Handout 21

Definition: For a rectangular-box domain $B=\{(x,y,z): x_1 \le x \le x_2, y_1 \le y \le y_2, z_1 \le z \le z_2\}$. We divide the domain into sub-boxes $B_{ijk}=[x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k]$ with volume $\Delta V=\Delta x\Delta y\Delta z$ and choose sample points $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$, the **triple Riemann sum** is given by:

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

Definition: **The triple integral** over $B=\{(x,y,z): x_1 \le x \le x_2, y_1 \le y \le y_2, z_1 \le z \le z_2\}$ is given by:

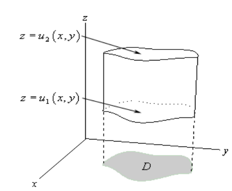
$$\iiint_B f(x,y,z)\,dV = \lim_{I,J,K\to\infty}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

Fubini's Theorem: If *f* is continuous on rectangular box $B=[x_1,x_2]\times[y_1,y_2]\times[z_1,z_2]$, then

$$\iiint_B f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx$$

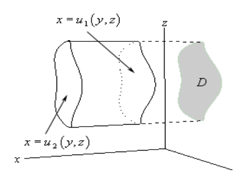
The above is the triple iterated integral, using this specific notations we first integrate in z, then in y and finally in x. However, the order can be changed; it has five different orders.

Definition: **The triple integral** over general shaped domain E:



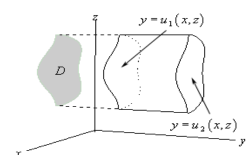
Type 1 $E=\{(x,y,z):(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\}$**:**

$$\iiint_B f(x,y,z)\,dV = \iint_D\left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)dA$$



Type 2 $E=\{(x,y,z):(y,z)\in D, u_1(y,z)\le x\le u_2(y,z)\}$**:**

$$\iiint_B f(x,y,z)\,dV = \iint_D\left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right)dA$$



Type 3 $E=\{(x,y,z):(x,z)\in D, u_1(x,z)\le y\le u_2(x,z)\}$**:**

$$\iiint_B f(x,y,z)\,dV = \iint_D\left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right)dA$$

The domain of the double integral D above can be of either **type I or II**. Note that D lies on different planes for **types 1,2,3** above, we will consider **type 1**, i.e. D is on xy plane, for the types 2 and 3 it is analogically. Thus, if D is of **type I** we have

$$E=\{(x,y,z): x_1\le x,\le x_2, g_1(x)\le y\le g_2(x), u_1(x,y)\le z\le u_2(x,y)\} \text{ and}$$

$$\iiint_B f(x,y,z)\,dV = \int_{x_1}^{x_2}\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Similarly, for D of **type II** we have

$$E=\{(x,y,z): y_1\le y,\le y_2, g_1(y)\le x\le g_2(y), u_1(x,y)\le z\le u_2(x,y)\} \text{ and}$$

$$\iiint_B f(x,y,z)\,dV = \int_{y_1}^{y_2}\int_{g_1(y)}^{g_2(y)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Cylindrical Coordinates: $(x,y,z)=(r\cos\theta, r\sin\theta, z)$

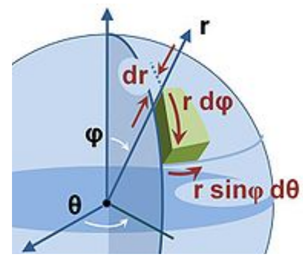
2) Let $D=\{(r,\theta):\alpha\le\theta\le\beta, h_1(\theta)\le r\le h_2(\theta)\}$, then $\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Consider $E=\{(x,y,z):(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\}$ where

$$D=\{(r,\theta):\alpha\le\theta\le\beta, h_1(\theta)\le r\le h_2(\theta)\},$$

then $\iiint_E f(x,y,z)\,dV=\iint_D\left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)dA=\int_\alpha^\theta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta,r\sin\theta)}^{u_2(r\cos\theta,r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

Spherical Coordinates: $(x,y,z)=(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$



In spherical coordinates with domain
$E=\{(\rho,\theta,\varphi):\rho_1\le\rho\le\rho_2, \theta_1\le\theta\le\theta_2, \varphi_1\le\varphi\le\varphi_2\}$
we "change" the **boxes** into **spherical wedges** E_{ijk} which volume is approximated by rectangular box with volume $\Delta\rho\times(\rho_i\Delta\varphi)\times(\rho_i\sin\varphi_k\Delta\theta)=\rho_i^2\sin\varphi_k\Delta\rho\Delta\theta\Delta\varphi$. The quantities $\rho_i\Delta\varphi$ and $\rho_i\sin\varphi_k\Delta\theta$ are appropriate arcs of the wedge. Thus we get

$$\iiint_E f(x,y,z)\,dV=\int_{\varphi_1}^{\varphi_2}\int_{\theta_1}^{\theta_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

General coordinates: $(u,v)=(u(s,t),v(st,y))$, $(u,v,w)=(u(s,t,p),v(s,t,p),w(s,t,p))$

Definition: The **Jacobian** is given by determinant of matrix of partial derivatives:

$$J_{(u,v)}(s,t)=\frac{\partial(u,v)}{\partial(s,t)}=\det\begin{pmatrix}u_s & u_t\\ v_s & v_t\end{pmatrix}=u_s v_t-u_s v_t \text{ and similarly}$$

$$J_{(u,v,w)}(x,y,z)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\det\begin{pmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{pmatrix}$$

Theorem: $\iint_A f(u,v)\,du\,dv=\iint_B f(u(s,t),v(s,t))\left|\frac{\partial(u,v)}{\partial(s,t)}\right|ds\,dt$ and similarly

$$\iiint_A f(u,v,w)\,du\,dv\,dw=\iiint_B f(u(s,t,p),v(s,t,p),w(s,t,p))\left|\frac{\partial(u,v,w)}{\partial(s,t,p)}\right|ds\,dt\,dp$$

Theorem: If $J_{(u,v)}(x,y)\ne 0$ then the coordinate system $(x(u,v),y(u,v))$ exists and $J_{(x,y)}(u,v)=\dfrac{1}{J_{(u,v)}(x,y)}$. Similarly, if $J_{(u,v,w)}(x,y,z)\ne 0$ then the coordinate system $(x(u,v,w),y(u,v,w),z(u,v,w))$ exists and $J_{(x,y,z)}(u,v,w)=\dfrac{1}{J_{(u,v,w)}(x,y,z)}$.