Cubic graphs with most automorphisms

Michael A. van Opstall, Răzvan Veliche

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1 Introduction

Let G be a connected simple cubic graph; Aut G denotes its automorphism group. Let n be half the number of vertices of G. We define the *arithmetic genus* of a (possibly disconnected) graph as $e-v+1$ where e is the number of edges, v the number of vertices of G). For a connected simple cubic graph, $g=n+1$. The definition of arithmetic genus is motivated by the following: to a projective nodal curve with rational components one may associate a so-called *dual graph*; the arithmetic genus of the curve is the arithmetic genus of this graph. We abbreviate arithmetic genus to "genus" everywhere in this article, although this is at variance with standard graph theory terminology. We trust that this will not actually be confusing.

A bound on the order of Aut G was obtained in [6], where it is shown that $|\text{Aut } G|$ divides $3n\cdot 2^n$. However, it can be easily checked by consulting a list of cubic graphs[1], that this bound is only rarely attained (in fact, it is only attained for graphs with four or six vertices). Thus a natural problem is to find a sharp bound for the order of Aut G. We solve this problem by the following:

Theorem 1.1. *Assume $g\geq 9$; set $l(g)=\min\{k|g=\sum_{i=1}^{k} a_i\cdot 2^{n_i}, a_i\in\{1,3\}\}$, and set $o(g)=g-l(g)$.*

- *if $g=9\cdot 2^m+s$ ($s=0,1,2$) ($m\geq 0$) except $g=10,11,19,20,38$, then* $|\text{Aut } G|\leq 3\cdot 2^{o(g)}$
- *if $g=3\cdot 2^m+s$ ($s=0,1,2$) ($m\geq 2$), or $g=9(2^m+2^p)$ (with $|m-p|\geq 5$) or if $g=10,11,19,20,38$, then* $|\text{Aut } G|\leq \frac{3}{2}\cdot 2^{o(g)}$
- *if $g=5\cdot a\cdot 2^m+1$ (where $a=1$ or 3, $m\geq 2$), then* $|\text{Aut } G|\leq \frac{5}{4}\cdot 2^{o(g)}$
- *otherwise,* $|\text{Aut } G|\leq 2^{o(g)}$

Moreover, these bounds are sharp; an explicit construction of graphs attaining the bounds in each case will be given in a subsequent section (see 4.1).

The graphs with maximal automorphism group for $g\leq 8$ will be listed in a table below.

This work was motivated by our earlier work [4] on maximal order automorphism groups of stable curves. Aaron Bertram asked us if we could bound the automorphism groups of stable curves with all rational components. This is equivalent to the problem of finding the maximal order automorphism groups of cubic multigraphs (with a slightly modified notion of graph automorphism). Such a result may indeed be obtained along the lines of this article and is pursued in [5].

The basic idea is as follows: once the genus is large enough (larger than eight), the graphs with the most automorphisms should be as nearly trees as possible. Of course a tree cannot be trivalent, and all trees have genus zero. Subject to these restrictions, we need to attach "appendages" of positive genus to trees in an optimal way. One sees that this is easiest when the appendages have the smallest possible genus. Restricting to simple graphs forces the appendages to have genus at least three, which in turn forces us to consider graphs slightly more general

[1] For example, Beresford's Gallery of Cubic Graphs, which can be viewed at `http://www.mathpuzzle.com/BeresfordCubic.html`. This gallery is complete for graphs of at most than twelve vertices.

than trees for the "cores" of our graphs. If we consider cubic multigraphs, then there are appendages of genus two (a triangle with one edge doubled) and genus one (a loop). The answers to Bertram's question are graphs with loops as appendages. We will not pursue these questions on non-simple graphs in this article.

Our appendages are shown in Figure 1, and have genus three and four. The "pinched tetrahedra" were used previously, for example, in an article of Wormald [6]. A graph of genus 16 with $8^5 = 2^{o(16)}$ automorphisms may be constructed by attaching four copies of the "pinched $K_{3,3}$" around the ends of a binary tree with four leaves (with the root vertex removed so the graph is cubic). To reach the bound given for genus 18, we factor 18 as $3 \cdot 6$. Our goal is to arrange six pinched tetrahedra around a core as symmetrically as possible. This is achieved by attaching them in three pairs to binary trees with two leaves, and then arranging these binary trees around a star with four vertices.

The main idea is to construct a candidate graph in each genus whose automorphism group has order equal to the bounds in the Main Theorem, thereby giving a lower bound for the upper bound we seek. For a graph G assumed to be optimal, we then remove the orbit of a well-chosen edge and attempt to proceed by induction. However, removing this orbit in general causes problems - the components of the remaining graph are not cubic, making them cubic may lead to a non-simple graph, some components may be cycles, which cannot be made cubic in any useful way. When one of these problems occurs, we will show that it constrains the automorphism group of the graph so that its order is smaller than that of the candidate graph's automorphism group.

Graphs are assumed to be connected, unless otherwise stated. We will actually have to work extensively with disconnected graphs, but this will be clearly stated.

It is a pleasure to thank Marston Conder for his assistance in pointing out the results of Goldschmidt's article, allowing us to significantly shorten the elimination of edge-transitive graphs from consideration. Professor Conder also made some helpful suggestions on an early version of the manuscript.

2 Technicalities

Definition 2.1. For a natural number g, define the functions

$$\begin{aligned} l(g) &= \min\{k : g = \sum_{i=1}^{k} a_i \cdot 2^{n_i}, a_i \in \{1,3\}\} \\ o(g) &= g - l(g). \end{aligned}$$

The function l may be computed as follows: expand g in binary and starting from the left, count the number of pairs 10 or 11, possibly adding one to the total if after pairing there is a 1 in the last digit. For example $l(15) = 2$, $l(21) = 3$.

We will make extensive use of various inequalities involving the function o; we collect them here for convenience:

Proposition 2.2.

- $\frac{3}{2}k < 2^{o(k+1)}$ *for all* k.
- $k < 2^{2+o(\lceil \frac{k}{6} \rceil + 1)}$ *for all* $k \geq 1$.
- $k < 2^{o(\lceil \frac{2k}{3} \rceil + 1)}$ *for all* k.
- $3k < 2^{o(2k-2)}$ *for* $k \geq 4$.
- $k \leq 2^{o(k-1)}$ *for* $k \geq 4$*, with strict inequality for* $k \geq 5$.
- $k \leq 2^{\lfloor \frac{k+1}{2} \rfloor}$*, with strict inequality for* $k \neq 2, 4$.
- $o(\lceil \frac{k}{2} \rceil + 1) - \lfloor \frac{k}{6} \rfloor \geq o(\lceil \frac{k}{8} \rceil + 1)$.
- $k \leq 2^{o(\lceil \frac{k}{2} \rceil + 1)}$*, with strict inequality for* $k \neq 2, 4, 8$.

Proof. Straightforward. ◻

We list also some properties of the function l which will be useful in what follows:

Lemma 2.3. *We have the following (in)equalities:*

- $l(a) = 1$ *if and only if* $a = 2^m$ *or* $a = 3 \cdot 2^m$ *for some* $m \geq 0$.
- $l(a) \leq \lceil \frac{\log_2(s)}{2} \rceil$.
- $l(a \cdot b) \leq 2l(a) \cdot l(b)$.
- $l(2a) = l(a)$.
- $o(2a) = o(a) + a$.
- $l(3a) \leq 2l(a)$.
- $l(a+1) \leq l(a) + 1$.
- $o(a) = 1$ *if and only if* $a = 2$*;* $o(a) \geq 2$ *for* $a \geq 3$.
- $2^{l(a)} \leq 2\sqrt{a}$.
- $l(a) = 2$ *and* $l(3a) = 4$ *if and only if* $a = 3 \cdot (2^m + 2^p)$ *with* $m, p \geq 0$ *and* $|m - p| \geq 5$.

Proof. Only the last two parts deserve some explanations: for u such that $2^u \leq a < 2^{u+1}$, the binary decomposition of a will have $u+1$ digits; writing a as sums in the definition of $l(a)$ effectively forms groups of at least two digits in this binary form, plus at most an extra one at the end; there are thus at most $\frac{u+2}{2}$ such groups, so $l(a) \leq \frac{u+1}{2}$; then $2^{l(a)} \leq 2\sqrt{2^u} \leq 2\sqrt{a}$. It is easy to see that the inequality is strict as soon as $a > 1$.

For the last part, note that $l(a) = 2$ means $a = b \cdot 2^m + c \cdot 2^m$ with $b, c \in \{1, 3\}$. It is immediate to check that if at least one of b, c is not 3, then $l(3a) \geq 3$, and moreover, even if both $b = c = 3$ one must have $|m - p| \geq 5$ for $l(3a) = 4$ to happen. ◻

Lemma 2.4. *The inequality*

$$sl(h) - l(sh) \geq \lfloor \frac{s+1}{2} \rfloor$$

is:

- *strict for any* h *if* $s = 4, 6$ *or* $s \geq 8$
- *strict for* $l(h) \geq 2$ *and any* $s \geq 2$, $s \neq 3$
- *strict for* $l(h) \geq 3$ *and* $s = 3$
- *equality for* $l(h) = 1$ *and* $s = 2, 5, 7$, *or* $l(h) = 2$ *and* $s = 3$
- *false for* $s = 1$ *or* $l(h) = 1$ *and* $s = 3$

Proof. We begin by noting that $l(sh) \leq 2l(s)l(h)$, so $sl(h) - l(sh) \geq l(h)(s - 2l(s)) = l(h)(2o(s) - s)$. Since $l(h) \geq 1$ we would like to see from what value of s we have $2o(s) - s \geq \lfloor \frac{s+1}{2} \rfloor$, or equivalently $2o(s) \geq \lfloor \frac{3s+1}{2} \rfloor$.

If $2^u \leq s < 2^{u+1}$ then $l(s) \leq \lceil \frac{u+1}{2} \rceil = \lfloor \frac{u}{2} \rfloor + 1$. Then $2l(s) \leq 2\lfloor \frac{u}{2} \rfloor + 2 \leq u + 2$, so $2o(s) = 2s - 2l(s) \geq 2s - u - 2$. The inequality we would like to prove becomes $2s - u - 2 \geq \lfloor \frac{3s+1}{2} \rfloor$ or $s - \lfloor \frac{s+1}{2} \rfloor \geq u + 2$. This in turn is implied by $\frac{s-1}{2} \geq u + 2$ or $s \geq 2u + 5$. Since $s \geq 2^u$ and $2^u > 2u + 5$ for $u \geq 4$ we see that the initial inequality is strict for $s \geq 16$.

Now one may construct a table of values for both sides of the inequality for values of s up to 15; we use the inequalities in (2.3) above:

s	$sl(h)-l(sh)$	$\lfloor \frac{s+1}{2} \rfloor$	comments
1	0	1	always false
2	$l(h)$	1	not strict for $l(h)=1$
3	$\geq l(h)$	2	strict for $l(h)\geq 3$
4	$3l(h)$	2	always strict
5	$\geq 3l(h)$	3	not strict for $l(h)=1$
6	$\geq 4l(h)$	3	always strict
7	$\geq 4l(h)$	4	not strict for $l(h)=1$
8	$7l(h)$	4	always strict
9	$\geq 7l(h)$	5	always strict
10	$\geq 8l(h)$	5	always strict
11	$\geq 8l(h)$	6	always strict
12	$\geq 10l(h)$	6	always strict
13	$\geq 10l(h)$	7	always strict
14	$\geq 11l(h)$	7	always strict
15	$\geq 11l(h)$	8	always strict

(1)

The lemma follows now immediately. $\square$

In the future, we will denote by $A(s,h)$ the quantity $sl(h)-l(sh)-\lfloor \frac{s-1}{2} \rfloor$, and by $B(s,h)$ the quantity $sl(h)-l(sh)$; the lemma may be interpreted as giving ranges of s and h for which $A(s,h)\geq 1$.

3 Eliminating edge-transitive graphs

For an edge e of a graph G, we will denote by $O(e)$ its orbit via the automorphism group of G. We use the word "edge" here to denote the graph with two vertices joined by an edge, so that $O(e)$ is a graph. Define the function $M(G)$ as the number of edges in a minimal orbit of an edge.

We refer to the star on four vertices simply as a "star", since we consider no stars with more vertices.

Lemma 3.1. *Let e be an edge of G such that $O(e)$ has minimal order among all orbits of edges of G. Then only the following possibilities occur:*

- *$G=O(e)$;*
- *$O(e)$ is a disjoint union of stars;*
- *$O(e)$ is a disjoint union of edges;*
- *$O(e)$ is a disjoint union of cycles; two such cycles are at distance at least two from each other.*

Proof. If is easy to see that if two stars in $O(e)$ have a common edge, then $G=O(e)$. Similarly, if two stars in $O(e)$ have a vertex in common, then either $G=O(e)$ or the third edge at that vertex will have an orbit of order smaller than that of $O(e)$ (which would be a contradiction to the choice of e).

Thus, if there is a star in $O(e)$, one of the first two possibilities occurs for G.

If no three edges in $O(e)$ share a common vertex, then either all edges in $O(e)$ are disjoint, or there are two edges e_1 (which may be assumed to be e, as $O(e)$ is acted upon transitively by Aut G) and e_2 in $O(e)$ with a common vertex v. Denote by f the third edge of G at v; f is then not in $O(e)$. Denote by w the other end of e.

If v and w are not in the same orbit of Aut G, then we see that $|O(e)|=2|O(v)|>|O(v)|\geq|O(f)|$, so we reach a contradiction to the choice of e. If however, $w\in O(v)$, then the existence of a cycle made of edges in $O(e)$ is immediate. Moreover, since $f\notin O(e)$, these cycles are disjoint.

Note that $|O(f)|\leq|O(e)|=|O(v)|$, with equality if and only if the ends of f are not in the same orbit; in particular two cycles in $O(e)$ cannot be at distance one from each other (the edge between them, necessarily in the orbit of f, would have both endpoints in the same orbit). $\square$

Note 3.2. If the fourth situation above occurs, we will actually choose the edge f and work with it in the arguments that follow; this is possible since $O(f)$ is also minimal, and may only be either a disjoint union of stars, or a disjoint union of edges. f (or more precisely, its orbit) in this case will be called *well-chosen*.

Theorem 3.3. *An edge-transitive graph G has at most* $384(g-1)$ *automorphisms.*

Proof. Tutte's papers [2][3] on symmetric graphs give a bound of $48(g-1)$ for such graphs, and Goldschmidt's results on semisymmetric graphs [1] imply the bound in the statement of the theorem. □

We will use a couple of other functions frequently. Define

$$\mu(g) = \max \frac{|\mathrm{Aut}\, G|}{2^{o(g)}},$$

the maximum taken over all simple cubic graphs of genus g. For an edge e of a graph, define $\mathrm{Aut}'_e G$ to be the group of automorphisms *preserving* (not necessarily fixing!) the edge e – that is, preserving the unordered pair of endpoints of e. Similarly, define

$$\mu_1(g) = \max \frac{|\mathrm{Aut}'_e G|}{2^{o(g)}},$$

the maximum here taken over all simple cubic graphs of genus g and all edges of these graphs. We are interested in the values of these functions for small g (when the optimal graphs are not our candidates). For a fixed graph G, we define $\mu_1(G)$ similarly by taking the maximum over all edges of G. Finally, set

$$\pi(G) = \max |\mathrm{Aut}'_e G|,$$

the maximum taken over all edges in G.

The following table may be compiled by inspection of lists of cubic graphs on a small number of vertices.

g	$l(g)$	$o(g)$	$\mu(g)$	$\mu_1(g)$
3	1	2	6	1
4	1	3	9	1
5	2	3	6	1
6	1	5	$\frac{15}{4}$	1
7	2	5	2	1
8	1	7	$\frac{21}{8}$	1

4 The candidate graphs

We describe now the candidates C_g for the cubic simple graphs with the most automorphisms when $g \geq 9$. The definitions make sense for smaller genus, but do not give the optimal graphs.

We need some non-standard terminology. If G is an edge-transitive graph, choose an edge e. Replace e by two edges with one endpoint in common and the other endpoints the former endpoints of e. We call this *pinching G*. This notion of pinching motivates the study of the function μ_1. If G is not edge transitive, we must specify an edge when pinching.

A tree has a unique edge or vertex through which all geodesics of maximal length pass; call this the *root*. If we attach a tree to another graph at its root, and this root is an edge, we pinch the edge and the new vertex is the point of attachment. If this attaching leads to an a vertex of higher valence, we tacitly introduce an edge to correct the problem. In most cases, the meaning of "attach" is not confusing, since we make the simplest attachment possible to keep the graph cubic. If there is possibility of confusion, we will be more explicit.

Definition 4.1.

- Define the graphs A_m as follows: attach a pinched tetrahedron to each leaf of a binary tree with 2^m leaves. A_m has genus $g = 3 \cdot 2^m$ ($m \geq 1$) and $2^{g-1} = 2^{o(g)}$ automorphisms. For $g = 3 \cdot 2^m$ ($m \geq 1$), define $C_g = A_m^{\mathrm{stab}}$ (see the next section for the definition of stabilization: in this case it ensures that the graph is cubic by shrinking the central edge pair to an edge). We note that $\pi(C_g) = 2^{o(g)}$ in this case.

A pinched tetrahedron A pinched $K_{3,3}$

Figure 1: Pinching illustrated.

- Define the graphs B_m for $m \geq 2$ as follows: attach a pinched $K_{3,3}$ to each leaf of a binary tree with 2^{m-2} leaves. B_m has genus $g = 2^m$ and $2^{g-1} = 2^{o(g)}$ automorphisms. For $g = 2^m$ ($m \geq 3$), define $C_g = B_m^{\text{stab}}$. Note also that $\pi(C_g) = 2^{o(g)}$ in this case.
- If $g = 3 \cdot 2^m$ ($m \geq 2$), C_g is defined by linking three copies of B_m at their roots, by three edges, to a common root vertex. This is easily seen to yield a simple cubic graph with $\frac{3}{4} \cdot 2^g = \frac{3}{2} \cdot 2^{o(g)}$ automorphisms. If $g = 3 \cdot 2^m + 1$, define C_g by expanding the root of the previous tree into a triangle (with the B_m attached at its vertices); this simple cubic graph has $\frac{3}{8} \cdot 2^g = \frac{3}{2} \cdot 2^{o(g)}$ automorphisms. Note that for this configuration, since $M(G) \geq 3$, we get $\pi(C_g) = \frac{1}{2} 2^{o(g)}$.
- If $g = 9 \cdot 2^m$, C_g is defined by linking three copies of A_m at their roots, by three edges, to a common root. This yields a simple cubic graph with $\frac{3}{4} \cdot 2^g = 3 \cdot 2^{o(g)}$ automorphisms. For $g = 9 \cdot 2^m + 1$ we proceed as above by inserting a triangle at the root of the previous tree; again $|\text{Aut}\, C_g| = \frac{3}{8} \cdot 2^g = 3 \cdot 2^{o(g)} (m \geq 2)$ or $|\text{Aut}\, C_g| = \frac{3}{2} \cdot 2^{o(g)} (m = 0, 1)$. We note that $\pi(C_g) = 2^{o(g)}$ for $m \geq 2$ and $\pi(C_g) = \frac{1}{2} 2^{o(g)}$ for $m = 0, 1$.
- If $g = 3 \cdot 2^m + 2$ ($m \geq 2$) or $g = 9 \cdot 2^m + 2$ ($m \geq 0$), we obtain C_g by attaching copies of B_m, respectively, A_m to the valence two vertices of a $K_{2,3}$. In both cases the genus two $K_{2,3}$ at the core yields extra symmetry for a total of $\frac{3}{8} \cdot 2^g$ automorphisms. If $g = 3 \cdot 2^m + 2$, this means $\frac{3}{2} \cdot 2^{o(g)}$ automorphisms, while if $g = 9 \cdot 2^m + 2$ one gets $3 \cdot 2^{o(g)}$ automorphisms for $m \geq 3$ and $\frac{3}{2} \cdot 2^{o(g)}$ automorphisms for $m = 0, 1, 2$. As above, we note that $\pi(C_g) = 2^{o(g)}$ for $m \geq 2$ and $\pi(C_g) = \frac{1}{2} 2^{o(g)}$ for $m = 0, 1, 2$.
- If $g = 9(2^m + 2^p) + s$ ($s = 0, 1, 2$) with $|m - p| \geq 5$, C_g is defined by attaching A_m to A_k at their roots, and arranging three copies of this configuration around a root which is a star, a triangle, or a $K_{2,3}$ depending on the value of s. This graph has $\frac{3}{2} 2^{o(g)}$ automorphisms. $\pi(C_g) = \frac{1}{2} 2^{o(g)}$ in this case.
- If $g = 2^m + 1$ ($m \geq 4$, as $m = 3$ is covered above), we attach four copies of B_m to the vertices of a square to obtain C_g; each of the quasi-trees will have genus 2^{m-2}; the order of the automorphism group of this graph is then $8 \cdot (2^{2^{m-2}-1})^4 = 2^{g-2} = 2^{o(g)}$. In this case $\pi(C_g) = \frac{1}{4} 2^{o(g)}$.
- If $g = 5 \cdot 2^m + 1$ ($m \geq 2$) or $g = 5 \cdot 3 \cdot 2^m + 1$ ($m \geq 0$), then C_g is a cycle of length five with copies of B_m, respectively A_m at its vertices. In this case one gets $\frac{5}{4} \cdot 2^{o(g)}$ automorphisms.
- C_7 is a pinched tetrahedron joined to a pinched $K_{3,3}$ by an edge.
- In all other cases, in the binary decomposition of g we have, counting from the left, at least two pairs 11 and/or 10, plus a possible 1 left over. We look at the binary decomposition of g and, from left to right, look for groups 11 and 10. We get a decomposition of g into a sum of powers of two with coefficients one or three; for each part of g of the form 2^m (with $m > 1$) we take a B_m of the corresponding genus, and for each part of g of the form $3 \cdot 2^m$ we take an A_m. If a 1 is left after this pairing, we replace the root of the last binary tree with a triangle attached the the two branches, with a free edge attached to its other branch. If a 10 is left, attach to the root of the last binary tree an edge connected to a pair of triangles with a common side.

  In the end, link each of these graphs to a distinct vertex of a path of length $l(g) - 1$ using an edge. It is easy to see that the graph such obtained is cubic, simple, and of genus g; moreover, it is easy to compute that the order of the automorphism group of this graph is precisely $2^{o(g)}$.

The final point in the definition will be called the *general case*, and the others *exceptional*.

Example 4.2. It is worth illustrating the general case with examples. Let $g = 57$, so the binary expansion of g is 111001. From left to right, there are two pairs – 11 and 10, and then a 1 is left over. We write $57 = 3 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^0$; $l(57) = 3$, so $o(57) = 54$. We are to take an A_4 and attach it to a B_3, inserting a triangle in the middle. A simpler way to describe the graph in this case is as follows: to two of the vertices of a triangle, attach an A_3, and attach a B_3 to the third.

C_{56} is C_{57} with the triangle collapsed to a point. There is no change in automorphism group. $o(56) = o(57) = 54$.

For $g = 58$, we have instead a pair of triangles sharing an edge, with an A_4 attached to the free vertex of one, and the B_3 attached at the other free vertex. $o(58) = 55$, and there is an extra involution of C_{58} coming from the configuration of two triangles.

Remark 4.3.

- We note that the candidates above yield no more automorphisms than $3 \cdot 2^{o(g)}$, and in most genera the bound is $2^{o(g)}$.
- We compare below the orders of the automorphism groups of the candidates constructed above with the theoretical bound of $48(g-1)$ obtained by Tutte for symmetric graphs (there are no semisymmetric graphs of genus smaller than 28). For $9 \le g \le 12$, methods similar to those of Tutte give a bound of $24(g-1)$ for such graphs which will be used in the table.

g	$o(g)$	$\lvert\mathrm{Aut}\, C_g\rvert$	edge − transitive bound	optimal
3	2	N/A	96	tetrahedron (24)
4	3	N/A	144	$K_{3,3}$ (72)
5	3	N/A	192	cube (48)
6	5	32	240	Petersen (120)
7	5	32	288	no − name (64)
8	7	128	336	Heawood (336)
9	7	**384**	192	
10	8	**384**	216	
11	9	**768**	240	
12	11	**3072**	264	
13	11	**3072**	576	
14	12	**6144**	624	
15	13	**8192**	672	
16	15	**32768**	720	

It is easy to see that for $g \ge 16$ we always have $2^{o(g)} > 384(g-1)$.

The conclusion so far: the graphs C_g have more automorphisms than any edge-transitive graph as soon as $g \ge 9$.

Thus, we will be concerned in what follows with the two cases in which the minimal orbit of an edge is a disjoint union of stars or edges inside G.

As a consequence of (2.3), we have the following:

Proposition 4.4. $\lvert\mathrm{Aut}\, C_{g+1}\rvert \ge \lvert\mathrm{Aut}\, C_g\rvert$ *except when* $g = 9 \cdot 2^m + 2$ *(*$m \ge 1$*) or* $g = 9(2^m + 2^p) + 2$ *(*$\lvert m-p\rvert \ge 5$*).* $\lvert\mathrm{Aut}\, C_{g+2}\rvert > \lvert\mathrm{Aut}\, C_g\rvert$ *for any* $g \ge 9$.

Proof. Lemma 2.3 shows that $o(g+1) \ge o(g)$.

- If C_g has $2^{o(g)}$ automorphisms, then C_{g+1} has at least $2^{o(g+1)} \ge 2^{o(g)}$.
- If C_g has $3 \cdot 2^{o(g)}$ automorphisms, then $g = 9 \cdot 2^m + u$, where $u = 0$ or $u = 2$; in the second case $m \ge 3$ also. But then C_{g+1}, for $g+1 = 9 \cdot 2^m + 1$, has the same number of automorphisms as C_g, while for $g+1 = 9 \cdot 2^m + 3$, C_{g+1} has $2^{o(g+1)} = 2^{o(g)+1}$ automorphisms. Thus for $g = 9 \cdot 2^m + 2$, $\lvert\mathrm{Aut}\, C_{g+1}\rvert < \lvert\mathrm{Aut}\, C_g\rvert$. Similar behavior occurs when $g = 9(2^m + 2^p) + 2$ (m and p as in the hypotheses) with p not too small.

- If C_g has $\frac{3}{2} \cdot 2^{o(g)}$ automorphisms, then $o(g+1) \geq o(g)+1$ would show that $|\mathrm{Aut}\, C_{g+1}| > |\mathrm{Aut}\, C_g|$. We have:
  - If $g = 3 \cdot 2^m$, then $o(g+1) = o(g)$, but also $|\mathrm{Aut}\, C_{g+1}| = \frac{3}{2} \cdot 2^{o(g+1)}$.
  - If $g = 3 \cdot 2^m + u$ ($u = 1,2$), then $o(g+1) = o(g)+1$.
  - If $g = 9 \cdot 2^m + 1$ ($n \geq 2$), then $o(g+1) = o(g)+1$.
  - If $g = 10, 11, 19$, in which case $g+1 = 11, 12, 20$ then again $o(g+1) \geq o(g)+1$.
  - If $g = 20, 38$, then as before we see that $|\mathrm{Aut}\, C_{g+1}| < |\mathrm{Aut}\, C_g|$.

The previous calculations then show that $|\mathrm{Aut}\, C_{g+1}| \geq |\mathrm{Aut}\, C_g|$ except for the noted exceptions.

Since $o(g+2) \geq o(g)+1$, the second part follows immediately except in the case $g = 9 \cdot 2^m + 2$; but a direct computation shows that $o(g+2) = o(g)+2$ for such g, so $|\mathrm{Aut}\, C_{g+2}| \geq 4 \cdot 2^{o(g)} > |\mathrm{Aut}\, C_g|$ so we are done. □

5 Some reductions

Definition 5.1. We will call a simple cubic graph *optimal* if its automorphism group has maximal order among all simple cubic graphs of the same genus. We call such a graph *strictly optimal* if it is optimal and the minimal orbit of edges in it has minimal order among all optimal graphs of that genus.

In this section we investigate the structure of a strictly optimal simple cubic graph of genus $g \geq 9$. The results of the previous sections show that G cannot be edge-transitive. Consequently, we will pick a minimal orbit of an edge and try to understand its structure, and the structure it brings to the whole graph G.

A first step in this direction has been made by (3.1).

We will denote by $G' = G \setminus O(e)$ and by g' the genus of G'. Due to the structure of $O(e)$, G' has valence either two or three at each of its vertices. Consequently, $g' \geq 1$ and $g' = 1$ if and only if G' is a disjoint union of cycles.

Remark 5.2. We will use a number of times a "local surgery" process, replacing subgraphs of the original graph G with other subgraphs; the surgery is to be done throughout the orbit of the replaced subgraph. These graphs will have valence three at each of their vertices except at those that are in the same well-chosen orbit. We will reattach the replacements at the same points, to preserve regularity and avoid multiple edges. Each time we will keep track of the genus lost, and of the order of decrease/increase in the number of automorphisms. Since we are targeting only graphs that are optimal (maximal order of the automorphism subgroup), if by chance the orbit of the newly introduced subgraph is larger than that of the original subgraph, then we must have effectively/strictly increased the automorphism subgroup of the graph, so the estimates we use to show that the original one was not optimal still hold up. Thus we will assume tacitly that we are in the worst case scenario, where the new subgraph has orbit "the same" as the original, and argue usually by the number of elements in a minimal orbit of edges to reach a contradiction. More precisely, one can simply mark the vertices where the original graph was disconnected; the subgroup of automorphisms of the graph (after surgery) required to preserve the marking (at most permuting marked vertices among themselves) is the one we are really estimating. Most times this extra marking is not needed; in the few cases where it is, we will make it clear.

Definition 5.3. For a graph G having valence at least two at each of its vertices and genus at least three we will denote by G^{stab} the *stabilization* of G (the terminology is motivated by algebraic geometry). This is the graph obtained by replacing each maximal path, with interior vertices all of valence two by a simple edge with the same endpoints as the path. It is clear that G^{stab} will have valence at least three at each of its vertices; however, it might have either loops or multiple edges.

The possibility of loops and/or multiple edges inside G^{stab} prohibits a direct induction; sharp bounds on such graphs may be obtained using the methods of this article; they are greater than the bound for simple graphs in every genus.

Theorem A.

- *If G is a (strict optimal) simple cubic graph of genus $g \geq 9$, then $\mu(g) = \frac{|\mathrm{Aut}\, G|}{2^{o(g)}} \leq 3$. Moreover, $\mu(g) \leq 1$ except when $g = a \cdot 2^m + b$, with $(a,b) \in \{(3,0),(3,1),(3,2),(9,0),(9,1),(9,2),(5,1),(15,1)\}$.*

- *If G is a (strict optimal) simple cubic graph of genus $g \geq 9$ with $|\mathrm{Aut}\, G| > 2^{o(g)}$, then $M(G) \geq 3$.*
- *If G is a simple cubic graph of genus $g \geq 9$, then $\pi(G) \leq 2^{o(g)}$ (or $\mu_1(g) \leq 1$).*

Note that the third assertion of the theorem follows easily from the first two. Also note that Theorem A immediately implies the Main Theorem of the article. We will call Theorem A restricted to graphs of genus g Theorem A_g.

The idea of the proof is to analyze the graph G' left after removing a well-chosen minimal orbit $O(e)$ from G. G' may be disconnected, $O(e)$ can be a disjoint union of stars or isolated edges, and G' (or its components) might have loops or multiple edges after stabilization. Overall, $2^{o(g)}$ acts as a filter in each genus: if a graph does not have at least as many automorphisms, then it is not optimal, as (4.1) shows. We will use estimates to determine precisely when (for what g, and for what structure of $O(e)$) a graph has more than $2^{o(g)}$ automorphisms. Moreover, we will determine and use in the inductive process the order of the minimal orbit of edges (or at least some useful estimate). In what follows, $O(e)$ will *always* refer to a minimal orbit of disjoint stars or edges.

We call a vertex of G' which has valence three a *stable* vertex. These might not exist. More precisely, these do not exist precisely in the components of G' which are cycles. However, in this instance G is easily seen not to be optimal; this will be shown inductively in Lemma 6.2.

In regard to the process of stabilization of the components of G', the following lemmas will be useful:

Lemma 5.4. *There can be at most two endpoints of edges in $O(e)$ on any unstable path with stable endpoints in G'.*

Proof. First note that if at least three such vertices exist on such a path, they cannot possibly be in the same orbit (there is in this case a "middle" edge distinguished from the others). Since the endpoints of stars are in the same orbit, this means that we need to discuss only the possibility of isolated edges in $O(e)$ having three or more endpoints on a path with stable endpoints in G'. For such paths of length five or more (four or more contact points), it is clear that one of the edges in this path (the most central) will have an orbit of order less than that of $O(e)$, which is a contradiction. For a path of order four, the only possibility is that the middle contact point (vertex) is not in the orbit of the other two contact points. But then it has an orbit of order half of the other contact points, which is impossible, since both are endpoints of edges in $O(e)$. □

Moreover:

Lemma 5.5. *Assume Theorem A_h for all $9 \leq h < g$. Let G be a strictly optimal simple cubic graph of genus g, and $O(e)$ a minimal orbit of edges in G. Then no two edges in $O(e)$ may have endpoints on a path with stable endpoints in $G' = G \setminus O(e)$.*

Proof. When two edges of $O(e)$ have contact points with a path with stable endpoints p in G', a naive (but effective) idea to try is to detach the two edges from the path, join their "free" ends to a common point, and join that common point to the middle of the original path (with the contact points removed) by a new edge f. This is illustrated in the first column of Figure 2. The procedure should be carried on throughout G, in the orbit of the path p. In this way, the graph obtained has the same genus and it is easily seen to have at least as many automorphisms as the original one, and in fact the $O(f)$ will have order less than that of the $O(e)$ (this is easily seen to be true regardless of the structure of $O(e)$), so G could not have been strictly optimal. There are only two possible cases when this procedure leads to double edges:

1. **Problem 1**: (does not happen when $O(e)$ is a disjoint union of stars): Two edges in $O(e)$ end up simultaneously with both ends on paths in the same orbit (see the middle column of Figure 2); assume the path has distinct stable endpoints (for the other case, see **Problem 2** below). However, in this instance replacing both edges by a single one yields a graph $\bar{G}$ which is simple, cubic, has no fewer automorphisms than G (we do this on the whole orbit of the path at once) and has genus $g - \frac{k}{2}$ (where $k = |O(e)|$). Moreover, the orbit of the replacement edge has fewer members, contradicting the strict optimality of G. More precisely, (4.4) shows that, except in case $g = 9 \cdot 2^m + 2$, C_{g+1} has at least as many automorphisms as C_g. Then $|\mathrm{Aut}\, G| \leq |\mathrm{Aut}(C_{g-\frac{k}{2}})| \leq |\mathrm{Aut}(C_g)|$; in case of equality throughout, the size of the minimal orbit helps establish the contradiction with the assumed strict optimality of G. This works except when $k = 2$ and

$g = 9 \cdot 2^m + 3$, when the last inequality in the sequence above fails. However, in that case a direct argument can show that G was not optimal; assuming $M(G) \geq 3$ for an optimal G with $g = 9 \cdot 2^m + 2$, and assuming that $|\mathrm{Aut}\, H| \leq 2^{o(g)}$ for every non-optimal graph H in the same genus, we see that $k = 2$ and/or the orbit of p having length two forces G to be non-optimal, so $|\mathrm{Aut}\, G| \leq 2^{o(g-1)} < 2^{o(g)} = |\mathrm{Aut}\, C_g|$.

2. **Problem 2**: The procedure yields a double edge when the path p has is a loop, starting and ending at the same stable point. Denoting by f the third edge around this stable point, we see that the orbit of f has order half that of $O(e)$ (contradiction!) unless the above-mentioned **Problem 1** also occurs. But in this case the picture is as in the third column of Figure 2 and the whole configuration: may be replaced (see the same figure). Since f was not in the orbit of e, this stabilizes the graph locally, and in fact globally, as it is easy to see. Moreover, the automorphism group of the new graph (of the same genus as the original one) increases at least twofold, which is a contradiction to the optimality of G. Thus this problem does not occur in the optimal graphs.

□

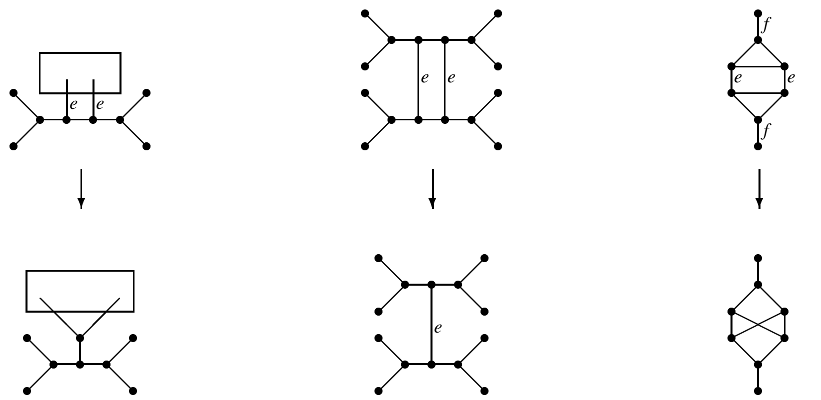

Figure 2: Stabilizations leading to a double edge. The problems are in the first row, and their solutions in the second.

Note that the previous lemma implies that any unstable path in G' has length two. Given this, we note, for future reference, what types of situations would lead to double (or triple) edges when stabilizing (the components of) G'. There are four classes, three labelled with roman numerals. A subscript on a roman numeral indicates the length of a stable path between the stable endpoints of an unstable path; if the subscript is a plus sign, the vertices are either connected by a stable path of length greater than two, or not connected by a stable path. The figures are drawn with the edges in $O(e)$ labelled "e".

We will discuss when can these structures occur in a strictly optimal graph G. All the constructions are assumed to be done throughout the orbit of the unstable path simultaneously. We denote by $\bar{G}$ the graph obtained as a result of the surgery. The following definition is convenient.

Definition 5.6. Let H be a graph which is cubic except for two vertices of valence two. A *pseudocycle* is a cubic graph obtained by replacing the vertices in a cycle with copies of H.

Lemma 5.7. *An optimal graph does not have a subgraph of type I_1.*

Proof. I_1 may only occur when the whole G is a pseudocycle (otherwise the middle edge would have a shorter orbit). Direct computation shows that these graphs are not optimal for $g \geq 9$. □

Lemma 5.8. *A strictly optimal graph has no subgraphs of type I_2.*

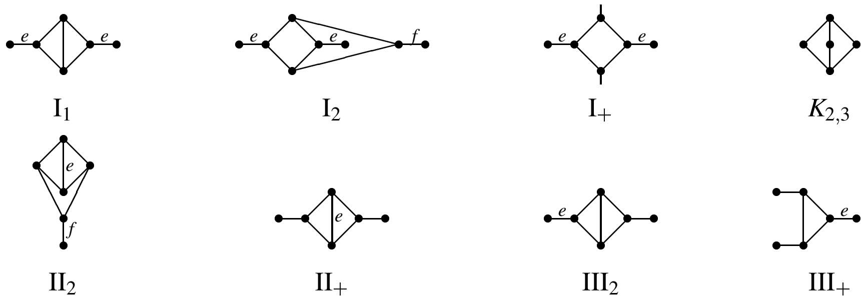

Figure 3: The subgraphs that lead to multiple edges and loops.

Proof. **Case 1:** A strictly optimal G with $O(e)$ a disjoint union of stars cannot have an I_2: it is clear that the orbit of the edge labelled f in Figure 3 will have an orbit with fewer elements than $O(e)$.

Case 2: When $O(e)$ is a disjoint union of $k \geq 2$ edges, the only possible way that an I_2 could occur in G is if members of $O(e)$ alternate with I_2 to form a pseudocycle (otherwise again the edge labelled f would have a smaller orbit). There are two subcases:

a The pseudocycles have exactly two I_2. We will create a new graph whose minimal orbit is smaller, contradicting strict optimality of G. Each pseudocycle is connected to the rest of G by the edges labelled f in the figure. If these two edges actually coincide, the graph G is one pseudocycle, and direct computation excludes it from optimality. So we may assume that each pseudocycle has two distinct edges connecting it to the rest of the graph. A pseudocycle has local genus five and local automorphism group of order eight (sixteen, if the two fs may be flipped). Remove the pseudocycle and replace it with a triangle connected by two vertices to the two fs and whose third vertex is connected to a graph A_1. This does not change the genus or number of automorphisms of the graph, but the edge connecting the triangle to the A_1 now has a smaller orbit than the previous e.

b If the pseudocycle has at least three I_2, we replace it with a cycle whose length is equal to the number of I_2. In this way the new graph, cubic and simple, has lost precisely 2^k automorphisms (permuting the stable endpoints of the unstable paths), but also lost $2k$ in genus. By induction, if $\bar{g} \geq 9$ we have $|\mathrm{Aut}\ \bar{G}| \leq 3 \cdot 2^{o(g-2k)}$ so $|\mathrm{Aut}\ G| \leq 3 \cdot 2^{k+o(g-2k)} \leq 3 \cdot 2^{o(g)-o(k)}$. Since $k \geq 3$, the last quantity is clearly less than $2^{o(g)}$ so G could not have been optimal. If however $\bar{g} \leq 8$, we must increase $\mu(G)$ to nine, so we can only derive the contradiction to the optimality of G when $k \geq 6$; if $\mu(G) \leq 6$, then the contradiction happens as soon as $k \geq 4$. So we are left with a few cases to consider:

- $\mu(G) = 9$ and $k = 4, 5$. Then $\bar{g} = 4$, so $g = 12$ or 14; $|\mathrm{Aut}\ \bar{G}| \leq 72$. In the first case, $|\mathrm{Aut}\ G| \leq 72 \cdot 2^4 < 6 \cdot 2^9 = |\mathrm{Aut}\ C_{12}|$ so G was not optimal; in the second case $|\mathrm{Aut}\ G| \leq 72 \cdot 2^5 < 2^{12} = |\mathrm{Aut}\ C_{14}|$ so again G was not optimal
- $\mu(G) \leq 6$ and $k = 3$. Then $g \leq 14$, and the table (4.3) shows that $|\mathrm{Aut}\ G| < |\mathrm{Aut}\ C_g|$ so G is not optimal.

□

Lemma 5.9. *Assume Theorem A_h for $9 \leq h < g$. Then an optimal graph of genus g does not have a subgraph of type I_+.*

Proof. **Case 1:** Suppose $O(e)$ is a disjoint union of k stars, so the components of G' are all isomorphic. If the components of G' pairs of squares connected by two edges, diagonally opposite each other, we may replace these components as in Figure 4 to gain automorphisms without changing the genus, contradicting optimality.

Otherwise, one may collapse each I_+ as in Figure 4. The resulting graph is cubic simple, and has at least as many automorphisms as G; however, its genus is $g - \frac{3k}{2}$, and then (4.4) and (3) show that G could not have been optimal.

Case 2: The same surgery as above may be done when $O(e)$ is a disjoint union of k isolated edges (under the same restriction as above).

- If the two unstable vertices of an I_+ are not in the same orbit, then the existence of a pseudocycle formed of edges in $O(e)$ and I_+ to which they are incident is immediate; in fact the whole orbit $O(e)$ will be partitioned in edges arranged in such isomorphic pseudocycles. If a pseudocycle contains at least three I_+, do the same surgery as above; the decrease in genus overall is precisely $k \geq 2$, so again G could not have been optimal. If a pseudocycle contains only two I_+, then the surgery is modified to unite by an edge the two vertices to which the eyes were contracted. This time the decrease in genus is at least three, so again G could not have been optimal.

- If the two unstable vertices of an eye are in the same orbit, but the other ends of the two incident edges are not in their orbit, then the surgery above can be done at the orbit of the unstable path without leading to double edges; the decrease in genus is $\frac{k}{2}$ (it is easy to see that k must be even), and the only case requiring consideration is when $k = 2$ (when the difference in genus is only one) and $\mu(\bar{G}) > 1$; but then $M(\bar{G}) \geq 3$ by induction, so we reach a contradiction.

- If the ends of the edges in $O(e)$ are in the same orbit, we see again the existence of pseudocycles in $O(e)$ and we continue the argument as above; G could not be optimal.

In the case of "cylinders" (as in Case 1, the left side of the figure) we may replace as in the case when $O(e)$ was a disjoint union of stars, and see that G was not optimal. This completes the proof. □

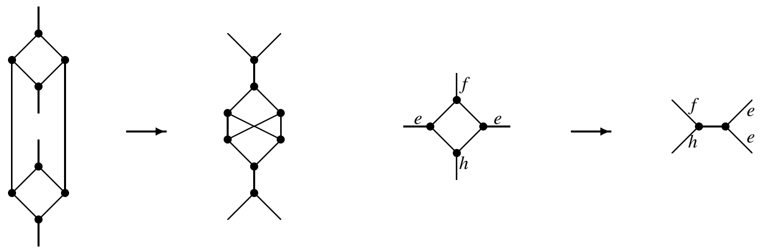

Figure 4: Surgeries for graphs of type I_+.

Lemma 5.10. *An optimal contains at most one subgraph of type III_+. Furthermore, if a graph of type III_2 occurs in an optimal graph, it is unique, so may be considered as a subgraph II_+.*

Proof. The argument for subgraphs of type III_+ follows the line of those above. In this case, the surgery is to collapse the triangle in each III_+ to a point. The resulting graph has the same automorphisms, and lower genus, which contradicts optimality as above if there were multiple III_+ in the graph (if there is only one, the genus does not decrease enough to apply (4.4)). Note that these subgraphs do appear in certain of the C_g.

The middle edge of the graph III_2 moves as much as the edge labelled e. If the orders of their orbits are equal, we may shift attention to the middle edge and think of the III_2 as a II_+. In this case, the result follows from the next lemma.

The only way that the orbit of e could be smaller than the orbit of the middle edge is if a configuration as in Figure 5 occurs, and the same figure gives a surgical solution to this problem. □

Lemma 5.11. *An optimal graph may have at most one II_+.*

Proof. First of all, replacing each II_+ by a single edge cannot produce a double edge or a loop (producing a loop would mean that we had a II_2). This is because either two such II_+ share the vertices at distance one from their stable ends, or there is a shortcut (edge) between those vertices (at distance one from their stable ends); we note that, if three such II_+ share the vertices of distance one from their stable endpoints, this configuration is the whole graph, of genus eight, and excluded from consideration. The surgeries in these two cases are depicted in Figure 6.

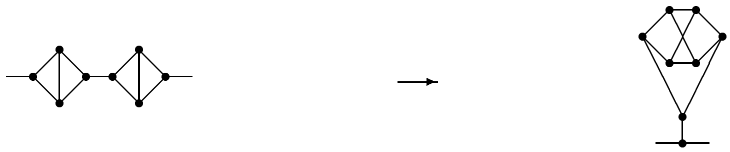

Figure 5: Surgery for graphs of type III_2.

In the first case, the order of the orbit of a minimal edge is decreased without changing genus or automorphisms, contradicting strict optimality. In the second case, the surgery produces a graph with a larger automorphism group.

Then, if $k = |O(e)| \geq 2$, the II_+ could not have adjacent vertices: more than three II_+ could only be adjacent if they form a cycle (the whole of G!) due to the requirement that their middle edges should be in the same orbit; then $|\mathrm{Aut}\, G| \leq 2k \cdot 2^k < 2^{o(2k+1)}$ for $k \geq 5$ and $|\mathrm{Aut}\, G| < |\mathrm{Aut}\, C_9|$ for $k = 4$. And if only two II_+ would be adjacent, the same surgery as in the previous lemma (see Figure 5) can be done.

Thus if more than two II_+ exist in a strictly optimal G, they are not adjacent, and the vertices at distance one from their stable endpoints are not neighbors; replace then the II_+ by single edges; the automorphism group decreased in order by a factor of 2^k, the resulting graph is simple and cubic, and of genus $g - 2k$. Then the same discussion as in the case of the I_2 shows that G could not have been optimal. $\square$

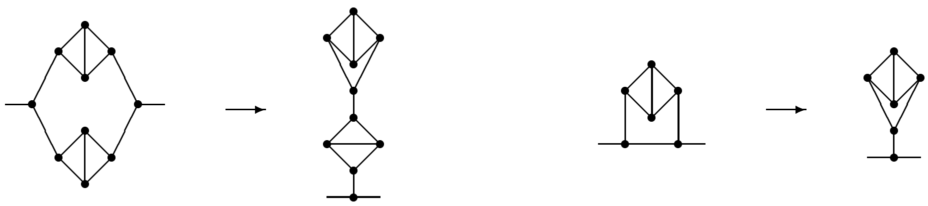

Figure 6: Surgeries for multiple II_2.

To eliminate most subgraphs of type II_2 from consideration, we refine our notion of well-chosen. Note that our candidate graphs contain many copies of II_2, but with e not minimal. We will avoid the multiple edges which occur as a result of stabilizing II_2 by choose the edge labelled f in the defining figure as the minimal edge. Arguments as above show that there is no problem in doing so. From henceforth, a well-chosen edge will also be subject to this restriction.

Lemma 5.12. *For any genus $g \geq 9$, a strictly optimal G and a well-chosen $O(e)$, G' contains at most one component which is a $K_{2,3}$.*

Proof. Let k be the number of $K_{2,3}$'s in a resulting G' (note that these must be connected components). The extra symmetry brought by these inside G is that even if their valence two vertices are fixed, there is still a swapping of their stable vertices possible. We proceed to eliminate this and discuss the result: replace therefore each $K_{2,3}$ by a vertex, to which the edges incident to the original $K_{2,3}$ will be linked. This results precisely in a 2^k times decrease in the order of the automorphism group, from the elimination of the swapping mentioned above. However, at the same time the genus has dropped by $2k$. The graph $\bar{G}$ thus obtained is easily seen to be simle cubic, except if two edges incident to one of the collapsed $K_{2,3}$'s have a common endpoints, or are incident to another collapsed $K_{2,3}$. In the first case $O(e)$ must have been a disjoint union of stars, and then actually only one of them, so $g = 4$; in the second case, all components of G' must have been isomorphic; the transitivity of the action of $\mathrm{Aut}\, G$ on $O(e)$ forces actually the two components linked by two edges in $O(e)$ to be linked by three edges in $O(e)$, and this is all of G. Then $g = 6$. Both cases are outside our considerations, therefore $\bar{G}$ is simple cubic.

Let $\bar{g}$ be the genus of $\bar{G}$. By induction $\mu(\bar{g}) \leq 9$, with equality only when $\bar{g} = 4$; moreover $4 \neq \bar{g} \leq 8$ implies $\mu(\bar{g}) \leq 6$ and $\bar{g} \geq 9$ implies $\mu(\bar{g}) \leq 3$. We are then studying the inequality $\mu(\bar{g}) \cdot 2^{o(g-2k)+k} \leq 2^{o(g)}$ or equivalently $\mu(\bar{g}) \leq 2^{k-l(g)+l(g-2k)}$; this is implied by $\mu(\bar{g}) \leq 2^{o(k)}$.

- $k \geq 6$ makes the last inequality strict even for $\mu(\bar{g}) = 9$, while $k \geq 4$ makes the last inequality strict for $\mu(\bar{g}) \leq 6$
- If $\bar{g} = 4$, $|\mathrm{Aut}\,\bar{G}| \leq 72$; we need to study the cases $k \leq 5$. If $k = 1,2$, then $g \leq 8$–too small. If $k = 5$, $g = 14$ and $|\mathrm{Aut}\,G| \leq 72 \cdot 2^5 < |\mathrm{Aut}\,C_{14}|$ so G was not optimal. If $k = 4$, $g = 12$ and $|\mathrm{Aut}\,G| \leq 72 \cdot 2^4 < |\mathrm{Aut}\,C_{12}|$ so again G was not optimal. If $k = 3$, $g = 10$ and since $|\mathrm{Aut}\,G| \leq 72 \cdot 2^3 > |\mathrm{Aut}\,C_{10}|$ we need to make use of the marking mentioned in (5.2): the three contracted $K_{2,3}$ must have been in the same orbit. Referring to a table of cubic graphs of low genus, we see that either $|\mathrm{Aut}\,\bar{G}| = 12$ in which case $|\mathrm{Aut}\,G| < |\mathrm{Aut}\,C_{10}|$, or $G = K_{3,3}$ so the three vertices representing the contracted $K_{2,3}$'s (which must not be neighbors, by the discussion on the simplicity of $\bar{G}$) fill one of the two sets of the partition. Then the marking of this partition cuts in half the number of automorphisms of $\bar{G}$, so $|\mathrm{Aut}\,G| \leq 36 \cdot 2^3 < |\mathrm{Aut}\,C_{10}|$ so again G could not have been optimal.
- If $4 \neq \bar{g} \leq 8$, we need to worry about $k \leq 3$.
  - $k = 1$ is only possible when $\bar{g} = 7,8$. In the first situation, $|\mathrm{Aut}\,G| \leq 64 \cdot 2 < |\mathrm{Aut}\,C_9|$ so G was not optimal. In the second situation, the marking of the unique vertex introduced by contracting the $K_{2,3}$ is easily seen to cut at least in half the order of $\mathrm{Aut}\,\bar{G}$; then $|\mathrm{Aut}\,G| \leq 168 \cdot 2 < |\mathrm{Aut}\,C_{10}|$ so again G was not optimal.
  - $k = 2$ is possible for $5 \leq \bar{g} \leq 8$. In all cases except when $\bar{G}$ is the Petersen's graph ($\bar{g} = 6$) one easily reaches the conclusion (using (4.3)) that G was not optimal. Using the marking (two marked points) in case of the Petersen's graph severely cuts the order of available automorphisms, to $|\mathrm{Aut}\,G| \leq 12 \cdot 2^2 < |\mathrm{Aut}\,C_{10}|$.
  - $k = 3$ and $4 \neq \bar{g} \leq 8$ is again easily shown using (4.3) to lead directly (without discussing markings) to the conclusion that G was not optimal.
- $\bar{g} \geq 9$; then by induction $\mu(\bar{g}) \leq 3$ and only the cases $k = 1$ and $k = 2$ do not lead to the immediate conclusion that G was not optimal (otherwise $o(k) \geq 2$).

  When $k = 2$, the graph G' has at most three components, and removing one of the two $K_{2,3}$'s and its incident edges the graph is still connected, simple and cubic (since there is at most one double edge that could occur in its stabilization, by the previous reductions); the surgery would drop the genus by four, while overall $|\mathrm{Aut}\,G|$ dropped fourfold (twice from the interchanging of the two $K_{2,3}$'s, and twice from the swapping of the two stable points of the removed $K_{2,3}$). Since $o(g) \geq o(g-4)+3$. Only now reducing the final $K_{2,3}$ would drop the genus by another two, while losing only a factor of two. Overall, we get a drop of six in genus, and a drop of eight in the order of the automorphism group. If $\bar{g} - 2 \geq 9$, then $o(g) \geq o(g-6)+5$ so $2^{o(g)} > \mu(\bar{G}') \cdot 2^{o(g-6)}$, so G was not optimal; otherwise, we discuss as above to reach the same conclusion.

□

To summarize the reductions so far:

Proposition 5.13. *In a strictly optimal G of genus $g \geq 9$, a minimal $O(e)$ may be chosen in such a way that G' may be stabilized to a simple graph, with the exception of the following situations:*

1. $k = |O(e)| = 1$ *and e is in the middle of a II_+; it is clear that the edges leaving from the stable ends of the II_+ can be at most swapped by* Aut G, *but cannot move anywhere else inside G. Moreover, in this case G' is clearly connected, but stabilizing it would produce a double edge; furthermore, the vertices at distance one from the endpoints of the II_+ are not connected by an edge.*
2. *(for $g \geq 10$) $k = |O(e)| = 1$ and this edge is incident to precisely one II_2; G' is disconnected, but we know precisely an isomorphism class of components of G' (it is easy to see that not all connected components of G' could be isomorphic; one would get two components and the genus would be seven, too low for our considerations).*

3. *G' contains cycles; due to the length of these cycles being at least three, $|O(e)| \geq 3$, so by the discussion above, the other components of G' must stabilize properly (or be cycles themselves).*

4. *G' may contain a unique $K_{2,3}$ as a component.*

As an immediate consequence, we may choose $O(e)$ minimal in such a way that one of the following is true:

- The components of G' of genus greater than two stabilize to simple cubic graphs.
- G' has components that are cycles, but the components that are not stabilize to simple cubic graphs.
- G' is connected, but stabilizing it leads to a (unique) double edge (this is when a unique II_+ occurs).
- G' is disconnected, and one component is a II_2 which stabilizes to a graph with a double edge.
- G' contains a unique $K_{2,3}$ as a component.

Remark 5.14. (Enforcing strictness) In certain situations, when the geometrical situation will allow, we will show that some graphs cannot be strictly optimal by the following constructions.

1. G' is made up of three components that stabilize to simple cubic graphs and $O(e)$ is a disjoint union of two stars, each incident exactly once to each component. The two points of contact of each component with the stars must be in the same orbit; for optimality of G it is necessary that fixing one of the points will fix the other (i.e. the two pinched edges where the incidence occurs must always move together under the action of Aut G). Then we detach the stars, link their ends incident to each component together to form a $K_{2,3}$, and link each of these new vertices with an edge to any one of the previous incidence points; we then stabilize (removing the pinch points at the other points of detachment). Then the automorphism group of the new graph is at least as large as that of the initial one, but clearly the minimal orbit has decreased in order; thus G was not strictly optimal.

2. G' is made up of two components that stabilize to simple cubic graphs, and $O(e)$ is a disjoint union of two or four edges. Regardless of whether the two components of G' are isomorphic, the incidence points of the edges in $O(e)$ with each component are in the same orbit, and their set is preserved by Aut G; moreover, the optimality of G will dictate that fixing one incidence point will necessarily fix the others on its component; this implies that once an edge in $O(e)$ is fixed, the others will be fixed as well.

   (a) If $k = |O(e)| = 2$, then we detach the ends from one component, join them, and link the resulting vertex with an edge to any either of the initial incidence points, while stabilizing the other.

   (b) If $k = 4$ we detach all the edges in $O(e)$, link the unstable vertices of a $K_{3,3}$ with an edge removed to the endpoints of one of the edges in $O(e)$ and stabilize the remaining endpoints.

   In both situations the new graph has the same genus and at least as many automorphisms, but the minimal orbit has strictly smaller order; thus again G was not strictly optimal.

3. G' is made up of three components, (all stabilizing to simple cubic graphs) two of which are isomorphic, linked each by two edges to the third one. Then as above we may detach the edges from the two isomorphic components, join their free ends, and link the resulting vertices to one of the initial incidence vertices (stabilizing the other). As above, this is easily seen to contradict the strict optimality of G.

6 Proof of the Main Theorem

We will repeatedly use an *exhausting subgraphs* argument. This entails choosing a connected component (star or edge) in $O(e)$, fixing its orientation (when the endpoints are in the same orbit) and then gradually enlarging the subgraph gotten at a certain stage by choosing one of its tails and adding whole components either of $O(e)$ or of G' reached by that tail. When a component of G' will be added, we will include in the new subgraph only the edges of

$O(e)$ incident to it, and of these, in case $O(e)$ is a union of stars, only those that do not lead to stars whose center is already a vertex of the previous subgraph (in order to avoid cutting unnecessarily the number of tails).

At each step we look at the relative gain in the automorphism group. If a star is included at that step, then one of its edges is already fixed by the initial subgraph, then there could be at most a twofold increase in the order of the automorphism group at such a stage; moreover, such an increase occurs only when none of the vertices of the star was part of the subgraph at the beginning of the stage.

If however, a component is included at a certain step, then one of its vertices (which has valence two in G') is already fixed, and that limits its symmetry; in other words, the automorphisms of the new subgraph fixing the previous one are precisely those fixing the incidence point. Once all these automorphisms are taken into account, all the edges incident to that component do not have extra freedom (they move where their incidence point moves), so may be added without further increase in the order of the automorphism group of the subgraph.

Unless otherwise noted, we will always expand the subgraphs by including whole components of G' if the possibility exists (i.e. when not all tails of the subgraph gotten so far are centers of stars in $O(e)$).

During the course of the proofs, it will sometimes be convenient to disallow certain automorphisms of a graph. In particular, sometimes we will collapse a cycle to a vertex, but we only want to remember the automorphisms of the resulting structure which come from the cycle. We will call the resulting vertex a *vertex with dihedral symmetry* to indicate that we do not allow the more general automorphisms in the contracted graph.

At this time, we introduce the following theorem, which will also be proved and used inductively in the course of this section:

Theorem B. *Suppose $g \geq 9$. If $|\mathrm{Aut}\, C_g| > 2^{o(g)}$ or g is a power of two, there is a unique strictly optimal graph of genus g, unless $g = 10$. Moreover, there is a unique graph G for which $\pi(G) = 1$ in the cases $g = 2^m, 3\cdot 2^m$ and $3(2^m + 2^p)$.*

Remark 6.1. The case $g = 10$ is a real exception. A graph different from C_{10} with the same number of automorphisms is depicted in Figure 7. In the cases where $|\mathrm{Aut}\, C_g| = 2^{o(g)}$, non-uniqueness is the norm: a simple example occurs for $g = 340$ (101010100 in binary). The candidate graph has four "tails" which can be arranged in three non-isomorphic ways around the edges of a binary tree with four ends.

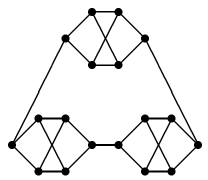

Figure 7: An optimal non-C_{10}

We use subscripts on B in the same way as they are used on A.

Throughout this section, we assume that Theorems A_h and B_h are true for all h less than the $g \geq 9$ under consideration.

As it relies on estimates based on the arithmetic of g, the proof of the Main Theorem is somewhat tedious. Here is the outline, with details filled in by the lemmas that occupy the rest of this section.

Proof of Theorems A and B. As usual, the proof is divided into two cases: when $O(e)$ is a disjoint union of stars, and when $O(e)$ is a disjoint union of edges.

Case 1: Suppose $O(e)$ is a disjoint union of stars. Then Lemma 6.4 will show that if $O(e)$ consists of more than one star, G is not strictly optimal. Then by the reductions of the previous section, we may remove the star, disconnecting the graph into subgraphs of lower genus. Lemma 6.4 will show furthermore that the genus of these subgraphs is quite restricted. The order of the automorphism group of G in this case will be six (for the star) times the automorphism groups of the components, so the stabilizations of the components must be optimal. We may then proceed by induction: the components are of smaller genus and we know the optimal pinched graphs in these genera, hence we get a bound for the automorphism group. The second part of Theorem A about the pinched graphs follows similarly. Theorem B will also follow by applying it inductively to the components when necessary. Thus, essentially, Case I is covered by the Lemma and induction hypotheses.

Case 2: If $O(e)$ is a disjoint union of edges, we are in a much more restricted situation. First of all, suppose that removing $O(e)$ and stabilizing results in a non-simple G'. We have classified the possible behaviors in the previous section: if G' remains connected, it has a unique subgraph II_+. Replacing this entire subgraph by an edge, we reduce the genus, are able to apply induction to the resulting graph (which no longer leads to an unstable G') and close this case.

If G' is disconnected, we either have the case of a unique II_2 (touching the minimal orbit; clearly there can be many II_2 in an optimal graph), in which case we may remove e and concentrate on the component which stabilizes well, again proving the theorems by induction, or the case of a unique $K_{2,3}$ component in G' (before stabilization). Again, we use induction on the components of G' that are not $K_{2,3}$ and obtain the result.

Therefore, we may assume that stabilizing G' does not introduce loops or multiple edges. This finally breaks into two subcases: either G' contains cycles or it does not. Lemma 6.2 shows that G' is not a disjoint union of cycles, and Lemma 6.3 then shows that a cycle in G' must be unique. After this is established, we see that G is a collection of isomorphic subgraphs that stabilize well arranged around a cycle. Again we may apply induction to the subgraphs to obtain the theorems.

If G' does not contain a cycle, then the orbit of a minimal edge must be small, and the estimates finish the work. The details are recorded in 6.5.

The proofs of the lemmas will show that if $g = 2^m$ there is only one optimal graph G for which $\pi(G) = 1$. If $g = 3\cdot 2^m$, then A_m is optimal, but $\pi(A_m) = \frac{1}{2}$. This part of Theorem B will be show by showing that the only other graphs with at least $2^{o(g)}$ automorphisms in these genera are those linking a B_{m+1} and a B_m by an edge; these graphs are not optimal (they have exactly $2^{o(g)}$ automorphisms), but they satisfy $\pi = 1$. A similar observation is true in the case $g = 3(2^m + 2^p)$. This proves the last part of Theorem B, which is essential in the induction. □

6.1 G' has components that are cycles

Lemma 6.2. *If G' is a disjoint union of cycles, then G cannot be optimal.*

Proof. **Case 1**: If G' is a single cycle, and $k = |O(e)|$, then the length of the cycle is $2k$ with $4k$ automorphisms; $g = k+1$ and $2^{k+1} > 4k$ for $k \geq 5$, i.e. for $g \geq 6$, which is certainly the case; such a G could not be optimal.

Case 2: If G' is disconnected and $O(e)$ is a disjoint union of k stars, then all the connected components of G' are isomorphic (since all the vertices of the stars are in the same orbit). Then G' having a connected component which is a cycle means that G' is a disjoint union of s cycles of the same length, say t; note that $t \geq 3$, otherwise G could not have been simple. We note that the transitivity of the action of $\mathrm{Aut}\, G$ on the orbit $O(e)$, coupled with the disconnectedness of G', prohibits the stars from having more than one contact point with any component of G'. The genus of G is $g = 2k+1$, and we have the "contact formula" $st = 3k$. In estimating $|\mathrm{Aut}\, G|$ we may simply replace the cycles in G' by (contract them to) vertices of valence t with dihedral symmetry. We get a graph with k vertices of valence three and s vertices of valence t.

We use growing trees: choose a star (k choices), fix its edges (at most six choices), then expand this tree and subsequent trees by reaching, from a tail, to adjacent vertices not included in the tree so far. Due to the at most dihedral symmetry around each vertex of the contracted graph, at each step in the process the size of the automorphism group increases at most two-fold; this occurs precisely when a tail of order three in G has only one neighbor in the tree gotten so far (say this happens a times), or when a tail of order t in G has at most two neighbors (one, if $t = 3$) in the tree (say this happens b times). Thus $|\mathrm{Aut}\, G| \leq 6k\cdot 2^{a+b}$ and (denoting by n the number of vertices of the contracted graph) $n = s+k \geq 4+2a+(t-2)b$ (if $t \geq 4$) respectively $n \geq 4+2a+2b$ (if $t = 3$). In the first instance we get $a \leq \frac{s+k}{2} - 2 - \frac{t-2}{2}b$, so $|\mathrm{Aut}\, G| \leq \frac{3}{2}k\cdot 2^{\frac{s+k}{2} - b\frac{t-4}{2}}$ (and note that $s \leq \frac{3k}{4}$); in the second instance we get directly $a+b \leq \frac{s+k}{2} - 2$ (but $s = k$) so $|\mathrm{Aut}\, G| \leq \frac{3}{2}k\cdot 2^k$. Both times we compare to $2^{o(2k+1)} = 2^{2k+1-l(2k+1)} \geq 2^{k+o(k)}$ (since $l(2k+1) \leq l(k)+1$; see (2.3)).

Now both inequalities are implied by $\frac{3}{2}k \leq 2^k$ which is strict by (2.2), since $k \geq o(k)+1 \geq o(k+1)$.

Case 3: If G' is disconnected and $O(e)$ is a disjoint union of isolated edges, then there are at most two isomorphism classes of connected components (according to whether both endpoints of e are in the same orbit or not).

We will deal directly with the general case, in which there are two isomorphism classes of cycles in G'; the simpler case may be dealt with by taking $s_1 = s_2$ below; all the estimates still work then. Thus there are s_1 cycles H_1 incident by t edges to each of n_1 neighbors (these have length n_1t), and s_2 cycles H_2 incident by t edges to each of n_2 neighbors (these have length n_2t). Then $k = s_in_it$ and $g = k+1$.

We need to show first that any two cycles may be linked by at most an edge in $O(e)$, i.e. $t = 1$.

If two cycles would have $t \geq 3$ common edges in $O(e)$ incident to both, then clearly fixing one cycle will fix the whole graph. Then $|\text{Aut } G| \leq 2s_1n_1 = 2k < 2^{o(k+1)}$ for $k \geq 5$ using (2.2). But $k \leq 4$ and $t \geq 3$ is only possible when G is made up of two isomorphic cycles sharing an edge between their vertices (one from each component); in that case the genus is too small (five or six) for our considerations. Thus such a G could not be optimal.

If two cycles would have precisely two edges in common, then $|\text{Aut } G| \leq 2k \cdot 2^{a+b}$, where a is the number of times we get an involution of a cycle of type H_1 by including it when expanding the subgraph at a tail incident to it; only one neighbor of this H_1 must have been included in the expanding subgraph previously, so $n_1 - 1$ new cycles of type H_2 will be incident to the newly increased subgraph afterwards; and similarly for b. Thus $s_2 \geq n_1 + a(n_1-1)$ and $s_1 \geq 1 + b(n_2-1)$.

Note that if $n_1 = 1$ we get $s_2 = 1$; then $t = 2$ would force the existence of double edges in G. Thus $\min(n_1,n_2) \geq 2$; as a consequence, since $t = 2$, $s_i \leq \frac{k}{2}$.

If $n_1 \geq 3$, then $a \leq \frac{s_2-1}{n_1-1} - 1 \leq \frac{s_2-1}{2} - 1 < \frac{k}{4} - 1$; similarly $b \leq \frac{s_1-1}{n_2-1} < s_1 = \frac{k}{n_1} \leq \frac{k}{3}$. Then overall $|\text{Aut } G| \leq 2k \cdot 2^{\frac{7k}{12}-1} < k \cdot 2^{\frac{k}{2}}$; now, since $k = 2s_1n_1 \geq 12$, this is easily seen to be strictly less than $2^{o(k+1)}$ by (2.2) so again G is not optimal.

If $n_1 = n_2 = 2$, then G is formed of cycles of length four, each with two neighbors with which it is linked by two edges; it is immediate that the edges between two adjacent cycles should be linked at their opposite vertices for maximum symmetry gain. These cycles are actually isomorphic, and there are precisely $\frac{k}{2}$ of them in G. However, in this case an involution in one cycle will force an involution in a neighboring cycle; overall $|\text{Aut } G| \leq k \cdot 2^{\frac{k}{2}} < 2^{o(k+1)}$ as soon as $k \geq 10$, so G is not optimal.

We note that for $g = 9$, i.e. $k = 8$, $|\text{Aut } G| \leq 8 \cdot 2^4 = 2^{o(g)} < |\text{Aut } C_9|$ so G cannot be optimal.

We also note that $g = 2^m$ is not possible in the above configuration.

Thus we have reduced to only the possibility $t = 1$.

We can estimate as above by first collapsing all cycles to vertices, then by expanding trees. So we get a graph $\bar{G}$ with s_1 vertices of order t_1 and s_2 vertices of order t_2, with dihedral symmetry around each vertex, and, most importantly, with $|\text{Aut } \bar{G}| = |\text{Aut } G|$. Note that $k = |O(e)| = s_1t_1 = s_2t_2$, $t_i \geq 3$ and $g = k+1$. Without loss of generality, we may assume $t_1 \geq t_2$.

Construct the exhausting trees by first choosing an edge in $O(e)$ (k choices), then fixing its orientation (at most two choices); afterwards, each tail which has no more than two neighbors in the trees constructed so far may bring at most an extra involution among the edges ending at it (since two of those are fixed necessarily); in case $t_i = 3$, there should be only one neighbor of that tail among the vertices touched by the tree so far in order for that involution to exist. Say this situation occurs a times for the cycles of length t_1 and b times for the cycles of length t_2. Overall $|\text{Aut } G| \leq 2k \cdot 2^{a+b}$. Then we have the following estimates:

1. $s_1 + s_2 \geq 2 + a(t_1-2) + b(t_2-2)$ when $t_i \geq 4$; thus $b \leq \frac{s_1+s_2-2}{t_2-2} - a\frac{t_1-2}{t_2-2}$, so $|\text{Aut}(\bar{G})| \leq 2k \cdot 2^{\frac{s_1+s_2-2}{t_2-2}}$
2. $s_1 + s_2 \geq 2 + a(t_1-2) + 2b$ when $t_1 > t_2 = 3$; thus $b \leq \frac{s_1+s_2-2}{2} - a\frac{t_1-2}{2}$, so $|\text{Aut}(\bar{G})| \leq k \cdot 2^{\frac{s_1+s_2}{2}}$.
3. $s_1 + s_2 \geq 2 + 2a + 2b$ when $t_1 = t_2 = 3$ (so $s_1 = s_2$), so $|\text{Aut}(\bar{G})| \leq k \cdot 2^{s_1}$.

In all instances we compare with $2^{o(k+1)}$.

In the first subcase, $s_1 = \frac{k}{t_1} \leq \frac{k}{t_2} = s_2$, so the inequality to prove is implied by $2k \cdot 2^{\frac{2(k-t_2)}{t_2(t_2-2)}} \leq 2^{o(k+1)} = 2^{k+1-l(k+1)}$; this in turn is implied, using (2.3) and the fact that $\frac{k-t_2}{t_2(t_2-2)} \leq \frac{k-4}{8}$, by $4k\sqrt{k+1} \cdot 2^{\frac{k-4}{4}} \leq 2^{k+1}$. This last one is equivalent to $k\sqrt{k+1} \leq 2^{\frac{3k}{4}}$, which is easily seen to be strict for $k \geq 5$; however, $g \geq 8$ implies $k \geq 7$ so in this first subcase we always get a strict inequality.

In the second subcase, $s_1 = \frac{k}{t_1} \le \frac{k}{3} = s_2$; as above, we reduce to $k\sqrt{k+1} \le 2^{\frac{2k}{3}}$, which is easy to be seen as true (strict inequality) for $k \ge 6$, which again is what we needed given that $k \ge 8$.

The third subcase is reduced to the same inequality as the second, so we are done. □

Lemma 6.3. *If G is a strictly optimal graph, G' cannot contain more than one component which is a cycle. Moreover, Theorem B holds for those g for which C_g contains an isolated cycle.*

Proof. Due to (6.2), we are left to analyze the case where some components of G' are cycles while the others are not.

First, (5.13) allows us to assume that the s_2 components H_i of G' which are not cycles may be stabilized without problems (no double edges or loops occur); in particular, the arithmetic genus, denoted by h, of these components is at least three. Then the induction hypothesis shows that $|\pi(H_i)| \le 2^{o(h)}$ for $h \ge 3$. G' has also s_1 components which are cycles, each with n_1 neighbors (components of G' at distance one); we denote by t the *incidence degree* of two components in different isomorphism classes, i.e. the number of edges in $O(e)$ joining two such components; we also denote by n_2 the number of neighbors of a component H_i. We have: the length of the cycles is $n_1 t$, $k = s_1 n_1 t = s_2 n_2 t$ and $g = s_2(h-1)+1+k$. Note that $n_1 t \ge 3$ (to avoid double edges in G) and since we want $s_1 = 1$ it is enough to show that $k \ge 6$ leads to contradictions. We will discuss also what happens when $s_1 = 1$.

Estimate $|\mathrm{Aut}\, G|$ by first contracting the cycles to vertices with dihedral symmetry at the edges around them, then using again an exhausting subgraphs argument; we start by choosing a cycle (s_1 choices), then fixing its orientation ($2n_1 t$ choices). We get $|\mathrm{Aut}\, G| \le 2k \cdot 2^a \cdot 2^{s_2 o(h)}$, and would like to compare this to $2^{o(g)}$. Due to the way we mentioned we expand the subgraphs, preferably at their tails incident to components H_i (when they exist at a certain stage) we see that the only possibility of gaining extra symmetry when forced to incorporate a cycle is when that cycle had at most two incident edges (and precisely one if its length $n_1 t = 3$). Including such a cycle will immediately yield $n_1 t - 2$ (respectively 2) tails incident to components H_i; since t was the incidence, we see that we have: if $t \ge 2$, only one component H_i was incident to this cycle, so $n_1 - 1$ new components will be reached by tails after the cycles is included in the newly increased subgraph; if $t = 1$ but $n_1 \ge 4$, at least $n_1 - 2$ new components will be reached; and finally if $t = 1$ but $n_1 = 3$, exactly $2 = n_1 - 1$ new components will be reached. This happens each of the a times. Thus we have:

- $s_2 \ge n_1 + a(n_1 - 1)$ if either $t \ge 2$ or $t = 1, n_1 = 3$
- $s_2 \ge n_1 + a(n_1 - 2)$ if $t = 1$ and $n_1 \ge 4$

Quick manipulations lead us to the inequality

$$2k \le 2^{s_2 h - s_2 + k + 1 - l(s_2 h - s_2 + k + 1) - a - s_2 h + s_2 l(h)}$$

Using (2.3) we get $l(s_2 h - s_2 + k + 1) \le l(s_2) l(h) + l(k + 1 - s_2)$ so the above inequality is implied by $2k \le 2^{B(s_2,h) - a + o(k+1-s_2)}$.

If $s_2 = 1$, then $n_1 t \ge 3$ (the cycles must have length at least three, otherwise G would have double edges); moreover, the cycles are connected to the "core" component H of G' by edges starting from all of their vertices. Thus (dispensing with the above bound on $|\mathrm{Aut}\, G|$) we have in fact $|\mathrm{Aut}\, G| = |\mathrm{Aut}\, H|$. But $g = h + k = h + s_1 n_1 t \ge h + 3$, so $o(g) \ge o(h) + o(3) = o(h) + 2$; now either $h \ge 9$, in which case the induction hypothesis says that $|\mathrm{Aut}\, H| \le 3 \cdot 2^{o(h)} < 4 \cdot 2^{o(h)} \le 2^{o(g)}$ (so G could not have been optimal), or $h \le 8$, in which case the table (4.3) and the estimates (4.4) show again that G could not have been optimal. Thus this case cannot occur for an optimal graph G.

Thus from now on $s_2 \ge 2$, which implies $n_1 \ge 2$.

Suppose $n_2 \ge 2$ (or, equivalently, $s_1 \ge 2$).

If $n_1 \ge 4$ (and any t), then $s_2 \ge 4$ (so by (2.4) $B(s_2,h) \ge \lfloor \frac{s_2+1}{2} \rfloor$) and $s_2 \le \frac{k}{n_2 t} \le \frac{k}{2}$; also $a \le \lfloor \frac{s_2-1}{2} \rfloor - 1$; then $B(s_2,h) - a \ge 2$. The inequality is then implied by $2k \le 2^{2+o(\lceil \frac{k}{2} \rceil + 1)}$, or equivalently $\frac{k}{2} \le 2^{o(\lceil \frac{k}{2} \rceil + 1)}$. Now (2.2) shows that this is strict, so G could not have been optimal.

If $n_1 = 3$ (and any t), then $s_2 \ge 3$, $k = 3s_1 \ge 6$ and $a \le \lfloor \frac{s_2-1}{2} \rfloor - 1$; in the same time $s_2 = \frac{k}{n_1 t} \le \frac{k}{3}$. (2.4) (more precisely, (1)) shows that $B(s_2,h) - \lfloor \frac{s_2+1}{2} \rfloor \ge -1$ (with equality if and only if $s_2 = 3$ and $l(h) = 1$) so

$B(s_2,h) - a \geq 1$; the inequality is then implied by $2k \leq 2^{1+o(\lceil \frac{2k}{3} \rceil+1)}$ which is strict by (2.2) so again G is not optimal.

Thus we must have both $s_1 = n_2 = 1$ and $s_2 = n_1 \geq 2$ in an optimal G; the lemma is proved.

Then $a = 0$ and $k = s_2 t$; we must have $k \geq 3$ since that is the length of the cycle.

If $t \geq 2$, $2 \leq s_2 \leq \lfloor \frac{k}{2} \rfloor$ and $B(s_2,h) \geq l(h) \geq 1$ (using (1) in (2.4)). The inequality is implied by $k \geq 2^{o(\lceil \frac{k}{2} \rceil+1)}$, which by (2.2) is strict except for $k = 2,4,8$. But $k = s_2 t \geq 4$ so only $k = 4,8$ need consideration. $k = 4$ may occur only when $s_2 = t = 2$ (and $l(h) = 1$) and then (5.14) shows that G was not optimal. $k = 8$ may occur for either $s_2 = 2, t = 4$ or $s_2 = 4, t = 2$; however the inequality $2k \leq 2^{B(s_2,h)+o(k+1-s_2)}$ is easily seen to be strict in these cases, so again G was not optimal.

Then we must have $t = 1$ so $k = s_2 \geq 3$; the inequality becomes $2k \leq 2^{B(k,h)}$. Then (2.4) and (2.2) show that the inequality is strict for all $k \geq 6$.

If $k = 5$, the inequality becomes $10 \leq 2^{B(5,h)}$ which is strict for $l(h) \geq 2$ by (1), and false for $l(h) = 1$

If $k = 4$, the inequality becomes $8 \leq 2^{B(4,h)}$ which is again strict for $l(h) \geq 2$ by (1), with equality for $l(h) = 1$.

If $k = 3$, the inequality becomes $6 \leq 2^{B(3,h)}$ which is strict for $l(h) \geq 3$ by (1), and fails for $l(h) = 1$ or when $l(h) = 2$ and $l(3h) = 4$.

Even if g would be 9, 2^n, or $3 \cdot 2^n$, a small calculation shows that the graphs with a cycle do not give an optimal graph.

The cases which remain are exactly the g for which C_g contains an isolated cycle, so Theorem B holds in these cases by induction. □

From now on, we may assume that G' contains no cycles. We first address the case that the minimal orbit is a disjoint union of stars.

Lemma 6.4. *Let G be a simple cubic graph of genus $g \geq 9$, with a minimal orbit $O(e)$ a disjoint union of k stars, with all the components of G' of genus $h \geq 3$ (by induction on Theorem A, $\pi(G_i) \leq 2^{o(h)}$). Then G is not strictly optimal as soon as $k \geq 2$; moreover, for $k = 1$, G is strictly optimal only if either $l(h) = 1$ or $h = 3 \cdot (2^m + 2^p)$ with $|m - p| \geq 5$. Therefore, by induction, Theorems A holds in these cases.*

Proof. The reduction (5.13) shows that in case $O(e)$ is a disjoint union of stars, the components of G' must stabilize without problems; thus we may use the induction hypothesis in this case.

If G' is connected, then $|\mathrm{Aut}\ G| \leq |\mathrm{Aut}\ G'^{\mathrm{stab}}| \leq |\mathrm{Aut}\ C_{g'}|$ where $g' = g - 2k$. Using (4.4) we see that G could not have been optimal for any $k \geq 1$.

Then let $s \geq 3$ be the number of components of G' (if $s \leq 2$ then G' is connected, as a star cannot be incident to a given component twice without having actually all tails in that component; thus G' disconnected implies that each star is incident to three distinct component of G'). Let t be the number of edges in $O(e)$ incident to a given component. Then $3k = st$, and $g = sh - s + 2k + 1$.

We also note that $t = 1$ implies $k = 1$ (the subgraph made up of a star and the three components to which it is incident would be a connected component of G, thus the whole G).

Using the exhausting subgraphs argument we get $|\mathrm{Aut}\ G| \leq 6k \cdot 2^{s \cdot o(h)} \cdot 2^a$, where a is the number of times we might have gained a twofold increase in the automorphism group of the subgraph by including a (new) star at a tail incident to it. Due to the way we construct these enhausting subgraphs, each inclusion of a star counted among the a ones will make the new subgraph incident to two other components to which the previous subgraph was not incident. Thus we see that $s \geq 3 + 2a$ (since at the beginning we already had a star incident to three components).

Thus we would like to show that $6k \cdot 2^{s \cdot o(h) + \lfloor \frac{s-3}{2} \rfloor} \leq 2^{o(g)}$, or equivalently (using (2.3))

$$3k \leq 2^{o(2k-s+1)+sl(h)-l(sh)-\lfloor \frac{s-1}{2} \rfloor} = 2^{A(s,h)+o(2k-s+1)}$$

If $t \geq 3$ then $s \leq k$ so the inequality is implied by $3k \leq 2^{A(s,h)+o(k+1)}$; since $s \geq 3$, by (2.4) $A(s,h) \geq 1$ and using (2.2) this is easily seen to be strict for all $k \geq 1$, so such a G cannot be optimal.

If $t = 2$, then setting $k = 2u$ we get $s = 3u$; we need to study when $6u \leq 2^{o(u+1)+A(3u,h)}$. For $u \geq 2$, $A(3u,h) \geq 2$ by (2.4) and then again (2.2) shows that the inequality is strict; such G cannot be optimal. If $u = 1$, there are three connected components in G', isomorphic and linked by two stars; it is apparent that already $M(G) \geq 3$. The inequality becomes $6 \leq 2^{o(2)+3l(h)-l(3h)-1} = 2^{3l(h)-l(3h)} \geq 2^{l(h)}$. Then clearly for $l(h) \geq 3$ the inequality is strict, as it is for $l(h) = 2$ but $l(3h) \leq 3$. Thus either $l(h) = 1$ or $l(h) = 2$ and $l(3h) = 4$; however, even in these cases (5.14) shows that G could not have been strictly optimal.

We are left with considering the case $t = 1$. Then $k = 1$ as remarked before, so the inequality becomes $6 \leq 2^{3l(h)-l(3h)}$. From (1), we see that the inequality is strict as soon as $l(h) \geq 3$. Moreover, for $l(h) = 2$ and $l(3h) \leq 3$ we get again a strict inequality.

Thus only $l(h) = 1$ or $l(h) = 2, l(3h) = 4$ are left. In all cases, $M(G) = 3$, and overall $|\mathrm{Aut}\ G| \leq 3 \cdot 2^{o(g)}$ as claimed; moreover, equality may occur only when $h = 3 \cdot 2^m$. We only need to show that any graph with more than $2^{o(g)}$ automorphisms is forced to have $M(G) \geq 3$, and it is clear that the above reductions prove just that, so we are done. $\square$

Finally, the case of $O(e)$ a disjoint union of edges must be analyzed.

Lemma 6.5. *Let G be a simple cubic graph with a minimal orbit $O(e)$ a disjoint union of k edges, with all components of G' of genus at least three and stabilizing to simple cubic graphs. Then G is not strictly optimal as soon as $k \geq 5$. Moreover, Theorems A and B hold in these cases.*

Proof. **Case 1: G' is connected** and (according to the hypothesis) it stabilizes without double edges or loops.

If $g = 9$ then $g' = g - k \leq 8$; it is clear that the inequality $|\mathrm{Aut}\ G| \leq |\mathrm{Aut}\ G'|$ and the table (4.3) show that G could not be optimal for any k.

If $g = 8$ then $g' \leq 7$; $2^{o(g)} = 128$ and it is clear that $|\mathrm{Aut}\ G| \leq |\mathrm{Aut}\ G'|$ and the table (4.3) show that G could not be optimal for any k.

If $g = 2^m \geq 16$, then for $k \geq 2$ we get $|\mathrm{Aut}\ G| \leq |\mathrm{Aut}\ G'| \leq |\mathrm{Aut}\ C_{g'}|$ and (4.4)) shows that G could not be optimal for such k; and if $k = 1$, then $l(g') \geq 2$ so by induction the Main Theorem shows that $|\mathrm{Aut}\ C_{g'}| \leq 3 \cdot 2^{g'-2} = 3 \cdot 2^{g-3} < 2^{o(g)}$; thus again G could not have been optimal.

Otherwise, $|\mathrm{Aut}\ G| \leq |\mathrm{Aut}\ G'| \leq |\mathrm{Aut}(G'^{\mathrm{stab}})| \leq |\mathrm{Aut}(C_{g'})|$. Now $g' = g - k$ so (4.4) shows that G is not optimal as soon as $k \geq 2$. If $k = 1$, (4.4) gives $|\mathrm{Aut}\ G| \leq |\mathrm{Aut}\ G'^{\mathrm{stab}}| \leq |\mathrm{Aut}\ C_{g'}|$; we would like to compare the last term with $2^{o(g)} \geq 2^{o(g')}$. If $|\mathrm{Aut}\ G'| \leq 2^{o(g')}$, we get $|\mathrm{Aut}\ G| \leq 2^{o(g)}$ which would prove Theorem A in this case. If however $|\mathrm{Aut}\ G'^{\mathrm{stab}}| > 2^{o(g')}$, then $M(G'^{\mathrm{stab}}) \geq 3$ by induction. Now $k = 1$ shows that the edges of G'^{stab} which are pinched by e must be in an orbit by themselves under the action of Aut G'. Thus Aut G' is the stabilizer of the two unstable paths (edges pinched by e). If $M(G') = 4$ we then get at least a twofold reduction in $|\mathrm{Aut}\ G'^{\mathrm{stab}}|$ vs. $|\mathrm{Aut}\ G'|$, while if $4 \neq M(G') \geq 3$ we get at least a threefold reduction in the same comparison. The induction shows that $M(G') = 4$ happens only for $g' = g - 1 = 5 \cdot 2^m + 1$ (and then $|\mathrm{Aut}\ G| < 2^{o(g)}$) and otherwise $|\mathrm{Aut}\ G'| \leq 2^{o(g')}$ as desired, with equality if and only if $g' = g - 1 = 9 \cdot 2^m + u$ $(u = 0, 2)$; if $u = 2$, $o(g) > o(g')$ so again G could not have been optimal, while if $u = 0$ the induction shows that the star disconnecting the G'^{stab} is the minimal orbit; inserting e produces actually a sixfold loss of symmetry, so again G could not have been optimal since in any case we get $|\mathrm{Aut}\ G| < 2^{o(g)}$. Note that the reasoning applies when $g = 9$ or $g = 2^m$ with $m \geq 4$ to yield that such a G could not have been optimal (strict or not).

Case 2: G' is disconnected and its components are split in two isomorphism classes, one of s_1 subgraphs isomorphic to H_1 and one with s_2 subgraphs isomorphic to H_2. Say each H_1 is at distance one from n_1 components isomorphic to H_2; define n_2 similarly. Define also t to be the number of edges linking two neighboring components H_1 and H_2. Note that $g = s_1(h_1 - 1) + s_2(h_2 - 1) + k + 1$, $k = s_1 n_1 t = s_2 n_2 t$; note also that $s_1 = 1$ if and only if $n_2 = 1$, and similarly $s_2 = 1$ if and only if $n_1 = 1$. We may assume, without loss of generality, that $s_1 \leq s_2$.

Start again with an exhausting subgraphs argument; the initial step is choosing an edge and its orientation (if all components of G' are isomorphic). We get $|\mathrm{Aut}\ G| \leq k \cdot 2^{s_1 o(h_1) + s_2 o(h_2)}$ and we would like to study the inequality $k \cdot 2^{s_1 o(h_1) + s_2 o(h_2)} \leq 2^{o(g)}$. This reduces to

$$k \leq 2^{s_1 l(h_1) + s_2 l(h_2) + k + 1 - s_1 - s_2 - l(s_1 h_1 + s_2 h_2 + k + 1 - s_1 - s_2)}.$$

This is implied, via (2.3), by the inequality

$$k \leq 2^{B(s_1, h_1) + B(s_2, h_2) + o(k + 1 - s_1 - s_2)}. \tag{*}$$

Assume first $t \geq 2$ and any $g \geq 9$.

If $n_1, n_2 \geq 2$, then $s_i \geq 4$ so by (2.4, (1)) we see that $B(s_i, h_i) \geq 2$; moreover, $s_i = \frac{k}{n_i t} \leq \frac{k}{4}$ so then the inequality is implied by $\frac{k}{16} \leq 2^{o(\lceil \frac{k}{2} \rceil + 1)}$ which is strict for all $k \geq 1$ in light of (2.2).

If $n_1 \geq 2$ and $n_2 = 1$, then actually $s_1 = 1$, $s_2 = n_1 \geq 2$ and $k = s_2 t \geq 4$. Then $B(s_2, h_2) \geq 1$ and $s_2 = \frac{k}{t} \leq \frac{k}{2}$ so the inequality is implied by $\frac{k}{2} \leq 2^{o(\lceil \frac{k}{2} \rceil + 1)}$ which is strict for all $k \geq 1$ by (2.2).

If finally $n_1 = n_2 = 1$ then $s_1 = s_2 = 1$, $k = t$ and the inequality becomes $k \leq 2^{o(k-1)}$. This is strict for $k \geq 5$ by (2.2), so we only need to consider the cases when G is made up of two (non-isomorphic) components linked by two, three or four edges in the same orbit of Aut G.

If $g = 9$, then if $k = 4$ the two components must have genus three each, so overall $|\text{Aut } G| \leq 4 \cdot 2^4$ (since $\mu_1(3) = 1$); if $k = 3$ then one component has genus three and the other genus four, so overall $|\text{Aut } G| \leq 3 \cdot 2^5$ (since $\mu_1(3) = \mu_1(4) = 1$); and if $k = 2$ we may have either two components of genus four each, or a component of genus three and one of genus five, in either case getting $|\text{Aut } G| \leq 2 \cdot 2^5$; then (4.3) shows that G could not have been optimal.

If $g = 8$, then only $k \leq 3$ is possible, otherwise one of the components will have genus less than two. If $k = 3$, the components are tetrahedra, so $|\text{Aut } G| \leq 3 \cdot 2^4$; if $k = 2$ then one component is a tetrahedron and the other one has genus four, so $|\text{Aut } G| \leq 2 \cdot 2^5$; in both cases we get less than $2^7 = 2^{o(g)}$ so such a G cannot be optimal.

If $g = 2^m \geq 16$, then $|\text{Aut } G| \leq k \cdot 2^{o(h_1)+o(h_2)} \leq k \cdot 2^{h_1+h_2-2} = k \cdot 2^{g-k-1} < 2^{g-1} = 2^{o(g)}$, so G cannot be optimal. Similar estimates show that such a G is not optimal if $g = 3 \cdot 2^m$ or $g = 3(2^m + 2^p)$ (with the usual restriction on m and p).

Otherwise, for $g > 9$ which is not a power of two, (5.14) reduces the search for strictly optimal G's to the case $|O(e)| = 3$. We note also that, since $g = h_1 + h_2 + 2$ and $l(g) \leq l(h_1) + l(h_2) + 1$, $|\text{Aut } G| \leq 3 \cdot 2^{o(h_1)+o(h_2)} = 3 \cdot 2^{o(g)-(l(h_1)+l(h_2)+2-l(g))} \leq \frac{3}{2} 2^{o(g)}$; we also have $M(G) = 3$. Fixing an edge in $O(e)$ must have as a result the fixing of all the edges in $O(e)$; regarding the components, fixing one of the three incidence points must fix all three (growing the subgraphs by choosing an edge, then including a component will automatically fix all the edges, therefore the other component's incidence points). Then, on one side, this implies $M(H_i^{\text{stab}}) \geq 3$; on the other it says $\pi(H_i) \leq \frac{1}{3}|\text{Aut } H_i|$. By induction we know that only for $h_i = 9 \cdot 2^{n_i} + u_i$ ($u = 0, 1, 2$) may we get the coefficient $\mu(H_i) = 3$; otherwise $\mu(H_i) \leq \frac{3}{2}$, so $\pi(H_i) \leq \frac{1}{2} 2^{o(h_i)}$ (for at least one i) so it is easy to see then that $|\text{Aut } G| \leq \frac{3}{4} 2^{o(g)}$ so G would not be optimal. However, a quick computation shows that even if $h_i = 9 \cdot 2^{n_i} + u_i$ one still gets $l(h_1) + l(h_2) \geq l(g)$ so $|\text{Aut } G| < 2^{o(g)}$ so G could not be optimal in this case.

Thus we must have $t = 1$ for a strictly optimal G.

If $n_1, n_2 \geq 3$, then $s_i \geq 3$ so again $B(s_i, h_i) \geq 2$, $s_i \leq \frac{k}{3}$ and the inequality is implied by $\frac{k}{16} \leq 2^{o(\lceil \frac{k}{3} \rceil + 1)}$, which is strict for all $k \geq 1$ by (2.2).

If $n_1 \geq 3$ and $n_2 = 2$ (similarly for $2 = n_1 < n_2$), then $s_2 \geq 3$, $s_1 \geq 2$ so $B(s_1, h_1) \geq 1$, $B(s_2, h_2) \geq 2$, while $s_2 \leq \frac{k}{2}$ and $s_1 \leq \frac{k}{3}$. Then the inequality is implied by $\frac{k}{8} \leq 2^{o(\lceil \frac{k}{6} \rceil) + 1}$, again strict for all $k \geq 1$ by (2.2).

If $n_1 = n_2 = 2$, then G is a pseudocycle formed by components linked to each of their two neighbours by an edge, with the alternate components isomorphic; then $s_1 = s_2$ (note that g is odd so $g = 2^m$, $3 \cdot 2^m$, and $3(2^m + 2^p)$ with $m, p > 0$ are impossible, and also that $g = 9$ would force all components to have genus two, which is again ruled out). If $s_1 \geq 3$ the cycle has length $2s_1 \geq 6$ and cannot be strictly optimal: grouping adjacent components two by two (linked by the common edge), pinching this edge and finally linking the new s_1 components to the vertices of a cycle of length s_1 produces a graph with the same number of automorphisms, but with fewer edges in the minimal orbit. If $s_1 = 2$ the inequality becomes $4 \leq 2^{l(h_1)+l(h_2)}$, so as soon as $\max(l(h_1), l(h_2)) \geq 2$ we get a strict inequality. Thus for an optimal G one must have $l(h_i) = 1$. Then $g = 2h_1 + 2h_2 + 1$ and $|\text{Aut } G| \leq 4 \cdot 2^{2o(h_1)+2o(h_2)} = 2^{o(g)-(2l(h_1)+2l(h_2)+1-l(2h_1+2h_2+1))}$; this is strictly less than $2^{o(g)}$ by (2.3) so G could not have been optimal.

If $n_1 = 1$ and $n_2 \geq 3$, then $s_2 = 1$ and $s_1 = n_2 = k \geq 3$; the inequality (*) becomes $k \leq 2^{B(k,h_1)}$.

- If $l(h_1) \geq 2$, (1) in the proof of (2.4) and (2.2) show that $2^{B(k,h_1)} \geq 2^{\lfloor \frac{k+1}{2} \rfloor} \geq k$, with at least one of these inequalities strict (since $k \geq 3$); thus G could not have been optimal.

- If $l(h_1) = 1$ a similar argument shows that only $k = 3$ should be considered in all cases. If $g = 9$ then $k = 3$ is not possible (a component would have genus less than two). If $g = 2^m$, $3 \cdot 2^m$, or $3(2^m + 2^p)$ and $k = 3$, $3h_1 + h_2 = g$ so $|\text{Aut } G| \leq 3 \cdot 2^{3o(h_1)+o(h_2)} = 3 \cdot 2^{3h_1+h_2-(3l(h_1)+l(h_2))} \leq 3 \cdot 2^{o(g)-(3l(h_1)+l(h_2)-l(g))} < 2^{o(g)}$

($l(g) \leq 2$) so G cannot be optimal. Otherwise, one first remarks that $M(H_i) \geq 3$, so $\mu_1(H_i) \leq \frac{1}{2}$ unless $h_i = 9 \cdot 2^{n_i} + u_i$ ($u_i = 0,1,2$); but that is impossible for h_1 since $l(h_1) = 1$. Then one has $|\mathrm{Aut}\ G| \leq 3 \cdot \frac{1}{2} \cdot 2^{3o(h_1)+o(h_2)} \leq \frac{3}{2} \cdot 2^{o(g)-(3l(h_1)+l(h_2)+1-l(3h_1+h_2+1))} \leq \frac{3}{4} \cdot 2^{o(g)}$ (due to (2.3)) so G could not have been optimal.

If $n_1 = 1$ and $n_2 = 2$, then $s_1 = 2$, $s_2 = 1$ and $k = 2$, then:

- If $g = 9$ then $2h_1 + h_2 - 3 + 1 + 2 = 9$ so only $h_1 = h_2 = 3$ is possible, i.e. all components are tetrahedra (after stabilization). Then $|\mathrm{Aut}\ G| \leq 2 \cdot 2^6 < |\mathrm{Aut}\ C_9|$ so G is not optimal.
- If $g = 2^m$ or $3 \cdot 2^m$ then $2h_1 + h_2 = g$ and $|\mathrm{Aut}\ G| \leq 2 \cdot 2^{2o(h_1)+o(h_2)} \leq 2^{o(g)-(2l(h_1)+l(h_2)-l(g))} < 2^{o(g)}$ so again G is not optimal.
- If $g = 3(2^m + 2^p)$ with $m \geq p+5$, then the previous estimate shows that G could only be optimal if $l(h_1) = l(h_2) = 1$. Inspecting the possibilities, one shows that $h_1 = 3 \cdot 2^{m-1}$ and $h_2 = 3 \cdot 2^p$. Now the induction shows that H_2, of genus $3 \cdot 2^p$, must be a A_p in order to reach equality above (otherwise $\pi(H_2) < 1$). But then one may compute $|\mathrm{Aut}\ G| < 2^{o(g)}$ so G is not optimal.
- In all other cases, G is made up of a core linked to each of two isomorphic components by an edge. Again pairing the isomorphic components, linking the ends of the edges incident to the core, and reattaching this to one of the original pinching points preserves (or increases) the number of automorphisms, but yields a smaller minimal orbit.

We are left then with $n_1 = n_2 = 1$, so $s_1 = s_2 = k = 1$, i.e. two (non-isomorphic) components linked by an edge, then $|\mathrm{Aut}\ G| \leq 2^{o(h_1)+o(h_2)} = 2^{o(g)-(l(h_1)+l(h_2)-l(g))}$.

- If $g = 9$, one can check this configuration cannot be optimal.
- If $g = 2^m$ or $g = 3 \cdot 2^m$, then $l(h_1) + l(h_2) - l(g)) \geq 1$, so G cannot be optimal.
- If $g = 3(2^m + 2^p)$ with $m \geq p+5$, then $2^o(g)$ automorphisms are obtained only if $l(h_1) = l(h_2) = 1$. This forces $h_1 = 3 \cdot 2^m$ and $h_2 = 3 \cdot 2^p$ (up to swapping h_1 and h_2). By induction, G may only be A_m and A_p linked by an edge (some checking rules out the case that $p \leq 1$).
- In all other cases, $l(h_1 + h_2) = l(h_1) + l(h_2)$ is forced by the optimality of G. We note that this prohibits $h_1 = h_2$; in particular, an optimal graph G cannot be built out of two non-isomorphic components of the same genus, linked by an edge. Thus, if $g = 3(2^m + 2^p)$, the bound $2^{o(g)}$ may only be obtained by linking an A_m and an A_p at their roots.

Case 3: G' is disconnected and all its components are isomorphic. Let s be the number of connected components of G', all of genus h, t the incidence degree between two neighbors, and m the number of neighbors of a given component; then $g = sh - s + k + 1$ and $2k = smt$; moreover, $s = 2$ if and only if $m = 1$. We bound, as before (but with the extra possibility of flipping the edges in $O(e)$) $|\mathrm{Aut}\ G| \leq 2k \cdot 2^{so(h)}$. The inequality $2k \cdot 2^{so(h)} \leq 2^{o(sh-s+k+1)}$ is equivalent to $2k \leq 2^{o(k+1-s)+sl(h)-l(sh)} = 2^{o(k+1-s)+B(s,h)}$.

If $m \geq 3$ (so $s \geq 3$) and $t \geq 2$, then $B(s,h) \geq 1$ by (1), and $s = \frac{2k}{mt} \leq \frac{k}{3}$; then the inequality is implied by $k \leq 2^{o(\lceil \frac{2k}{3} \rceil + 1)}$ which is strict for all k by (2.2).

If $m = 2$ and $t \geq 2$, then $k = st \geq 6$, $B(s,h) \geq 1$, $s \leq \frac{k}{2}$ and the inequality is implied by $k \leq 2^{o(\lceil \frac{k}{2} \rceil + 1)}$ which is strict for all k except $k = 2,4,8$ by (2.2). $k = 2,4$ are not possible here, while $k = 8$ forces $s = 4, t = 2$. Then $g = 4h+5$ impossible for $g = 9$ or $g = 2^n$; in other genera, $B(4,h) \geq 3$ so the inequality is easily seen to be strict anyway; no such G may be optimal.

If $t = 1$ and $m \geq 4$, then $s \geq 4$ so $B(s,h) \geq 3$ and $s \leq \frac{2k}{m} \leq \frac{k}{2}$ so the inequality is implied by $\frac{k}{4} \leq 2^{o(\lceil \frac{k}{2} \rceil + 1)}$ which is strict for all k by (2.2).

If $t = 1$ and $m \geq 3$ then $s \geq 4$ so $B(s,h) \geq 3$ and $s \leq \frac{2k}{3}$; we readily get a strict inequality by (2.2).

If $t = 1$ and $m = 2$ then G is a cycle of components of genus h (each component is incident to precisely other two by an edge); $s = k \geq 3$ (otherwise $t = 2$, in fact), $g = sh + 1$.

- This is easily seen to be impossible for $g \leq 9$ ($h \leq 2$ is forced).

- If $g = 2^m, 3 \cdot 2^m$, or $g = 3(2^m + 2^p)$, $|\mathrm{Aut}\, G| \leq 2k \cdot 2^{ko(h)} \leq 2k \cdot 2^{o(g)-(kl(h)+1-l(kh+1))} \leq 2^{o(g)} \cdot \frac{2k}{2^{B(k,h)}}$. If $k \geq 8$ or $k = 6$, the last fraction is strictly less than one (because at least one of the inequalities $2^{B(k,h)} \geq 2^{\lceil \frac{k+1}{2} \rceil + 1} \geq 2k$ is strict), so G is not optimal. For $k = 7$, one sees directly that $2^{B(k,h)} > 14$ by (1). For $k = 4, 5$, the inequality $2^{B(k,h)} > 2k$ is easily seen to be true except for $l(h) = 1$. However, in that case, the induction shows that the components have to be graphs of type A_m or B_p. In this case, $|\mathrm{Aut}\, G| \leq k \cdot 2^{ko(h)}$, and this is easily seen to be less than $2^{o(g)}$. For $k = 3$ and $l(h) \geq 3$, we have $2^{B(k,h)} > 6$. Moreover, $g = 3 \cdot 2^m$ and $g = 3(2^m + 2^p)$ cannot be written in the form $3h + 1$, so only the case $g = 2^m = 3h + 1$, $k = 3$, $l(h) = 1$ is left. This is easily seen to be impossible (numerically).
- In all other genera, we study $2k \leq 2^{B(k,h)}$. $l(h) \geq 3$ would imply strict inequality, so only $l(h) \leq 2$ is possible. $l(h) \geq 2$ and $s \geq 3$ imply by (2.4) and (2.2) that G is not optimal (split in $s \geq 4$ and $s = 3$). So only $l(h) = 1$ is left, for which again (2.4) and (2.2) show that the only hope for optimality is $k = 3$. This is only possible for $h = 3, g = 10$ - a pseudocycle of $K_{3,3}$s with two edges pinched of length three. given below.

  If the two ends of edges in $O(e)$ incident to a given component are in the same orbit, then all ends of edges in $O(e)$ are in the same orbit. These ends cannot pinch the same edge of H by (5.5); also, (5.13) says that we may have at most one "problem" when trying to stabilize a strictly optimal graph G; since we have six incidence loci, H is a simple cubic graph. Then we must have $M(H) \geq 2$. Then $h \geq 9$ must be one of the special genera of the Main Theorem, and for $l(h) = 1$ (and $M(H) \geq 2$) we must have $\pi(H) \leq \frac{1}{2} 2^{o(h)}$. Then $|\mathrm{Aut} G| \leq 6 \cdot \frac{1}{8} 2^{3o(h)} = \frac{3}{4} \cdot 2^{o(g)-(3l(h)+1-l(3h+1))} < 2^{o(g)}$ so G is not optimal. If $l(h) = 1$ and $h \leq 8$, then $h = 3, 4, 6, 8$; we note that for H it must be true that fixing one of the two pinched edges is the same as fixing both pinched edges; it is easy now to determine that the only genus for which there exists a graph H with the maximum number of automorphisms preserving two edges equal to $\pi(H) = 2^{o(h)}$ and $M(H) \geq 2$ is $g = 3$, and H must be a tetrahedron (pinching this in two opposite edges gives the $K_{3,3}$ with an edge removed). Thus this last situation occurs indeed only for $g = 10, h = 3$.

If $t = 1$ and $m = 1$ then G is made up of two isomorphic components linked by an edge. If $g = 9$, this is not possible since nine is odd. If $g = 8$, $2^{o(g)}$ automorphisms may only be obtained when the two components of G' are tetrahedra each with a pinched edge, i.e. a graph B_3 (note that this graph is not optimal). In the other cases, we have $|\mathrm{Aut}\, G| \leq 2 \cdot 2^{2o(h)} = 2 \cdot 2^{o(g)-(2l(h)-l(g))} \leq 2^{o(g)}$ with equality only if $l(h) = 1$. If $g = 2^m$ or $3 \cdot 2^m$, then the inductive argument yields the unique shape of G as B_m, respectively A_m, proving Theorem B in this case. If $g = 3(2^m + 2^p)$, such a graph is not optimal since $l(h) = 2$. □

This completes the proof of Theorems A and B.

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