Variation of hyperplane sections

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Throughout this article we assume $n > 2$ and $\epsilon^2 = 0$. All varieties are defined over an algebraically closed field k.

In [HMP98], Harris, Mazur, and Pandharipande ask: given a smooth, degree d hypersurface X in $\mathbf{P}^n$, does the family of all smooth l-plane sections of X vary maximally in moduli? They answer this question when d and l are quite small compared to n, and when the characteristic of the base field is zero. The purpose of this article is to establish that the variation of smooth hyperplane sections of X is maximal when X is a general smooth hypersurface of degree d, over an algebraically closed field of arbitrary characteristic.

Beauville proves that the variation in moduli is not zero (as long as $d > 2$, of course), except for Fermat hypersurfaces of degree d such that $(d-1)$ is a power of the characteristic [Bea90]. The proof is elegant, and we follow the same line here, at least initially. We also show that the variation of hyperplane sections of the Fermat is maximal except in those cases excluded by Beauville.

The main theorem of this article is

Theorem 1. *If X is a general hypersurface in $\mathbf{P}^n$, then the hyperplane sections of X vary maximally in moduli.*

First of all, note that the degree one and two cases are trivial, since there are no moduli, so we assume from now on that $d > 2$. Geometric invariant theory then provides a moduli space $M(d,n)$ of smooth degree d hypersurfaces in $\mathbf{P}^n$, at least when char $k \not| d$ (since the methods employed in this paper are infinitesimal, we need not rely on the existence of a global moduli space, but we use it for motivation). This is the quotient of some open set of the projective space parameterizing all degree d hypersurfaces by the action of $\mathrm{Psl}(n+1)$, so the dimension $m(d,n)$ of the moduli space is

$$m(d,n) = \binom{n+d}{d} - (n+1)^2.$$

The space of hyperplane sections is n dimensional, so $n > m(d, n-1)$ if and only if $n = 3$ and $d = 3$. That is, for a cubic surface, there is a three parameter family of hyperplane sections, but only one modulus of cubic curves. In this case we have

Theorem 2. *Suppose* char $k \neq 2$*, and let X be a smooth cubic surface. Then the smooth hyperplane sections have maximal variation in moduli. If* char k *is two, then if X is a general cubic surface the smooth hyperplane sections have maximal variation in moduli.*

Proof. This follows from the main result of Beauville quoted above. □

Therefore, in what follows, we may assume that $n > 2$, $d > 2$, and $(n,d) \neq (3,3)$. Then we are asking that the map from the complement of the discriminant locus in the dual projective space parameterizing hyperplane sections of X to the moduli space of hypersurfaces is generically finite. Note that this is in a sense the opposite of the considerations of Harris, Mazur, and Pandharipande, in that their lower bound for n in terms of d and k ensures that maximal variation is checked by checking *surjectivity* of the differential of the variation, whereas we will check *injectivity.*

1 Infinitesimal study

In this section, we reduce the problem to linear algebra by considering it only to first order.

1.1 Preliminaries

As above, let X be a smooth degree d hypersurface in $\mathbf{P}^n$. Let us briefly recall some basics about deformations of X. Since X is smooth, its first order deformations up to isomorphism are classified by $H^1(X, T_X)$. The deformations of X as a subvariety of $\mathbf{P}^n$ are classified by $H^0(X, N_{X/\mathbf{P}^n})$. In this case, $N_{X/\mathbf{P}^n} \cong \mathcal{O}_X(d)$, and choosing a degree d polynomial g in this space, the corresponding first order deformation is given by $f + \epsilon g = 0$. The short exact sequence

$$0 \to T_X \to T_{\mathbf{P}^n}|_X \to N_{X/\mathbf{P}^n} \to 0$$

induces a morphism $H^0(N_{X/\mathbf{P}^n}) \to H^1(T_X)$ taking an embedded deformation to its isomorphism class as an abstract deformation. Recall that the Jacobian ring $R(X)$ of a hypersurface X with defining equation f is the ring

$$k[x_0, \ldots, x_n]/(f, \partial f/\partial x_0, \ldots, \partial f/\partial x_n).$$

Note that if char $k \not| d$, then f is automatically in the ideal generated by the partials by the Euler formula. We will use the following result of Beauville (*loc. cit.*), whose proof is elementary:

Lemma 3. *The morphism $H^0(N_{X/\mathbf{P}^n}) \to H^1(T_X)$ factors through $R(X)_d$, and the induced morphism $R(X)_d \to H^1(T_X)$ is injective.*

Remark 4. This lemma is false in the case $n = 2$ and $d = 3$, which is excluded from our consideration.

1.2 Criterion for maximal variation

Now let H be a hyperplane such that $X \cap H$ is smooth. Suppose f is the defining equation of X and that H is given by $x_0 = 0$. Then to first order, a deformation of H is given by $x_0 = \epsilon l(x_1, \ldots, x_n)$ where l is a linear form. The equation for the corresponding first order deformation of $X \cap H$ is given by

$$f(\epsilon l(x_1, \ldots, x_n), x_1, \ldots, x_n) = 0.$$

We may expand this in ϵ to obtain

$$f(0, x_1, \ldots, x_n) + \epsilon \frac{\partial f}{\partial x_0}(0, x_1, \ldots, x_n) l(x_1, \ldots, x_n) = 0$$

From the results of the last section, we conclude:

Proposition 5. *Notation as above. Suppose $\frac{\partial f}{\partial x_0}(0, x_1, \ldots, x_n) \neq 0$. An embedded deformation of $X \cap H$ corresponding to a linear form l in $x_1, \ldots, x_n$ is trivial to first order (as an abstract deformation) if and only if*

$$\frac{\partial f}{\partial x_0}(0, x_1, \ldots, x_n) l(x_1, \ldots, x_n)$$

is zero in the Jacobian ring $R(X \cap H)$.

Remark 6. Note that this first order criterion must be applied with care. For example, for the hyperplane section $x_0 = 0$ of the Fermat hypersurface, the left hand side will be zero regardless of the choice of l. On the other hand, in general, the hyperplane sections of the Fermat have some variation (and in characteristic zero, in fact, maximal variation; see the examples below). We must show that l vanishes, so choosing a hyperplane section as above with $\frac{\partial f}{\partial x_0}(0, x_1, \ldots, x_n) = 0$ gives us no information. When this derivative vanishes, we must perturb our hyperplane section a little and then check the criterion.

2 Openness of maximal variation

Theorem 7. *The set of smooth hypersurfaces whose variation of hyperplane sections is maximal is Zariski open in the space of all smooth hypersurfaces.*

Proof. Denote by $D(d,n)$ the open subset of projective space parameterizing smooth hypersurfaces of degree d in $\mathbf{P}^n$. Let U be the set in $D(d,n)$ whose hyperplane sections vary maximally in moduli. Let X be such a hypersurface. Then there exists a hyperplane section H such that the map

$$H^0(N_{X\cap H/H}) \to H^1(T_{X\cap H})$$

is injective. But $H^0(N_{X\cap H/H})$ can be identified with the tangent space to $D(d,n-1)$ at the point corresponding to $X\cap H$. Let $\mathcal{X}$ be the universal hypersurface over $D(d,n)$ and $\mathcal{H}$ the constant hyperplane $D(d,n)\times H$. Let $\mathcal{Y}$ be the intersection $\mathcal{X}\cap\mathcal{H}$ and π the projection of $\mathcal{Y}$ onto $D(d,n-1)$. Note that since X will have singular hyperplane sections, π is not everywhere defined, but is defined in a Zariski open neighborhood of $X\cap H$. Then the map of cohomology spaces above is just the restriction of the map of vector bundles

$$T_{D(d,n-1)} \to R^1\pi_* T_{\mathcal{Y}/D(d,n-1)}$$

to the point in $D(d,n-1)$ corresponding to X. Since this map of vector bundles is injective at this point, it is injective in an open neighborhood. Hence U is open. $\square$

Theorem 1 follows from this theorem if the set U is nonempty. This is shown in the next section.

3 Examples

Example 8. The hyperplane sections of the smooth hypersurface defined by

$$\sum_{i=0}^{n} x_i^d + \sum_{j=0}^{n-1} x_j^{d-1}x_{j+1} + x_n^{d-1}x_0 = 0$$

vary maximally in moduli.

Proof. We will use the criterion from Proposition 5. Small deformations of the hyperplane section $x_0 = 0$ vary maximally in moduli if when we write

$$l_0 x_n^{d-1} = \sum_{i=1}^{n-1} l_i(dx_i^{d-1} + (d-1)x_i^{d-2}x_{i+1} + x_{i-1}^{d-1}) + l_n(dx_n^{d-1} + x_{n-1}^{d-1}), \tag{1}$$

where the l_i are linear forms in the variables $x_1,\ldots,x_n$, we can conclude that $l_0 = 0$.

First assume char $k \mid (d-1)$. Then by considering terms in which x_n occurs to the $(d-1)$st or dth power, we conclude $l_0 = dl_n$. Repeating for terms where x_{n-1} occurs to the $(d-1)$st or dth power we obtain $l_n = -dl_{n-1}$. Continuing likewise, we see that all the l_i are multiples of l_1. We can cancel l_1 from the rewritten form of (1). In this way, we obtain a linear relation among the partial derivatives of f with x_0 set equal to zero, which contradicts the smoothness of X.

The second case is when char $k \mid d$. Considering terms with high powers of x_n as above, if $d > 3$ it follows immediately that $l_0 = 0$. If $d = 3$, we see that necessarily

$$l_{n-1} = ax_n + bx_{n-1} + cx_{n-2}, l_0 = 2ax_{n-1},$$

but this introduces a term on the right hand side $ax_nx_{n-2}^2$ which cannot be cancelled by any other term, so $a = 0$.

So we may assume that char k divides neither $(d-1)$ nor d. Again, for simplicity, assume first that $d > 3$. Then consindering terms with high powers of x_n, we see that $l_0 = dl_n$. Passing on to terms with x_{n-1} occuring to the power of at least $d-2$, cancelling x_{n-1}^{d-2} we obtain

$$0 = dl_{n-1}x_{n-1} + (d-1)l_{n-1}x_n + l_nx_{n-1}$$

from which it follows that

$$l_{n-1} = ax_{n-1}$$

and that $l_n = 0$ (and therefore $l_0 = 0$) if $l_{n-1} = 0$. Considering terms with x_{n-2} occuring to a high power, we get

$$0 = dl_{n-2}x_{n-2} + (d-1)l_{n-2}x_{n-1} + ax_{n-1}x_{n-2}$$

and conclude that $l_{n-2} = \alpha x_{n-2} + \beta x_{n-1}$ for some choice of constants. Plugging this back in shows that in fact l_{n-2} has to be zero, which also implies that l_{n-1} is zero, so we are done.

The case where $d = 3$ is similar and left to the reader. $\square$

The second example is superfluous in proving the main result, but shows that in some sense the "most probable counterexample" to the conjecture that hyperplane sections vary maximally (at least in "good" characteristics) is in fact not a counterexample.

Example 9. Assume char $k \neq d$ and that $d-1$ is not a power of char k. Then the hyperplane sections of the Fermat hypersurface defined by $\sum x_i^d = 0$ vary maximally in moduli.

Proof. Since char $k \neq d$, the Fermat is smooth. As noted above, the hypersurface section defined by $x_0 = 0$ is not a good choice for applying our criterion. Let $a = \sum_{i=1}^n a_i x_i$ be a general linear form. Then we apply our criterion to the hypersurface defined by

$$(x_0 + a)^d + \sum_{i=1}^n x_i^d = 0,$$

which is equivalent to considering variation near the hypersurface section $x_0 + a = 0$ of the Fermat (which is smooth by Bertini, since a is general). We assume there is a relation of the form:

$$l_0 \frac{\partial f}{\partial x_0} |_{x_0=0} = \sum_{i=1}^n l_i \frac{\partial f}{\partial x_i} |_{x_0=0}$$

where l_i are linear forms in $x_1, \ldots, x_n$ as above. That is,

$$dl_0 a^{d-1} = \sum dl_i(a_i a^{d-1} + x_i^{d-1}).$$

Set $m = l_0 - \sum_{i=1}^n a_i l_i$, so that

$$ma^{d-1} = \sum l_i x_i^{d-1}.$$

Since a is a general linear form, the polynomial on the left hand side has monomials with three or more distinct variables (as long as $d-1$ is not a power of the characteristic, so "freshman exponentiation" does not hold), but the right hand side does not, so $m = 0$, from which it follows that all the l_i are zero. $\square$

4 Conclusion

Consider the hypersurface defined by

$$0 = x_0^3 + x_1^3 + x_0x_1^2 + x_1x_2^2 + x_3^3 + x_2x_4^2.$$

It is smooth, and the hyperplane section $x_0 = 0$ is also smooth. Furthermore, since $\frac{\partial f}{\partial x_0}|_{x_0=0} \neq 0$, the criterion is not vacuous. However, if $l_0 = ax_1 + bx_4$, one can solve for $l_1, \ldots, l_4$ in the criterion above. So the variation of hyperplane sections is not maximal near this hyperplane. However, one can check that variation is still maximal near some other hyperplane. Here, the two tangent vectors which are killed must be tangent to some two-dimensional subvariety which is blown down by the map to the moduli space. So one must check the criterion for all possible linear perturbations of x_0. This is computationally quite complex, since the computations must be done symbolically, and at present, the computation for sections of a cubic threefold seems too intense for Maple (at least for the authors' patience), even when the form of the equation can be simplified using coordinate changes in certain characteristics. For example, in characteristic zero, the equation of a general cubic threefold can be written in the form

$$x_0^3 + x_0\left(\sum_{i=1}^4 a_i x_i^2\right) + g(x_1, x_2, x_3, x_4)$$

where g is a cubic form in four variables and the a_i are constant.

References

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