

Math 2270-6

Notes of 11/13/2019

Announcement

- **Undergraduate Colloquium**

today, 12:~~45~~⁵⁵-1:45, LCB 225

- Ryleigh Moore will present:

Adventures in the Arctic

Mathematics Ph.D. student, Ryleigh Moore, was one of three American graduate students invited to participate in the Multidisciplinary drifting Observatory for the Study of Arctic Climate (MOSAiC) expedition out of Tromsø, Norway, from September 20 - October 28, 2019. The flagship German icebreaker, RV Polarstern, will be frozen in ice and drift for a full year through the Central Arctic following in the footsteps of an earlier 19th century expedition under Norwegian explorer, Fridtjof Nansen. MOSAiC will mark the first time a modern research icebreaker will study near the North Pole throughout the polar winter; it is hailed as the largest Central Arctic expedition ever, with 19 countries, over 600 people, and a budget exceeding 155 million US dollars.

In this talk, Ryleigh will discuss the science goals of MOSAiC and her experiences while on

the Russian research vessel, Akademik Fedorov. She will discuss life on a research vessel, how the expedition identified ice floes that would be used for instrument deployments, and her role of leading the installation of three seasonal ice mass balance (SIMB3) buoys in the Central Arctic.

More on Least Squares

- Suppose we have the overdetermined linear system

$$A\mathbf{x} = \mathbf{b}$$

where A is $m \times n$ and $m > n$.

- We discussed solving instead the Least Squares Problem

$$\|A\mathbf{x} - \mathbf{b}\| = \min \quad (1)$$



Note that any solution \mathbf{x} of (1) is also a solution of the problem

$$\|A\mathbf{x} - \mathbf{b}\|^2 = \min \quad (2)$$

- We saw that we can find \mathbf{x} by solving the **normal equations**

$$A^T A\mathbf{x} = A^T \mathbf{b}.$$

- Moreover, we saw that
0. Geometrically we get $A\mathbf{x}$ by projecting \mathbf{b} into the column space of A and we get the normal equations by observing that $A\mathbf{x} - \mathbf{b}$ is in the orthogonal complement of the column space of A .
 1. $A\mathbf{x}$ exists and is unique for every \mathbf{b} .
 2. \mathbf{x} is unique if the columns of A are linearly independent.
 3. $A^T A$ is invertible if and only if the columns of A are linearly independent.
 4. In that case we get

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

although we would not normally actually compute the inverse of $A^T A$

- Example: Suppose you have a some data (x_i, y_i) , $i = 1, 2, \dots, n$ and you have reason to believe that the answer is some constant plus a 2π -periodic function that looks something like a sine curve. In other words,

$$y_i \approx f(x_i) = \alpha + \beta \sin x_i + \gamma \cos x_i.$$

- In equation form this is

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} 1 & \sin x_1 & \cos x_1 \\ 1 & \sin x_2 & \cos x_2 \\ \vdots & \vdots & \vdots \\ 1 & \sin x_n & \cos x_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- This is an overdetermined linear system. As we discussed in class yesterday, its least squares solution \mathbf{x} is given by the normal equations

$$A^T A \mathbf{x} = A^T \mathbf{b}$$

where, in this case, with all sums running from 1 to n ,

$$A^T A = \begin{bmatrix} \sum 1 & \sum \sin x_i & \sum \cos x_i \\ \sum \sin x_i & \sum \sin^2 x_i & \sum \sin x_i \cos x_i \\ \sum \cos x_i & \sum \cos x_i \sin x_i & \sum \cos^2 x_i \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} \sum y_i \\ \sum y_i \sin x_i \\ \sum y_i \cos x_i \end{bmatrix}$$

- Example: Temperatures in Salt Lake City.
- You'd expect (low and high daily) temperatures $T(t)$ in SLC to follow (roughly) some kind of sin curve such as

$$T(t) = a + b \sin \frac{2\pi t}{12} + c \cos \frac{2\pi t}{12} \quad (3)$$

where t is the time of the year, measured in months. a is the mean temperature over the whole year. b and c are suitable coefficients. I found the following data at

<http://www.rssweather.com/climate/Utah/Salt%20Lake%20City/>

Month	Low	High
Jan	21.0F	37.0F
Feb	25.5F	43.F
Mar	33.4F	52.8F
Apr	39.0F	60.9F
May	46.9F	70.6F
Jun	55.8F	82.2F
Jul	63.4F	90.6F
Aug	62.4F	88.7F
Sept	52.4F	77.6F
Oct	41.0F	64.0F
Nov	30.4F	48.7F
Dec	22.4F	38.0F

- Suppose you want to find the Least Squares approximation of the form (3) that best approximates the data.

- It would be unreasonable to do this problem by hand. Numerous facilities (e.g., matlab, maple, wolfram alpha, any number of programming languages) exist to get a computer to do the necessary calculations.
- The following matlab code and its outputs show the results of doing a Least Squares Approximation.

```

1 % approximate low and high temperatures
in SLC by sin and cos
2 % curve
3 % see http://www.rssweather.com/climate/Utah/Salt%20Lake%2
4
5 low=[21.3;25.5;33.4;39.0;46.9;55.8;63.4;62.4;52.4;41.0;30.
6
7 pi = 2.0*acos(0.0);
8
9 for i =1:12
10     O(i,1) = 1.0;
11     S(i,1) = sin(2*i/12*pi);
12     C(i,1) = cos(2*i/12*pi);
13 end
14
15 A=[O'*O,O'*S,O'*C;S'*O,S'*S,S'*C;C'*O,C'*S,C'*C]
16 B=[O'*low;S'*low;C'*low];
17
18 X= A\B
19
20 for i = 1:12
21     app = X(1) + X(2)*S(i,1) + X(3)*C(i,1);
22     err = low(i)-app;
23     res = [i,low(i), app, err];
24     disp(res)
25 end

```

$$"A^T A" = A$$

1 A =
 2
 3 12.0000 -0.0000 -0.0000
 4 -0.0000 6.0000 0.0000
 5 -0.0000 0.0000 6.0000

6
 7 X =
 8
 9 41.1583
 10 -10.9147
 11 -16.9332

12
 13 1.0000 21.3000 21.0364 0.2636
 14 2.0000 25.5000 23.2393 2.2607
 15 3.0000 33.4000 30.2436 3.1564
 16 4.0000 39.0000 40.1725 -1.1725
 17 5.0000 46.9000 50.3655 -3.4655
 18 6.0000 55.8000 58.0915 -2.2915
 19 7.0000 63.4000 61.2803 2.1197
 20 8.0000 62.4000 59.0774 3.3226
 21 9.0000 52.4000 52.0731 0.3269
 22 10.0000 41.0000 42.1442 -1.1442
 23 11.0000 30.4000 31.9511 -1.5511
 24 12.0000 22.4000 24.2252 -1.8252
 25

```

1 % approximate low and high temperatures
in SLC by sin and cos
2 % curve
3 % see http://www.rssweather.com/climate/Utah/Salt%20Lake%2
4
5 high=[37.0;43.4;52.8;60.9;70.6;82.2;90.6;88.7;77.6;64.0;48
6
7 pi = 2.0*acos(0.0);
8
9 for i =1:12
10     O(i,1) = 1.0;
11     S(i,1) = sin(i/6*pi);
12     C(i,1) = cos(i/6*pi);
13 end
14
15 A=[O'*O,O'*S,O'*C;S'*O,S'*S,S'*C;C'*O,C'*S,C'*C]
16 B=[O'*high;S'*high;C'*high];
17
18 X= A\B
19
20 for i = 1:12
21     app = X(1) + X(2)*S(i,1) + X(3)*C(i,1);
22     err = high(i)-app;
23     res = [i,high(i), app, err];
24     disp(res)
25 end

```

$$\begin{array}{r}
 1 \\
 2 \text{ A} = \\
 3 \\
 4 \quad 12.0000 \quad -0.0000 \quad -0.0000 \\
 5 \quad -0.0000 \quad 6.0000 \quad 0.0000 \\
 6 \quad -0.0000 \quad 0.0000 \quad 6.0000 \\
 7 \\
 8
 \end{array}$$

$$\begin{array}{r}
 9 \text{ X} = \\
 10 \\
 11 \quad 62.8750 \\
 12 \quad -13.7609 \\
 13 \quad -21.7808 \\
 14
 \end{array}$$

$$\begin{array}{r}
 15 \quad 1.0000 \quad 37.0000 \quad 37.1318 \quad -0.1318 \\
 16 \quad 2.0000 \quad 43.4000 \quad 40.0673 \quad 3.3327 \\
 17 \quad 3.0000 \quad 52.8000 \quad 49.1141 \quad 3.6859 \\
 18 \quad 4.0000 \quad 60.9000 \quad 61.8481 \quad -0.9481 \\
 19 \quad 5.0000 \quad 70.6000 \quad 74.8573 \quad -4.2573 \\
 20 \quad 6.0000 \quad 82.2000 \quad 84.6558 \quad -2.4558 \\
 21 \quad 7.0000 \quad 90.6000 \quad 88.6182 \quad 1.9818 \\
 22 \quad 8.0000 \quad 88.7000 \quad 85.6827 \quad 3.0173 \\
 23 \quad 9.0000 \quad 77.6000 \quad 76.6359 \quad 0.9641 \\
 24 \quad 10.0000 \quad 64.0000 \quad 63.9019 \quad 0.0981 \\
 25 \quad 11.0000 \quad 48.7000 \quad 50.8927 \quad -2.1927 \\
 26 \quad 12.0000 \quad 38.0000 \quad 41.0942 \quad -3.0942 \\
 27
 \end{array}$$

Perhaps more impressive is a graphic display of our data and its approximations:

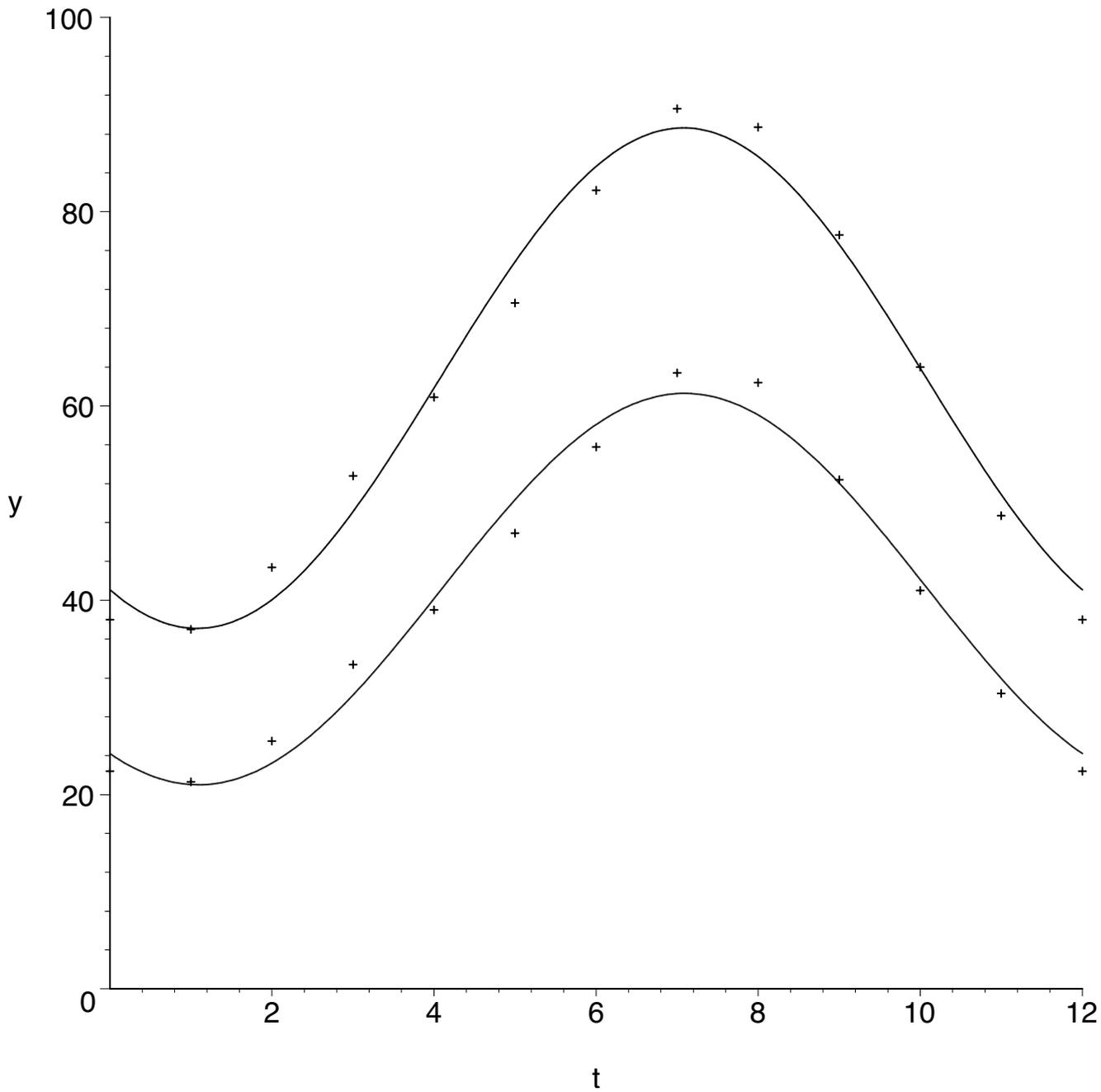


Figure 1. Temperature in Salt Lake City.

- As we've seen, things get more convenient if we have matrices with orthonormal columns. Let's first look at the case that A itself have orthonormal columns.

$$A = [a_1, \dots, a_n]$$

$$A^T A = [a_i^T a_j] = \underline{I}$$

- Next let's see what happens if we have a QR factorization of A .
- Suppose

$$A = QR$$

where R is upper triangular and Q has orthonormal columns.