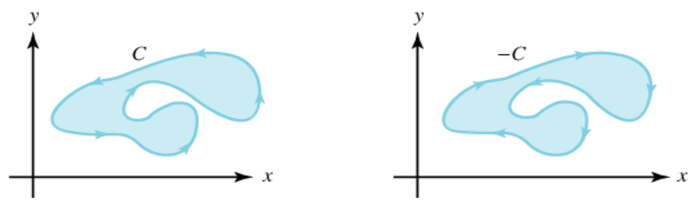
Handout 25

Definition: A simple closed curve is said to be **positive oriented** if it traversed **counterclockwise**.



(a) A positively oriented contour. (b) A negatively oriented contour.

Green's Theorem: Let C be positively oriented piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\,dA$$

Note: The circle on the line integral is sometime related to the positive oriented curve and sometime even drawn with an arrow on the circle.

One views the Green's theorem as a counterpart of Fundamental Theorem of Calculus $\left(\int_a^b F'(x)dx = F(b) - F(a)\right)$ for double integral, i.e. $\iint_D \tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\,dA = \oint_{\partial D} P\,dx + Q\,dy$.

Theorem: Let D be a domain. Rewrite D as union of subdomains, e.g. $D = D_1 \cup D_2$, let $\partial D = C_1 \cup C_2$ and $C_3 = D_1 \cap D_2$, i.e. $D_1 = C_1 \cup C_3$ and $D_2 = C_2 \cup (-C_3)$.

Now $\iint_D \tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\,dA = \oint_{C_1 \cup C_3} P\,dx + Q\,dy + \oint_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy = \oint_{C_1 \cup C_2} P\,dx + Q\,dy$

Note: One use the theorem above to use Greens theorem for domain with holes.

Definition: Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on $\mathbb{R}^3$ and partial derivatives of P,Q,R all exists, then

- the curl of $\mathbf{F}$ is defined as

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{d}{dx} & \dfrac{d}{dy} & \dfrac{d}{dz} \\ P & Q & R \end{pmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

- the divergence of $\mathbf{F}$ is defined as

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Theorem: Suppose f(x,y,z) has continuous second-order partial derivatives, then $\operatorname{curl}(\nabla f) = 0$.

Theorem: If $\mathbf{F}$ is vector field defined on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

Theorem: Suppose $\mathbf{F}=\langle P,Q,R\rangle$ is a vector field on $\mathbb{R}^3$ and has continuous second-order partial derivatives, then $\operatorname{div}\operatorname{curl}\mathbf{F}=0$.

Green's Theorem (vector form): Let $\mathbf{F}=\langle P,Q,0\rangle=P\mathbf{i}+Q\mathbf{j}+0\mathbf{k}$, then $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_D(\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA$.

Theorem: Let $\vec{r}(t)=\langle x(t),y(t)\rangle$ then $\mathbf{T}(t)=\dfrac{\langle x'(t),y'(t)\rangle}{|\vec{r}'(t)|}$ and $\mathbf{n}(t)=\dfrac{\langle y'(t),-x'(t)\rangle}{|\vec{r}'(t)|}=\mathbf{N}=\dfrac{\mathbf{T}'}{|\mathbf{T}'|}$.

Theorem: Let $\mathbf{F}=\langle P,Q\rangle$ be a vector field and $\vec{r}'(t)=\langle x'(t),y'(t)\rangle$, then $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds=\iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$

Definition: Let S be a surface $\vec{r}(u,v)=\langle x(u,v),y(u,v),z(u,v)\rangle, \qquad (u,v)\in D$.

For a **rectangular *D*** the surface integral is defined by $\oiint_S f(x,y,z)\,dS=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(P_{ij}^*\right)\Delta S_{ij}$

where P_{ij}^* is a sample point on the patch S_{ij} which area is $\Delta S_{ij}\approx|r_u\times r_v|\Delta u\Delta v$.

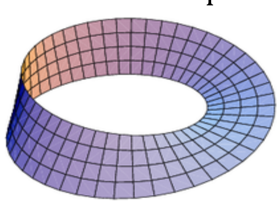
For a **non rectangular domain** $\oiint_S f(x,y,z)\,dS=\iint_D f(r(u,v))|r_u\times r_v|\,dA$

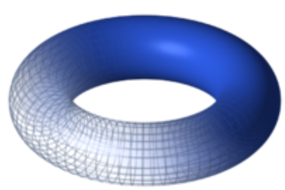
Example: for $z=g(x,y)$ we have $\iint_S f(x,y,z)\,dS=\iint_D f(x,y,g(x,y))\sqrt{(g_x)^2+(g_y)^2+1}\,dA$

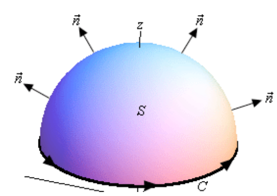
Definition: A two-sided surface *S* is called oriented if the unit normal vector **n** is defined at every point (except the boundary points). The orientation is chosen by direction (positive or negative) of the unit normal, in other words by choosing the side.

Example: A Mobius strip is one-sided surface, therefore non orientable. A torus is two-sided, so it can be oriented inward or outward.

Definition: A closed surface is a boundary of solid region. A closed surface is considered positive oriented if the unit normal points outward.







Definition: Flux of **F** across surface S is defined by $\oiint_S \mathbf{F}\cdot d\mathbf{S}=\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS$. Note the difference between $d\mathbf{S}$ and dS and the similarity with $d\mathbf{r}$.

Corollary: $\oiint_S \mathbf{F}\cdot d\mathbf{S}=\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS=\oiint_S \mathbf{F}\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,dS=\iint_D\left(\mathbf{F}(\vec{r}(u,v))\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right)|\vec{r}_u\times\vec{r}_v|\,du\,dv=\iint_D \mathbf{F}\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$

Examples:

1) For z=g(x,y) one get $\oiint_S \mathbf{F}\cdot d\mathbf{S}=\iint_D\langle P,Q,R\rangle\cdot\langle -g_x,-g_y,1\rangle\,dA=\iint_D -Pg_x-Qg_y+R\,dA$

2) Rate of flow of a fluid with density ρ and velocity field $\vec{v}$ is given by $\oiint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS$.

3) Electric flux of an electric field **E** through the surface *S* is given by $\oiint_S \mathbf{E}\cdot d\mathbf{S}$.

Furthermore, **Gauss's Law** says that a net charge enclosed by a closed surface *S* is $Q=\varepsilon_0\oiint_S \mathbf{E}\cdot d\mathbf{S}$ where ε_0 is a permittivity of free space.

4) Given a conductivity constant K of a substance and a temperature of a body given by $u(x,y,z)$, the rate of heat flow is given by $-K\nabla u$ and the rate of heat flow by $-K\oiint_S \nabla u\cdot d\mathbf{S}$.

Stoke's theorem (a sort of higher dimensional version of Green's theorem): Let *S* be am oriented piecewise-smooth surface that is bounded by a simple closed piecewise-smooth boundary curve *C* with positive orientation. Let **F** be a vector field whole components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains *S*. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oiint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Note: The when *S* is a special degenerated surface, a curve in xy-plane with upward orientation, then $\mathbf{n}=\mathbf{k}$ and Stoke's theorem become a vector form of Green's theorem.

One of the important uses of Stoke's Theorem is in calculating surface integrals over "non convenient" surface using surface integral over more convenient surface with the same boundary:

$$\oiint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}=\oint_{\partial S}\mathbf{F}\cdot d\mathbf{r}=\oiint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$$

Recall: $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds=\iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$

The Divergence Theorem: Let *E* be a simple solid region and let *S* be the boundary surface of *E*, given with positive (outward) orientation. Let **F** be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\oiint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_E \operatorname{div}\mathbf{F}\,dV$$