $= 2$ | $\implies c =$ | -1 |
| 5 \implies | | | $b - 3 + \frac{7}{3}$ | $= 2$ | $\implies b =$ | $\frac{8}{3}$ |
| 1 \implies | $a + \frac{8}{3} - 1 + \frac{1}{3}$ | | | $= 2$ | $\implies a =$ | 0 |

Checking your Answer

- You **always check your answers!**
- In this case we could substitute our values for a , b , c and d in our original linear system.
- But we are not really interested in the linear system as such. It only served as a stepping stone to finding the cubic polynomial that interpolates the exponential. So it is better to check that it actually accomplishes its purpose.
- We get

$$p(x) = \frac{8}{3}x - x^2 + \frac{x^3}{3}$$

and hence

$$p(1) = \frac{8}{3} - 1 + \frac{1}{3} = 2$$

$$p(2) = \frac{16}{3} - 4 + \frac{8}{3} = 4$$

$$p(3) = \frac{8 \times 3}{3} - 9 + \frac{27}{3} = 8$$

$$p(4) = \frac{32}{3} - 16 + \frac{64}{3} = 16$$

as required.

- Figure 1 provides an even more impressive demonstration of the correctness of our calculation. It shows the graphs of p and of f .
- In the interval $[1, 4]$ the two graphs are essentially indistinguishable.

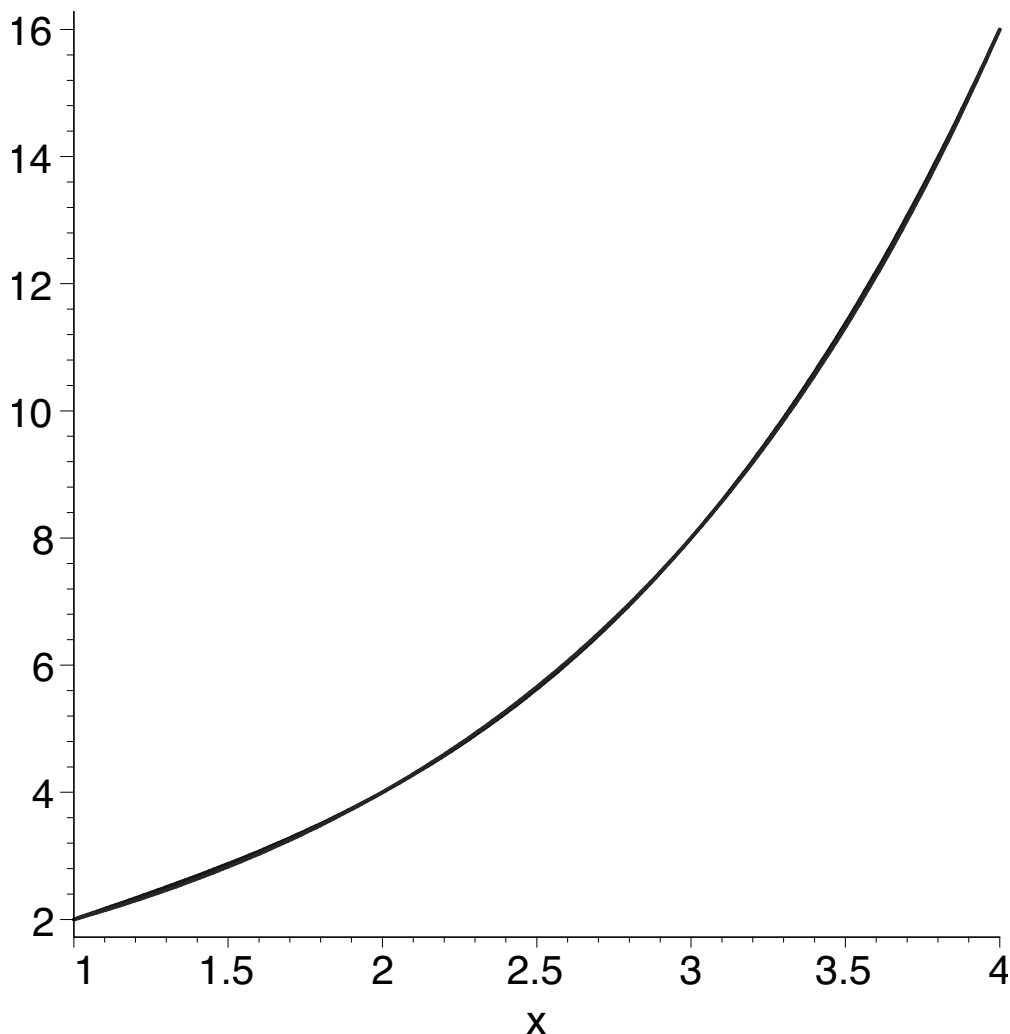


Figure 1. f and its cubic interpolant.

- Plotting the error, i.e., the graph of $f - p$, as in Figure 2, shows that in fact the two functions are distinct. Note the vertical scale!
- Figure 3, showing the graphs of p and f , illustrates that interpolation works well only in the interval containing the interpolation points. Again, notice the vertical scale.

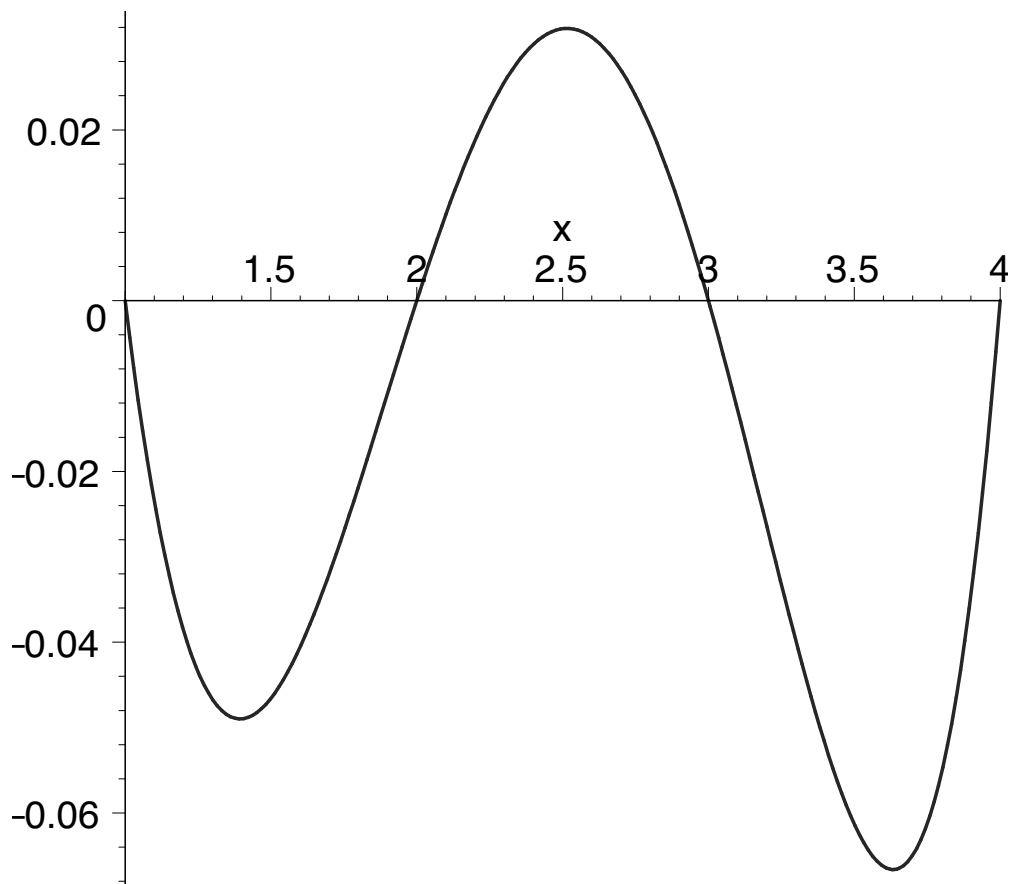


Figure 2. Interpolation Error, $f - p$.

- You may wonder why we keep track of how we actually obtain the individual rows in this process.
- It happens frequently that we need to solve several linear systems, all of which have the same coefficient matrix. In such a case we want to process the coefficients only once!
- For example, suppose we want

$$p(1) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad p(4) = 5.$$
- Then we just modify the previous calculation.

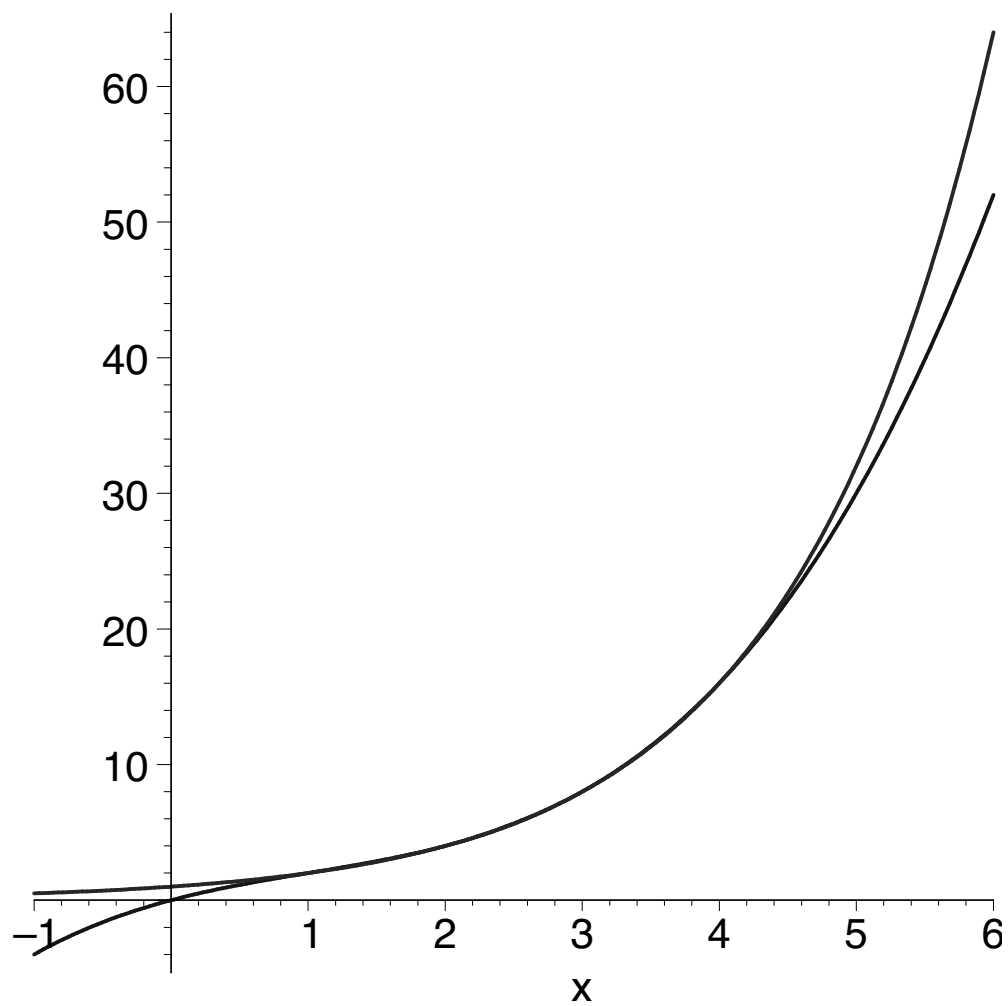


Figure 3. Extrapolation, graphs of f and p .

| # | <i>a</i> | <i>b</i> | <i>c</i> | <i>d</i> | RHS | RS |
|-----------------------|----------|----------|----------|----------|-----|-------|
| 1 | 1 | 1 | 1 | 1 | 2 | 6 1 |
| 2 | 1 | 2 | 4 | 8 | 4 | 19 2 |
| 3 | 1 | 3 | 9 | 27 | 8 | 48 3 |
| 4 | 1 | 4 | 16 | 64 | 16 | 101 5 |
| 5 = 2 - 1 | | 1 | 3 | 7 | 2 | 13 1 |
| 6 = 3 - 1 | | 2 | 8 | 26 | 6 | 42 2 |
| 7 = 4 - 1 | | 3 | 15 | 63 | 14 | 95 4 |
| 8 = 6 - 2 × 5 | | | 2 | 12 | 2 | 16 0 |
| 9 = 7 - 3 × 5 | | | 6 | 42 | 8 | 56 1 |
| 10 = 9 - 3 × 8 | | | | 6 | 2 | 8 1 |

$$\begin{array}{lclclcl}
 10 & \implies & 6d & = & 1 & \implies & d & = & \frac{1}{6} \\
 8 & \implies & 2c + 2 & = & 0 & \implies & c & = & -1 \\
 5 & \implies & b - 3 + \frac{7}{6} & = & 1 & \implies & b & = & \frac{17}{6} \\
 1 & \implies & a + \frac{17}{6} - 1 + \frac{1}{6} & = & 1 & \implies & a & = & -1
 \end{array}$$

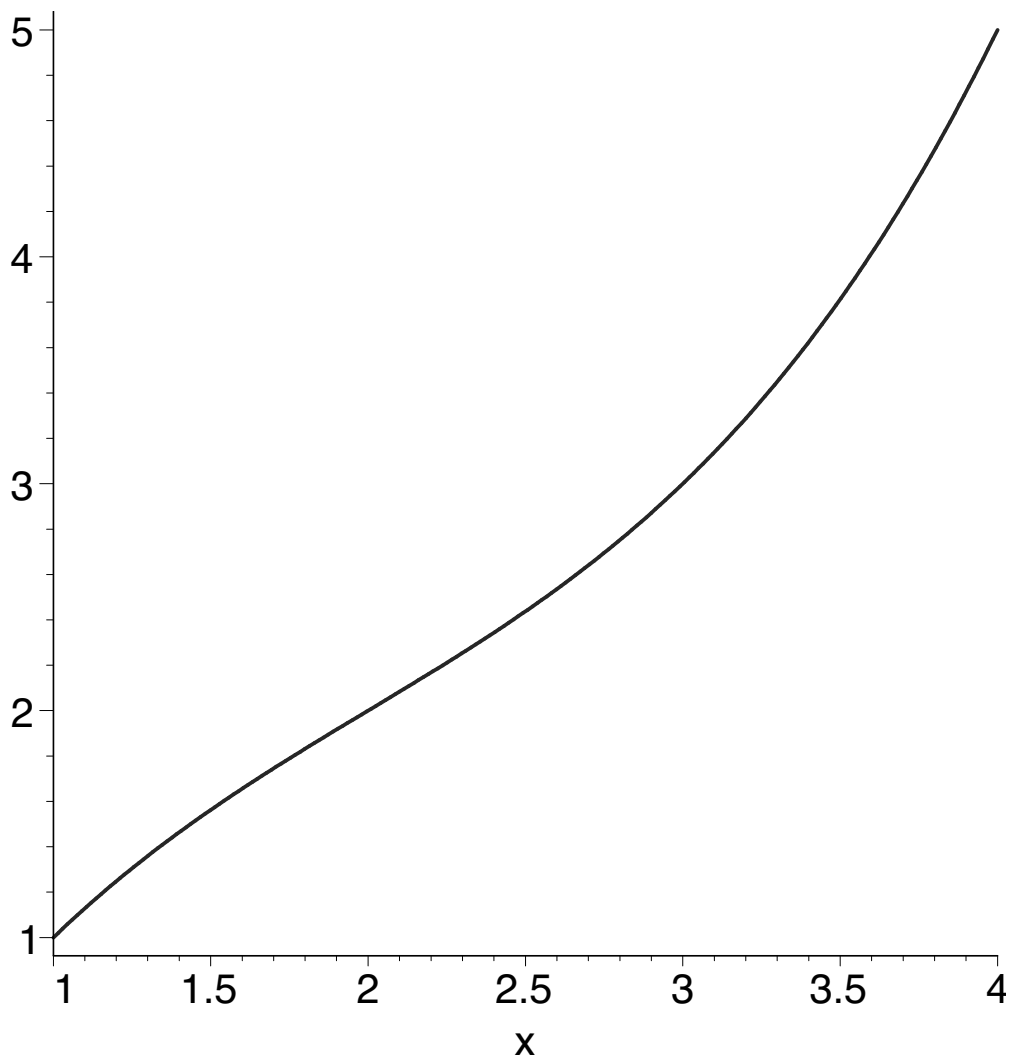


Figure 4. $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$.

- Of course we check our answer. We get

$$p(x) = -1 + \frac{17}{6}x - x^2 + \frac{x^3}{6}$$

and

$$p(1) = -1 + \frac{17}{6} - 1 + \frac{1}{6} = 1$$

$$p(2) = -1 + \frac{17 \times 2}{6} - 4 + \frac{8}{6} = 2$$

$$p(3) = -1 + \frac{17 \times 3}{6} - 9 + \frac{27}{6} = 3$$

$$p(4) = -1 + \frac{17 \times 4}{6} - 16 + \frac{64}{6} = 5$$