Math 2270-1

Notes of 9/24/2019

Review

- A is square, $n \times n$.
- The determinant of A is given by

$$\det A = |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

  where

$$C_{ij} = (-1)^{i+j} |A_{ij}|$$

  and A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and the j-th column.

- C_{ij} is the (ij)-**cofactor** and the formula is the **cofactor expansion** of the determinant.

$$|A| = \sum_{\sigma} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

- The determinant is linear in each row and column
- In particular, multiplying a row with a scalar multiplies the determinant with that scalar.

- Interchanging two rows of A changes the sign of the determinant.
- Adding a multiple of a row to another row does not change the determinant.
- A is invertible if and only if $\det A \neq 0$.
- Transposing does not change the determinant:

$$\det A^T = \det A.$$

- The determinant is multiplicative:

$$|AB| = |A||B|$$

3.3 Cramer's Rule etc.

- Cramer's Rule (named after Gabriel Cramer, 1704–1752) is useful for some theoretical calculations. It's not a competitive numerical method!

- Consider the linear system

$$A\mathbf{x} = \mathbf{b}$$

  where once again A is an invertible square $(n \times n)$ matrix.

- define

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \dots & \mathbf{a}_n \end{bmatrix}$$

- In other words, $A_i(\mathbf{b})$ is the matrix formed by replacing the i-th column of A by $\mathbf{b}$.

- Cramer's rule states that

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}$$

- Examples:

- Proof of Cramer's Rule:

A Formula for A^{-1}

- remember our definition of cofactors:

$$C_{ij} = (-1)^{i+j}|A_{ij}|$$

- Next remember that the j-th column of A^{-1} is the solution of

$$A\mathbf{x} = \mathbf{e}_j$$

  where, as usual, $\mathbf{e}_j$ is the j-th column of the identity matrix.

- By Cramer's Rule, the (i,j)-entry x_{ij} of A^{-1} is

$$x_{ij} = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

- Finally, expand $A_i(e_j)$ by cofactors about the i-th column to see that

$$x_{ij} = \frac{\det A_i(\mathbf{e}_j)}{\det A} = \frac{C_{ji}}{\det A}.$$

- We get the formula

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- The matrix

$$\mathrm{adj}A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

  is called the **adjugate** of A.

note that the adjugate is the **transpose** of the matrix of cofactors!

- Examples

Geometric Interpretation of the determinant

- Recall these three properties:

1. The determinant of the identity matrix is 1.

2. If you interchange two rows of the matrix the determinant changes it sign.

3. The determinant is a function that is linear in each row separately. (In other words, if you think of the determinant as a function of a specific row, keeping everything else constant you get a linear function.)

- Aside: Strang uses these three properties to define determinants.

- Ignoring the sign change, these properties define the area or volume of the rectangle (in R^2) or parallelepiped (in R^3) formed by the 2 or 3 columns of the given matrix.

- The corresponding object in higher dimensions is called a **parallelotope**.

- The volume V of a parallelotope defined by the columns of A is therefore given by

$$V = |\det A|$$

where the vertical bars in this case do denote the absolute values.

- Examples

Linear Transformations

- The textbook discusses in some detail that applying a linear transformation to a geometric object multiplies the volume of the object by the absolute value of the determinant of the linear transformation.
- You actually already saw this principle in action when discussing the change of variable formula in multivariable calculus.