Handout 23(22a)

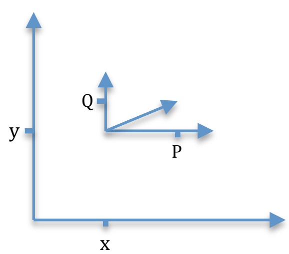
Definition: Let D be a subset of $\mathbb{R}^2/\mathbb{R}^3$. A **vector field** in $\mathbb{R}^2/\mathbb{R}^3$ is a function **F** that assign to each point (x,y)/(x,y,z) a 2/3 dimensional vector **F(**x,y)/ **F(**x,y,z).

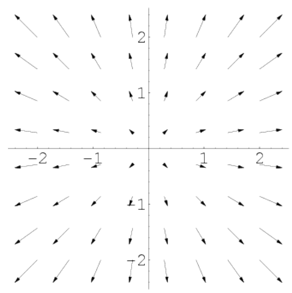
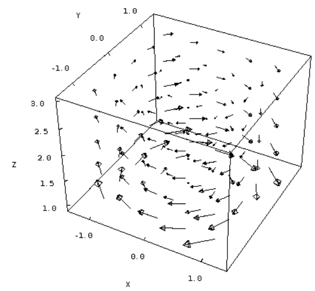
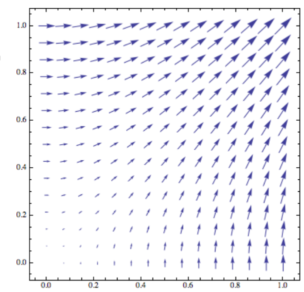
$$\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle = P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$$

$$\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle = P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$$

Note: In past we defined vector valued functions $\mathbf{r}:\mathbb{R}\to\mathbb{R}^n$ (n=2,3) and parametric surfaces $\mathbf{r}:\mathbb{R}^2\to\mathbb{R}^3$. A vector field is a function $\mathbf{F}:\mathbb{R}^n\to\mathbb{R}^n$ (with n=2 or n=3 in this course), i.e. is just a special (and very usable) case of vector-valued function.

We draw of vector field as a vector which starts at (x,y)\(x,y,z) and points into direction of the vector $\mathbf{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$ / $\mathbf{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$.



Line Integrals

Recall: A curve described by $\vec{r}(t)=\langle x(t),y(t)\rangle$ for $t\in[a,b]$ is a smooth curve, if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t)\neq 0$.

Note: For a function f(x,y) defined on all points of the curve *C* described $\vec{r}(t)=\langle x(t),y(t)\rangle$. Given a sample point (x_j^*,y_j^*) on an arc s_j (between $\vec{r}(t_{j-1})=\langle x_{j-1},y_{j-1}\rangle$ and $\vec{r}(t_j)=\langle x_j,y_j\rangle$) of the curve *C* between points with length Δs_j.

The following sum looks very similar to Reimann sum: $\sum_{j=1}^{n} f(x_j^*,y_j^*)\Delta s_j$.

Definition: If f is defined on a smooth curve C given by $\vec{r}(t)=\langle x(t),y(t)\rangle$ or $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$, then the line\contour\path\curve integral is defined by:

$$\int_C f(x,y)ds=\oint_C f(x,y)ds=\lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*,y_j^*)\Delta s_j \text{ or } \oint_C f(x,y,z)ds=\lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*,y_j^*,z_j^*)\Delta s_j$$ if the limit exists.

Reminder: The length of the curve C is given by

$$L=\int_a^b\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt \quad\text{or}\quad L=\int_a^b\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2+\left(\tfrac{dz}{dt}\right)^2}\,dt$$

Thm: If f is continues then

$$\oint_C f(x,y)ds = \int_a^b f(x(t),y(t))\frac{ds}{dt}dt = \int_a^b f(x(t),y(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt \quad \text{or}$$

$$\oint_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\frac{ds}{dt}dt = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2+\left(\tfrac{dz}{dt}\right)^2}\,dt$$

regardless of the parameterization as long as the curve traversed exactly once between a and b, i.e. it doesn't wind circles.

Note: One can rewrite the formulas above as $L=\int_a^b |\vec{r}\,'(t)|dt$ and $\int_C f(x,y,z)ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|dt$

Definition: Given $C_j, 1\le j\le n$ are smooth curves the curve $C = C_1 \cup C_2 \cup \ldots \cup C_n$ called piecewise smooth.

Theorem: For a piecewise smooth $C = C_1 \cup C_2 \cup \ldots \cup C_n$ we have $\oint_C f(\vec{r}(t))ds = \sum_{j=1}^{n}\oint_{C_j} f(\vec{r}(t))ds$.

Recall: The formula of the line segment that starts at point r_0 and ends at point r_1:

$$\vec{r}(t) = (1-t)r_0 + tr_1 \text{ where } 0\le t\le 1.$$

Theorem: $\oint_C f(x,y)ds = \oint_{-C} f(x,y)ds$ where C and $-C$ is the same curve with different direction.

Similarly $\oint_C f(x,y,z)ds = \oint_{-C} f(x,y,z)ds$.

Definition: The line integral $\oint_C f(x,y)ds$ is called a line integral with respect to arc length.

There is also a line integral with respect to x $\oint_C f(x,y)dx = \oint_C f(x(t),y(t))x'(t)dx$ and a line integral with respect to y $\oint_C f(x,y)dy = \oint_C f(x(t),y(t))y'(t)dy$. They are often occur together:

$$\oint_C f(x,y)dx + \oint_C g(x,y)dy = \oint_C f(x,y)dx + g(x,y)dy$$

Recall:

1) A work done by variable force f(x) in moving a particle from a to b along the x-axis is $W=\int_a^b f(x)dx$ (see section 6.6).
2) A work done by a constant force **F** in moving object from point P to point Q in space is $W = \mathbf{F}\cdot\overrightarrow{PQ}$ (example 6 section 9.3)

Theorem: The Work done by variable force $\mathbf{F}(x,y,z)$ along a smooth curve C is given by:

$W = \oint_C \mathbf{F}(x,y,z)\cdot\mathbf{T}(x,y,z)ds = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds$, furthermore:

$$W = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C\left(\mathbf{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)|\vec{r}\,'(t)|dt = \oint_C \mathbf{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt = \oint_C \mathbf{F}(\vec{r}(t))\cdot d\vec{r}(t) = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

Theorem: (A connection between line integrals of vector fields and the line integrals of scalar fields). Consider $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ and $\vec{r}(t) = \langle x(t),y(t),z(t)\rangle$ then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt = \int_a^b P(x,y,z)dx + Q(x,y,z)dy + Rv\,dz$$