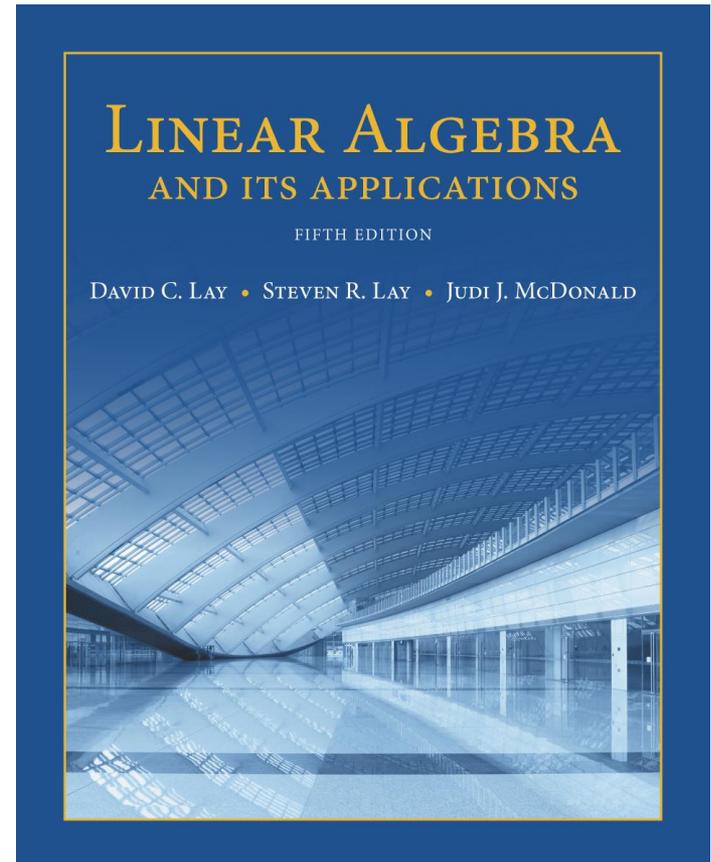


2

Matrix Algebra

2.3

CHARACTERIZATIONS OF INVERTIBLE MATRICES



THE INVERTIBLE MATRIX THEOREM

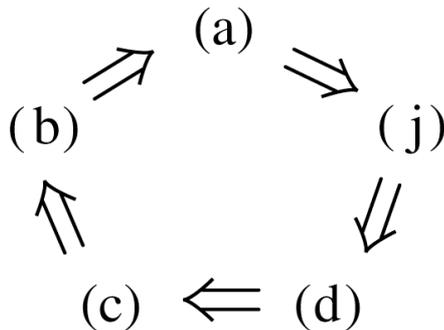
- **Theorem 8:** Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.
 - a. A is an invertible matrix.
 - b. A is row equivalent to the $n \times n$ identity matrix.
 - c. A has n pivot positions.
 - d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
 - e. The columns of A form a linearly independent set.

THE INVERTIBLE MATRIX THEOREM

- f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
- h. The columns of A span \mathbb{R}^n .
- i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- j. There is an $n \times n$ matrix C such that $CA = I$.
- k. There is an $n \times n$ matrix D such that $AD = I$.
- l. A^T is an invertible matrix.

THE INVERTIBLE MATRIX THEOREM

- First, we need some notation.
- If the truth of statement (a) always implies that statement (j) is true, we say that (a) *implies* (j) and write $(a) \Rightarrow (j)$.
- The proof will establish the “circle” of implications as shown in the following figure.



- If any one of these five statements is true, then so are the others.

THE INVERTIBLE MATRIX THEOREM

- Finally, the proof will link the remaining statements of the theorem to the statements in this circle.
- **Proof:** If statement (a) is true, then A^{-1} works for C in (j), so $(a) \Rightarrow (j)$.
- Next, $(j) \Rightarrow (d)$.
- Also, $(d) \Rightarrow (c)$.
- If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n .
- Thus $(c) \Rightarrow (b)$.
- Also, $(b) \Rightarrow (a)$.

THE INVERTIBLE MATRIX THEOREM

- This completes the circle in the previous figure.
- Next, $(a) \Rightarrow (k)$ because A^{-1} works for D .
- Also, $(k) \Rightarrow (g)$ and $(g) \Rightarrow (a)$.
- So (k) and (g) are linked to the circle.
- Further, (g) , (h) , and (i) are equivalent for any matrix.
- Thus, (h) and (i) are linked through (g) to the circle.
- Since (d) is linked to the circle, so are (e) and (f) , because (d) , (e) , and (f) are all equivalent for *any* matrix A .
- Finally, $(a) \Rightarrow (l)$ and $(l) \Rightarrow (a)$.
- This completes the proof.

THE INVERTIBLE MATRIX THEOREM

- Theorem 8 could also be written as “The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each \mathbf{b} in \mathbb{R}^n .”
- This statement implies (b) and hence implies that A is invertible.
- The following fact follows from Theorem 8.
Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.
- The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices.

THE INVERTIBLE MATRIX THEOREM

- Each statement in the theorem describes a property of every $n \times n$ invertible matrix.
- The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix.
- For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n , does *not* have n pivot position, and has linearly *dependent* columns.

THE INVERTIBLE MATRIX THEOREM

- **Example 1:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution:**

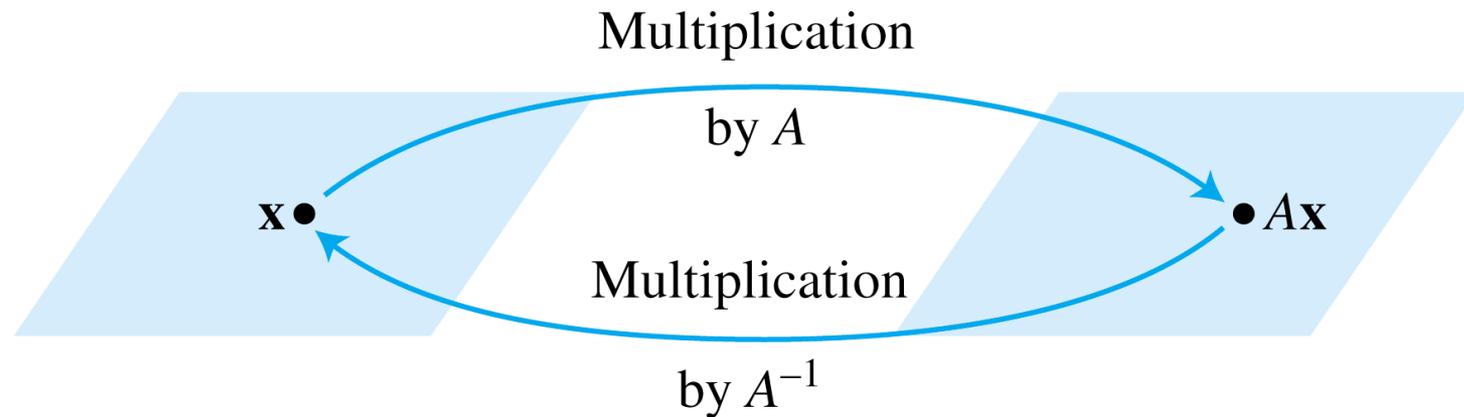
$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

THE INVERTIBLE MATRIX THEOREM

- So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).
- The Invertible Matrix Theorem *applies only to square matrices*.
- For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $Ax = b$.

INVERTIBLE LINEAR TRANSFORMATIONS

- Matrix multiplication corresponds to composition of linear transformations.
- When a matrix A is invertible, the equation $A^{-1}Ax = x$ can be viewed as a statement about linear transformations. See the following figure.



A^{-1} transforms $A\mathbf{x}$ back to \mathbf{x} .

INVERTIBLE LINEAR TRANSFORMATIONS

- A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (1)$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \quad (2)$$

- **Theorem 9:** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T . Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equation (1) and (2).

INVERTIBLE LINEAR TRANSFORMATIONS

- **Proof:** Suppose that T is invertible.
- Then (2) shows that T is onto \mathbb{R}^n , for if \mathbf{b} is in \mathbb{R}^n and $\mathbf{x} = S(\mathbf{b})$, then $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$, so each \mathbf{b} is in the range of T .
- Thus A is invertible, by the Invertible Matrix Theorem, statement (i).
- Conversely, suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}\mathbf{x}$. Then, S is a linear transformation, and S satisfies (1) and (2).
- For instance, $S(T(\mathbf{x})) = S(A\mathbf{x}) = A^{-1}(A\mathbf{x}) = \mathbf{x}$.
- Thus, T is invertible.